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Plateau's Problem for Integral Currents

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Introduction

The Plateau's problem is a fundamental topic in geometric measure theory, it was named after the Belgian physicist *Joseph Plateau* (1801-1833) who was interested in the study of soap bubbles. The classical Plateau's problem aims to find a surface in $\mathbb{R}^3$ of minimal area which spans a given curve Γ. But today, we can consider a more general case, that is given an $M-1$-dimensional manifold Γ in an M-dimensional Riemannian manifold $\mathcal{M}^N(M<N)$, find an M-dimensional surface $\Sigma \subset \mathcal{M}^N$ of minimal area such that $\partial\Sigma = \Gamma$.

Over the years, several approaches have been developed to solve Plateau's problem. The first one is the parametrized approach, which was developed by Garnier, Douglas and Radó in [14], [15] and [16]. The method is to use disk parameterizations. When dealing with a smooth simple loop Γ in $\mathbb{R}^3$, we try to represent the surface using functions f that map the unit disk in $\mathbb{R}^2$ to $\mathbb{R}^3$. The area of the surface is calculated by the integral of its Jacobian determinant. Douglas was able to prove the existence of a function f that minimizes this area by using the harmonic extension under certain regularity conditions. However, this approach has some drawbacks. The first one is that getting reasonably normalized parameterizations will be much harder for higher dimensional sets, thus making existence results in these dimensions much less likely. The second one is that many physical solutions of Plateau's problem are not parameterized by disks. And also the solutions obtained from this method may cross themselves in ways that are not seen in real world soap films, which means they don't accurately represent the physical situation, see page 116 in chapter 8 of [23] by G.David.

The second one is the set theoretical approach, which was developed by Reifenberg in [17]. Here, for an M-dimensional surface, the area is defined using the M-dimensional Hausdorff measure $\mathcal{H}^M$. A closed set E is considered to be bounded by a given $(M-1)$-dimensional set Γ based on homology conditions, that is requiring the homomorphism $i_* : H_{M-1}(\Gamma,\mathbb{Z}) \to H_{M-1}(E,\mathbb{Z})$ induced by the inclusion i to be trivial. This is a good framework to study soap bubbles, but there are still many open questions. For instance, we don't know much about the existence of solutions for other homologies and groups like $\mathbb{Z}$.

The third one is the distributional approach, which is also the one that is studied in this thesis. This approach was developed by Federer and Fleming [5] in the 1960s, where they invented a powerful tool: *Currents*. Currents are the dual of differential forms and have proven to be a natural framework for formulating extremal problems in geometry. Let $\mathbb{R}^N$ be our ambient space, an M-dimensional ($M \leq N$) current T is a linear functional on $\mathcal{D}^M(\mathbb{R}^N)$, the space of M-dimensional differential forms with compact support. The boundary ∂T of T is defined by

$$\partial T(\omega) = T(d\omega) \quad \forall \omega \in \mathcal{D}^M(\mathbb{R}^N).$$

Then Plateau's problem is to find an M-current with minimal "area" (here the notion of area will be the *mass*) such that $\partial T = S$ for some given $M-1$-dimensional current S. A natural idea to prove the existence of solutions to the Plateau's problem is to use the direct method from calculus of variation. Due to the definition of mass

$$M(T) = \sup_{\substack{\omega \in \mathcal{D}^M(\mathbb{R}^N) \\ |\omega| \leq 1}} T(\omega),$$

the lower semicontinuity of mass is obvious. Then by using the Banach-Alaoglu theorem, the existence of solutions to the Plateau' problem is obtained. However, general currents don't have that much geometric information, more precisely, they are too far away from smooth surfaces of $\mathbb{R}^N$. In this case, we have to find a class of currents that are closer to these surfaces. The right objects will be *Integral currents*.

Integral currents are Integer-Multiplicity (rectifiable) M-currents with finite mass and finite boundary mass. Roughly speaking, integral currents are the countable union of "pieces" of C^1-manifolds with integer multiplicity. Let U be an open set of $\mathbb{R}^N$, T is an Integer-Multiplicity (rectifiable) M-current if there exist S, θ, ξ such that

$$T(\omega) = \int_S \langle \omega(x), \xi(x) \rangle \theta(x) d\mathcal{H}^M(x) \quad \forall \omega \in \mathcal{D}^M(U),$$

where

1. S is a $\mathcal{H}^M$-measurable and M-rectifiable subset of U with $\mathcal{H}^M(S \cap K) < +\infty$ for all $K \subseteq U$ compact;

2. θ is a locally $\mathcal{H}^M$-integrable, nonnegative, integer-valued function;

3. $\xi : S \to \bigwedge_M \mathbb{R}^N$ is a $\mathcal{H}^M$-measurable function such that for $\mathcal{H}^M$-almost every point $x \in S$, $\xi(x)$ is a simple unit M-vector on the approximate tangent space $T_x S$ of S.

Our goal is to prove the compactness theorem of $I_M(\mathbb{R}^N)$, the space of Integral currents, which is stated as the following: Let $\{T_j\} \subset \mathcal{D}_M(\mathbb{R}^N)$ be a sequence of uniformly bounded Integral currents, Then there is an Integral current $T \in \mathcal{D}_M(\mathbb{R}^N)$ and a subsequence $\{T_{j'}\}$ such that $T_j \to T$ weakly. Using again the direct method, we obtain the existence of solutions to the Plateau's problem. The proof of the compactness theorem is complicated and will be divided into several steps.

The first step will be the deformation theorem, which is one of the fundamental results of the theory of currents. It provides a useful approximation of a current T by a polyhedral chain P lying on a certain M-skeleton such that the error is of the form $T - P = \partial R + S$. The main error term is ∂R, where R is the $(M+1)$-dimensional surface through which T is deformed to P. The other error term S arises in moving ∂T into the skeleton, this is called the weak polyhedral approximation. The isoperimetric inequality is also an important result yielded by the deformation theorem.

The next step of the proof of the compactness theorem will be an induction argument on the dimension of the currents. In the case $M = 0$, the compactness is just a result of Bolzano-Weierstrass theorem. Now we assume the compactness theorem is valid for the dimension of $M-1$, then we can use the induction assumption and the weak polyhedral approximation to get that every ∂T_j is an Integral current (this result is called the *boundary rectifiability theorem*), and so is ∂T. Then by *Homotopy Formula*, we can assume that $\partial T = 0$. Finally, using some preliminary lemmas such as the *Density Lemma* and the *Rectifiability Criterion*, we can conclude that $T \in \mathcal{D}_M(\mathbb{R}^N)$ is indeed an Integral current.

The Integral currents approach has the advantage of providing existence results in all dimensions, and solutions are very regular away from a small singular set. See [18], [19], [20], [21], [22], [24] and [25]. However, mass-minimizers may not be a perfect model for soap films as the mass may not be the right notion of area, and some soap films with interior singularities cannot be described by mass-minimizers. Also, the fact that the notion of current inherently involves an orientation is problematic for certain examples, such as Möbius films.

The thesis will focus on the theory of current in Euclidean spaces, but in fact, currents could be generalized to metric spaces. In the 1990s, De Giorgi's paper [13] formulated a generalized Plateau's problem in any metric space E using only the metric structure, having done so, he raised some natural questions about the existence of solutions to the generalized Plateau problem in metric or in Banach and Hilbert spaces. See also [26]. In metric spaces, the concept of currents is extended in a more abstract way. Since metric spaces lack a differentiable structure, currents are no longer defined as the dual of differential forms. Instead, following De Giorgi's approach, currents are defined in terms of metric functionals. Metric functionals are functions T defined on $(M+1)$-tuples $\omega = (f_0, f_1, \cdots, f_M)$, where M is the dimension, f_i are Lipschitz functions in the metric space E, and f_0 is also bounded, the space of these $(M+1)$-tuples is denoted by $\mathcal{D}_M(E)$. Then, an M-currents T is a function $T : \mathcal{D}_M(E) \to \mathbb{R}$ satisfying the following three conditions:

1) T is $(M+1)$-linear;

2) continuity with respect to pointwise convergence in the last M arguments with uniform Lipschitz bounds;

3) locality, that is $T(f_0, f_1, \cdots, f_M) = 0$ whenever some f_i $(i \neq 0)$ is a constant on a neighborhood of the support of f_0.

The mass of a current T denoted by $||T||$ in this context is defined as the least measure μ satisfying

$$|T(f_0, f_1, \cdots, f_M)| \leq \prod_{i=1}^{M} \mathrm{Lip}(f_i) \int_E |f_0| d\mu.$$

In metric spaces, the class of rectifiable currents $\mathcal{R}_M(E)$ can be defined as

$$\mathcal{R}_M(E) = \{T : ||T|| \ll \mathcal{H}^M \text{ and is concentrated on a countably } \mathcal{H}^M\text{-rectifiable set}\},$$

and the class of Integer-Multiplicity rectifiable currents $\mathcal{I}_k(E)$ is defined based on the property that the pushforward $\varphi_\#(T \lfloor A) \in \mathcal{D}_M(\mathbb{R}^M)$ has Integer-Multiplicity for any Borel set $A \subset E$, and Lipschitz map $\varphi : E \to \mathbb{R}^M$. Similar to the Euclidean case, the pushforward is defined by

$$\varphi_\# T(f_0, f_1, \cdots, f_M) = T(\varphi \circ f_0, f_1 \circ \varphi, \cdots, f_M \circ \varphi),$$

and the boundary of T is defined by

$$\partial T(\omega) = T(d\omega) \quad \forall \omega \in \mathcal{D}^M(E),$$

where $d\omega$ is the exterior differential defined by

$$d\omega = d(f_0, f_1, \cdots, f_M) := (1.f_0, f_1, \cdots, f_M) \in \mathcal{D}_{M+1}(E).$$

In [11], Ambrosio and Kirchheim proved that the closure theorem and boundary rectifiability theorem for Integer-Multiplicity rectifiable currents hold in any complete metric space, which is a significant result as it shows that these are general phenomena independent of the Euclidean-like homogeneous structure. Finally, in chapter 8 and chapter 10 of [11], the existence of solutions to the generalized Plateau's problem

$$\min\{||T||(E) : T \in \mathcal{I}_{M+1}(E), \partial T = S\}$$

was proven.

The theory of currents remains an active area of research. As we mentioned before, integral currents can be approximated by *polyhedral chains*. A natural question arises: *Can integral currents be approximated by smooth manifolds?* A recent work [27] by De Lellis and his collaborators provides an answer: Each integral cycle T (integral current with $\partial T = 0$) in a Riemannian manifold $\mathcal{M}$ can be approximated by an integral cycle in the same homology class which is a smooth submanifold Σ of nearly the same area, up to a singular set of codimension 5. Moreover, if the homology class τ is representable by a smooth submanifold (there exists a smooth embedding $f : \Sigma \to \mathcal{M}$ such that the fundamental class of Σ equals τ), then Σ can be chosen free of singularities.

Assume $N, M \in \mathbb{N}^+$ are positive integers, $\mathcal{M}$ is a connected smooth oriented closed Riemannian manifold of dimension $M+N$, τ is a nonzero element of the M-dimensional integral homology group $H_M(\mathcal{M}, \mathbb{Z})$ and T is an integral current (hence a cycle) representing τ. Then there is a sequence of smooth triangulations $\mathcal{K}_j$ of $\mathcal{M}$ and a sequence of smooth embedded oriented M-dimensional submanifolds $(\Sigma_j)_j$ in $\mathcal{M} \setminus \mathcal{K}_j^{M-5}$ such that

1. $[|\Sigma_j|] \to T$ in the sense of currents,
2. $\lim_{j\to\infty} \mathcal{H}^M(\Sigma_j) = \mathbb{M}(T)$,
3. $\partial[|\Sigma_j|] = 0$ and $[|\Sigma_j|]$ is in the same homology class as T.

This theorem provides a stronger approximation result than polyhedral chains.

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I also appreciate our department for accepting me into the Master's program. Since my bachelor's degree was not in pure mathematics, I lacked foundational knowledge. However, the professors were patient and supportive, guiding me through my first year. Moreover, the department also provided me with the opportunity to participate in the Erasmus program at the University of Helsinki. These experiences greatly contributed to my academic growth over the past two years.

Chapter 1

Preliminary Tools

1.1 Analytical Tools

Denote the set of continuous functions with compact support in $\mathbb{R}^N$ by

$$C_c(\mathbb{R}^N) = \{f \in C(\mathbb{R}^n) : \operatorname{supp} f \subset \mathbb{R}^N \text{ compact}\}$$

where $C(\mathbb{R}^N)$ is the space of continuous functions $f : \mathbb{R}^N \to \mathbb{R}$ and

$$\operatorname{supp} f = \overline{\{x \in \mathbb{R}^N : f(x) \neq 0\}}$$

is the support of f. Because all elements of $C(\mathbb{R}^N)$ are bounded functions, we may equip $C(\mathbb{R}^N)$ with the norm

$$\|f\| = \sup_{x \in \mathbb{R}^N} |f(x)|.$$

Linear functionals on $C_c(\mathbb{R}^N)$ are described by the following

Theorem 1.1.1 (Riesz Representation Theorem). *Let $L : C_c(\mathbb{R}^N) \to \mathbb{R}$ be a linear functional satisfying*

$$M = \sup\left\{|L(\phi)| : \phi \in C_c(\mathbb{R}^N), \sup_{x \in \mathbb{R}^N} |\phi(x)| \leq 1\right\} < \infty.$$

Then there exists a Radon measure λ on $\mathbb{R}^N$ and a λ-measurable function $g : \mathbb{R}^N \to \mathbb{R}$ such that
1. $\lambda(\mathbb{R}^N) = M$, 2. $L(\phi) = \int_{\mathbb{R}^N} \phi g d\lambda$, for all $\phi \in C_c(\mathbb{R}^N)$.

One can find the proof on page 116 of [1].
We call φ a *mollifier* if

- $\varphi \in C^\infty(\mathbb{R}^N)$;
- $\varphi \geq 0$;
- $\operatorname{supp} \varphi \subseteq B(0, 1)$;
- $\displaystyle\int_{\mathbb{R}^N} \varphi(x) dx = 1$;
- $\varphi(x) = \varphi(-x)$.

For $\sigma > 0$ we set $\varphi_\sigma(x) = \sigma^{-N}\varphi(x/\sigma)$ and we call $\{\varphi_\sigma\}_{\sigma>0}$ a family of mollifiers. In case $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $\sigma > 0$, we define

$$f_\sigma(x) = f * \varphi_\sigma(x) = \int_{\mathbb{R}^N} f(z)\varphi_\sigma(x-z)dz = \int_{\mathbb{R}^N} f(x-z)\varphi_\sigma(z)dz.$$

Theorem 1.1.2. *We have $f_\sigma \in C^\infty$ and f_σ converges to f as $\sigma \to 0^+$ in the following senses:*

- *$f_\sigma \to f$ pointwise almost everywhere;*
- *$f_\sigma \to f$ in the L^1_{loc} topology;*
- *If f is continuous then f_σ converges uniformly on compact sets to f.*

The reference[7] contains details of these assertions.

Next, we will introduce the notion of weak topology, which is very important in functional analysis.

Let V be a normed space, the space of continuous linear functional on V is denoted by V^*, it is equipped with the operator norm. A sequence $(T_n) \subset V^*$ is said to converge weakly-$*$ to $T \in V^*$ if

$$T_n(v) \to T(v) \quad \text{for all } v \in V.$$

In this case, we use the notation: $T_n \stackrel{*}{\rightharpoonup} T$.

The proof of the following propsition is easy:

Proposition 1.1.1. *Let $T, T_n \in V^*$, $n \in \mathbb{N}$. Then*

1. *If $T_n \to T$, then $T_n \stackrel{*}{\rightharpoonup} T$.*
2. *If $T_n \stackrel{*}{\rightharpoonup} T$, then $\|T\| \le \liminf_{n\to\infty} \|T_n\|$.*

The most important fact about the weak-$*$ topology is the following compactness result:

Theorem 1.1.3 (Banach-Alaoglu)**.** *Let V be a separable normed space and $(T_n) \subset V^*$ a sequence satisfying*

$$\sup_{n\in\mathbb{N}} \|T_n\| < \infty.$$

Then there exists a subsequence (T_{n_k}) and $T \in V^*$ such that

$$T_{n_k} \stackrel{*}{\rightharpoonup} T.$$

For the proof, see Theorem 3.17 in [9].

Next we will introduce the BV functions, let $U \subset \mathbb{R}^N$ be an open set and $u \in L^1(U)$. We can define

$$\int_U |Du| := \sup_{|g(x)|\le 1} \left\{ \int_U u \operatorname{div} g dx : g = (g_1, \cdots, g_N) \in C^1(U, \mathbb{R}^N), \operatorname{supp} g \subset U \right\}.$$

Then u is said to have bounded variation in U(u is a BV function in U) if $\int_U |Du| < \infty$, and the space of these functions is denoted by $BV(U) := \{u \in L^1(U) : \int_U |Du| < \infty\}$.

Moreover, the total variation measure $|Du|$ of u is defined by

$$|Du|(A) = \int_A |D(u)|$$

for $A \subset \mathbb{R}^N$ open. We also have the local version which is $BV_{loc}(U) := \{u \in L^1_{loc}(U) : \int_{U'} |Du| < \infty \quad for\ U' \subset\subset U\}$.

The space $BV(U)$ equipped with the BV norm

$$||u||_{BV} := ||u||_{L^1(U)} + \int_U |Du|$$

is a Banach space.

Theorem 1.1.4 (Compactness theorem for BV functions). *Let $U \subset \mathbb{R}^N$ be open, bounded with Lipschitz boundary and assume $\{f_k\}_{k=1}^{+\infty}$ is a sequence in $BV(U)$ such that $||f_k||_{BV} < +\infty$. Then there exists a subsequence $\{f_{k_j}\}_{j=1}^{+\infty}$ and a function $f \in BV(U)$ such that*

$$f_{k_j} \to f \ \text{in}\ L^1(U).$$

One can find the content of BV functions and this theorem in [10]. The next theorem gives us some information on smooth approximations to BV functions.

Theorem 1.1.5. *Let $\Omega \subset \mathbb{R}^N$ be open and $f \in BV(\Omega)$. Then there exists a sequence of functions $f_1, f_2, \ldots$ in $C^\infty(\Omega)$ such that*

1. $f_i \to f$ *in* $L^1(\Omega)$,
2. $\int_\Omega |Df_i| \to \int_\Omega |Df|$,
3. $Df_i \to Df$.

For the proof, see Theorem 3.6.12 in[10].

Next we introduce the Poincaré Inequalities. We begin with a version for smooth functions. Let $\mathcal{L}^N$ denote the standard N-dimensional Lebesgue measure. If f is a Lebesgue measurable function and U is a subset of the domain of f such that $\mathcal{L}^N(U)$. Then the average of f over U is defined by

$$f_U = \frac{1}{\mathcal{L}^N(U)} \int_U f(t)dt. \tag{1.1}$$

Lemma 1.1.1. *Let U be a bounded, convex, open subset of $\mathbb{R}^N$. Let f be a continuously differentiable function on U. Then there is a constant $c = c(U)$ such that*

$$\int_U |f(t) - f_U(t)|\, dt \leq c \cdot \int_U |Df(t)|\, dt.$$

Next we wish to replace the average f_U in the statement of the lemma with a more arbitrary constant.

Lemma 1.1.2. *Let $\beta \in \mathbb{R}$ and $0 < \theta < 1$ be constants. Let f and U be as in Lemma 1.1.1, and let f_U be as in (1.1). Assume that*

$$\mathcal{L}^N\{x \in U : f(x) \geq \beta\} \geq \theta \mathcal{L}^N(U)$$

and

$$\mathcal{L}^N\{x \in U : f(x) \leq \beta\} \geq \theta \mathcal{L}^N(U).$$

Then there is a constant $C = C(\theta)$ such that

$$\int_U |f(x) - \beta|\, dx \leq \theta^{-1}(1+\theta) \cdot \int_U |f(x) - f_U|\, dx.$$

Theorem 1.1.6. *Let U be a bounded, convex, open subset of $\mathbb{R}^N$. Let β, θ be as in Lemma 1.1.2. Let f be a continuously differentiable function on U. Then*

$$\int_U |f(x) - \beta|\, dx \leq c \cdot \int_U |Df(x)|\, dx.$$

Theorem 1.1.7. *Let U be a bounded, convex, open subset of $\mathbb{R}^N$. Let β, θ be as in Lemma 1.1.2. Let $u \in BV(U)$, then*

$$\int_U |u - \beta|\, d\mathcal{L}^N \leq c \cdot \int_U |Du|.$$

Theorem 1.1.8. *Let $U \subseteq \mathbb{R}^N$ be a bounded, open, and convex domain. If $u \in BV_{loc}(\mathbb{R}^N)$ with* $\operatorname{supp} u \subseteq \overline{U}$*, then there is a constant $c = c(U)$ such that*

$$\int_{\mathbb{R}^N} |Du(x)|\, dx \leq c \cdot \left(\int_U |Du| + \int_U |u(x)|\, dx \right).$$

One can find the proof of these lemmas and theorems in section 5.5 of [1].

1.2 Algebraic Tools

Current are the dual of differential forms. To define differential forms, we need some exterior algebra. We first introduce the space of M-vectors in $\mathbb{R}^N$.

M-vectors are a kind of "products" of vectors. Given $v_1, v_2 \in \mathbb{R}^N$, a geometric interpretation of the 2-vector $v_1 \wedge v_2$ is the oriented parallelogram spanned by vectors v_1 and v_2. If $v_1 = \lambda v_2$ for some $\lambda \in \mathbb{R}$, then the parallelogram is degenerate, and we have $v_1 \wedge v_2 = 0$. Similarly, a 3-vector $v_1 \wedge v_2 \wedge v_3$ can be interpreted as an oriented parallelepiped spanned by vectors $v_1, v_2, v_3 \in \mathbb{R}^N$.

We generalize this observation:

1. Define an equivalence relation $\sim$ on

$$(\mathbb{R}^N)^M = \underbrace{\mathbb{R}^N \times \mathbb{R}^N \times \cdots \times \mathbb{R}^N}_{M-factors}$$

by requiring, for all $a \in \mathbb{R}$, $1 \leq i < j \leq m$ and $u_i \in \mathbb{R}^N$,

(a)

$$(u_1, ..., au_i, ..., u_j, ..., u_M) \sim (u_1, ..., u_i, ..., au_j, ..., u_M),$$

(b)

$$(u_1, ..., u_i, ..., u_j, ..., u_M) \sim (u_1, ..., u_i + au_j, ..., u_j, ..., u_M),$$

(c)
$$(u_1, ..., u_i, ..., u_j, ..., u_M) \sim (u_1, ..., -u_j, ..., u_i, ..., u_M).$$

Extend the resulting relation to be symmetric and transitive.

2. The equivalence class of $(u_1, u_2, ..., u_M)$ under $\sim$ is denoted by $u_1 \wedge u_2 \wedge \cdots \wedge u_M$. We call $u_1 \wedge u_2 \wedge \cdots \wedge u_M$ a simple M-vector, the symbol $\wedge$ is called exterior product (or wedge product).

3. On the vector space of formal linear combinations of simple M-vectors, we define the equivalence relation $\approx$ by extending the relation defined by requiring
   (a) $a(u_1 \wedge u_2 \wedge \cdots \wedge u_M) \approx (au_1) \wedge u_2 \wedge \cdots \wedge u_M$.
   (b) $(u_1 \wedge u_2 \wedge \cdots \wedge u_M) + (v_1 \wedge u_2 \wedge \cdots \wedge u_M) \approx (u_1 + v_1) \wedge u_2 \wedge \cdots \wedge u_M$.

4. The equivalence classes of formal linear combinations of simple M-vectors under $\approx$ are the M-vectors in $\mathbb{R}^N$. The vector space of M-vectors in $\mathbb{R}^N$ is denoted by $\bigwedge_M(\mathbb{R}^N)$, and one can observe that it is also the space of all linear combinations
$$\sum_{1 \le i_1 < \cdots < i_M \le N} a_{i_1 \dots i_M} e_{i_1} \wedge \cdots \wedge e_{i_M},$$
   where $a_{i_1 \dots i_M} \in \mathbb{R}$, and $\{e_1, \dots, e_N\}$ is the standard basis of $\mathbb{R}^N$.

5. The exterior product $\wedge$ defined by the following:
$$\wedge : \bigwedge_K(\mathbb{R}^N) \times \bigwedge_M(\mathbb{R}^N) \to \bigwedge_{K+M}(\mathbb{R}^N)$$
$$(u_1 \wedge \cdots \wedge u_K) \wedge (v_1 \wedge \cdots \wedge v_M) \mapsto u_K \wedge \cdots \wedge u_K \wedge v_1 \wedge \cdots \wedge v_M$$
   is an anticommutative, multilinear multiplication, and the exterior algebra of $\mathbb{R}^N$, denoted by $\bigwedge_*(\mathbb{R}^N)$, is the direct sum of $\bigwedge_i(\mathbb{R}^N)$, i.e.
$$\bigwedge_*(\mathbb{R}^N) = \bigwedge_0(\mathbb{R}^N) \oplus \bigwedge_1(\mathbb{R}^N) \oplus \cdots .$$

One can show that $\{e_{i_1} \wedge \cdots \wedge e_{i_M}\}$ is the basis of $\bigwedge_M(\mathbb{R}^N)$, and since it is defined by the strictly increasing sequences $i_1 < \cdots < i_M$, then
$$\dim \bigwedge_M(\mathbb{R}^N) = \binom{N}{M}.$$

If $M = N$, $e_1 \wedge \cdots \wedge e_N$ is the only basis vector, and therefore
$$\dim \bigwedge_N(\mathbb{R}^N) = 1.$$

So we identify $\bigwedge_N(\mathbb{R}^N) = \mathbb{R}$. Similarly, $\bigwedge_1(\mathbb{R}^N) = \text{span}(e_1, \dots, e_N) = \mathbb{R}^N$. We also define $\bigwedge_0(\mathbb{R}^N) = \mathbb{R}$ and $\bigwedge_K(\mathbb{R}^N) = \{0\}$ for $K > N$.

For $a, b, c \in \mathbb{R}, u, v \in \bigwedge_K(\mathbb{R}^N), w \in \bigwedge_M(\mathbb{R}^N)$, one can easily check that the exterior product has the following properties:

(a) Multilinearity:
$$(au + bv) \wedge cw = ac(u \wedge w) + bc(v \wedge w),$$

$$au \wedge (bv + cw) = ab(u \wedge v) + ac(u \wedge w),$$

(b) Associativity:

$$u \wedge (v \wedge w) = (u \wedge v) \wedge w,$$

(c) Anticommutativity:

$$u \wedge w = (-1)^{KM} v \wedge w$$

Since the M-vectors $e_{i_1} \wedge \cdots \wedge e_{i_M}$, $1 \leq i_1 < \cdots < i_M \leq N$, form a basis of $\bigwedge_M(\mathbb{R}^N)$, we may equip it with an inner product $\langle \cdot, \cdot \rangle$ such that these M-vectors form an orthonormal basis. This peoduct can defined in the following way:

Denote

$$\bigwedge(N, M) = \{(i_1, ..., i_M) \in \mathbb{N}^M : 1 \leq i_1 < ... < i_M \leq N\},$$

and $e_I = e_{i_1} \wedge ... \wedge e_{i_M}$ for $I = (i_1, ..., i_M) \in \bigwedge(N, M)$. Then for $a_I, b_I \in \mathbb{R}$, the inner product is defined by letting

$$\langle \sum_{I \in \bigwedge(N,M)} a_I e_I, \sum_{J \in \bigwedge(N,M)} b_J e_J \rangle = \sum_{I \in \bigwedge(N,M)} a_I b_I,$$

and the norm is defined by

$$|v| = \sqrt{\langle v, v \rangle},$$

for $v \in \bigwedge_M(\mathbb{R}^N)$.

If v is a simple M-vector, that is $v = v_1 \wedge \cdots \wedge v_M$, then

$$|v| = |v_1 \wedge \cdots \wedge v_M|$$

is the M-dimensional volume of the parallelepiped spanned by $v_1, \ldots, v_M$. In particular,

$$|v_1 \wedge \cdots \wedge v_M| = 0$$

if and only if $v_1, \ldots, v_M$ are linearly dependent.

Once we have M-vectors, we can define the M-covectors by the following way.

Let $\bigwedge^1(\mathbb{R}^N)$ denote the dual of $\mathbb{R}^N$ ($\bigwedge^1(\mathbb{R}^N) = (\mathbb{R}^N)^*$) and let $dx_1, \ldots, dx_N$ denote the dual basis of $e_1, \ldots, e_N$. That is,

$$dx_i(e_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Then we define the vector space

$$\bigwedge^M(\mathbb{R}^N) = \bigwedge_M \left(\bigwedge^1(\mathbb{R}^N) \right)$$

as above by replacing e_i with dx_i. The elements

$$\sum_{i_1 < \cdots < i_M} a_{i_1 \ldots i_M} dx_{i_1} \wedge \cdots \wedge dx_{i_M} = \sum_{I \in \bigwedge(N,M)} a_I dx_I$$

of $\bigwedge^M(\mathbb{R}^N)$ are called M-covectors. The space $\bigwedge^M(\mathbb{R}^N)$ has the induced inner product defined by

$$\left\langle \sum_{I\in\bigwedge(N,M)} a_I dx_I, \sum_{J\in\bigwedge(N,M)} b_J dx_J \right\rangle = \sum_{I\in\bigwedge(N,M)} a_I b_I,$$

such that the M-covectors $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_M}$, $1 \leq i_1 < \cdots < i_M \leq N$, form an orthonormal basis. We have the norm induced by this inner product.

$$|\omega| = \sqrt{\langle \omega, \omega \rangle},$$

for $\omega \in \bigwedge^M(\mathbb{R}^N)$.

$\bigwedge^M(\mathbb{R}^N)$ is the dual space of $\bigwedge_M(\mathbb{R}^N)$, $\bigwedge^0(\mathbb{R}^N) = \mathbb{R}$, $\bigwedge^1(\mathbb{R}^N) = \mathbb{R}^N$ and $\bigwedge^M(\mathbb{R}^N) = \{0\}$ if $M > N$.

Chapter 2

Currents in Euclidean space

2.1 Basic Facts

We start with the definition of an M-differential form.

Definition 2.1.1. Let $U \subset \mathbb{R}^N$ be an open set and $a_I(x)$ be a function. The mapping

$$\begin{aligned} \xi : U &\to \bigwedge\nolimits_M(\mathbb{R}^N) \\ x &\mapsto \sum_{I \in \bigwedge(N,M)} a_I(x) e_I \end{aligned}$$

is an M-vector field in U, and the mapping

$$\begin{aligned} \alpha : U &\to \bigwedge\nolimits^M(\mathbb{R}^N) \\ x &\mapsto \sum_{I \in \bigwedge(N,M)} a_I(x) dx_I \end{aligned}$$

is an M-differential form (or M-covector field) in U.

We also define the norms on the spaces of M-vector fields and M-differential forms by:

$$||\xi|| = \sup_{x \in U} \sqrt{\langle \xi(x), \xi(x) \rangle}$$

and

$$|\alpha| = \sup_{x \in U} \sqrt{\langle \alpha(x), \alpha(x) \rangle},$$

respectively.

If $U \subset \mathbb{R}^N$ is open and

$$\alpha = \sum_{I \in \bigwedge(N,M)} a_I(x) dx_I,$$

where the functions a_I are C^∞-smooth, we say that α is a C^∞-smooth differential M-form in U.

The space of all C^∞-smooth differential M-forms in U will be denoted by $\mathcal{E}^M(U)$.

Since $\bigwedge^0(\mathbb{R}^N) = \mathbb{R}$, we have $\mathcal{E}^0(U) = C^\infty(U, \mathbb{R})$. If $f : U \to \mathbb{R}$ is C^∞, $f \in \mathcal{E}^0(U)$, its differential $df : U \to \bigwedge^1(\mathbb{R}^N)$ is a C^∞ smooth differential 1-form such that at a point

$x \in U$, $df(x) : \mathbb{R}^N \to \mathbb{R}$ is the linear mapping defined by

$$df(x)v = \langle \nabla f(x), v \rangle, \quad v \in \mathbb{R}^N,$$

that is

$$df = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i} dx_i.$$

Moreover, dx_i is the differential of the i^{th} coordinate function $x \mapsto x_i$.

Definition 2.1.2. Let

$$\alpha = \sum_{I \in \bigwedge(N,M)} \alpha_I dx_I$$

be a C^∞-smooth differential M-form. The exterior derivative of α is the $(M+1)$-form

$$d\alpha = \sum_{I \in \bigwedge(N,M)} d\alpha_I \wedge dx_I = \sum_{I \in \bigwedge(N,M)} \sum_{i=j}^{N} \frac{\partial \alpha_I}{\partial x_j} dx_j \wedge dx_I.$$

In particular, df is the exterior derivative of a 0-form f.

Using the facts that

$$\frac{\partial^2 \alpha_I}{\partial x_i \partial x_j} = \frac{\partial^2 \alpha_I}{\partial x_j \partial x_i}$$

and $dx_i \wedge dx_j = -dx_j \wedge dx_i$, we obtain $d^2\alpha = d(d\alpha) = 0$.

Definition 2.1.3 (Pull-Back)**.** Let $U \subset \mathbb{R}^N$ and $V \subset \mathbb{R}^D$ be open sets and $f = (f^1, \dots, f^D) : U \to V$ a C^∞-smooth mapping. The pull-back of the differential M-form α in V,

$$\alpha = \sum_{1 \le i_1 < \cdots < i_M \le D} \alpha_{i_1 \cdots i_M} dx_{i_1} \wedge \cdots \wedge dx_{i_M},$$

is the differential M-form $f^*\alpha$ in U defined by

$$f^*\alpha = \sum_{1 \le i_1 < \cdots < i_M \le D} (\alpha_{i_1 \cdots i_M} \circ f) df^{i_1} \wedge \cdots \wedge df^{i_M},$$

where

$$df^j = \sum_{i=1}^{N} \frac{\partial f^j}{\partial x_i} dx_i.$$

Notice that we do not require α being smooth. The pull-back and the exterior derivative commute for smooth α, that is

$$f^*(d\alpha) = d(f^*\alpha).$$

For $U \subset \mathbb{R}^N$, let $\mathcal{D}^M(U) \subset \mathcal{E}^M(U)$ denote the space of all C^∞-smooth differential M-forms in U with compact support, that is, if

$$\alpha = \sum_{I \in \bigwedge(N,M)} \alpha_I dx_I,$$

then each α_I is C^∞-smooth and there exists a compact set $K \subset U$ such that $\operatorname{supp} \alpha_I \subset K$ for every I, i.e., $\alpha_I \in C_c^\infty(K)$.

The normed space $(\mathcal{D}^M(U), |\cdot|)$ is separable and the topology induced by the norm is different from the locally convex topology which is defined by the following.

We endow $\mathcal{D}^M(U)$ with the locally convex topology by saying that a sequence $\alpha^k \in \mathcal{D}^M(U)$, $k \in \mathbb{N}$,

$$\alpha^k = \sum_{I \in \bigwedge(N,M)} \alpha_I^k dx_I$$

converges to

$$\alpha = \sum_{I \in \bigwedge(N,M)} \alpha_I dx_I \in \mathcal{D}^M(U)$$

if there exists a compact set $K \subset U$ such that

$$\operatorname{supp} \alpha^k := \bigcup_{I \in \bigwedge(N,M)} \operatorname{supp} \alpha_I^k \subset K \quad \forall k$$

and

$$\frac{\partial^{|J|} \alpha_I^k}{\partial x_J} \to \frac{\partial^{|J|} \alpha_I}{\partial x_J}$$

uniformly as $k \to \infty$ for every multi-index $J = j_1 \cdots j_N$.

Once we have the differential forms, we could define the currents.

Definition 2.1.4. An M-current T in an open set $U \subseteq \mathbb{R}^N$ is a continuous(with respect to the locally convex topology) linear functional on on $\mathcal{D}^M(U)$:

$$T : \mathcal{D}^M(U) \to \mathbb{R}.$$

The space of M-currents in U is denoted by $\mathcal{D}_M(U)$, and the support of T is the set:

$$\operatorname{spt} T = U \setminus \bigcup \{\text{V: } V \subseteq \mathbb{R}^N \text{ open, } T(\omega) = 0 \ \forall \omega \in \mathcal{D}_M(U), \operatorname{supp} \omega \subseteq V\}$$

Each M-current $T \in \mathcal{D}_M(U)$ is continuous with respect to the locally convex topology, but not necessary continuous with respect to the norm topology of $\mathcal{D}^M(U)$.

Definition 2.1.5. If $M \geq 1$, the boundary of an M-current $T \in \mathcal{D}_M(U)$ is the $(M-1)$-current $\partial T \in \mathcal{D}_{M-1}(U)$ defined by

$$\partial T(\omega) = T(d\omega),$$

for all $\omega \in \mathcal{D}^{M-1}(U)$. Since $d^2 = 0$, we have $\partial^2 T = \partial(\partial T) = 0$, we also define $\partial T = 0$ for all $T \in \mathcal{D}_0(U)$.

Definition 2.1.6. Let $T \in \mathcal{D}_M(U)$, if $\phi \in \mathcal{E}^k(U)$ and $k \leq M$, then we can define

$$T \lfloor \phi \in \mathcal{D}_{M-k}(U)$$

by letting for all $\omega \in \mathcal{D}^{M-K}(U)$

$$(T \lfloor \phi)(\omega) = T(\phi \wedge \omega).$$

Now, let ξ be the a p-vector field with C^∞-coefficients on U. Then we define

$$T \wedge \xi \in D_{M+p}(U)$$

by letting

$$(T \wedge \xi)(\omega) = T(\xi \rfloor \omega)$$

for all $\omega \in \mathcal{D}^{M+p}(U)$, where $\xi \rfloor \omega \in \mathcal{D}_M(U)$ is the interior product, characterized by $\langle \xi \rfloor \omega, \alpha \rangle = \langle \omega, \alpha \wedge \xi \rangle$ for each $\alpha \in \bigwedge_M(\mathbb{R}^N)$.

Since T is a linear functional on $\mathcal{D}^M(U)$, we can define the partial derivatives of T in the sense of distribution.

Definition 2.1.7. Let $f \in C_c^\infty(\mathrm{U})$ and $T \in \mathcal{D}_M(U)$, the partial derivatives $D_{x_j}T \in \mathcal{D}_M(U)$ of T are defined by

$$D_{x_j}T(f dx_I) = -T[(D_{x_j} f) dx_I],$$

where $I \in \bigwedge(N, M)$, $1 \leq j \leq N$ and $D_{x_j} f$ is the classical partial derivative of the function f.

Proposition 2.1.1. *Suppose that ϕ and ξ have C^∞-coefficients on U, where ϕ is a k-form and ξ is a p-vector field. Then*

(1) $d(\partial T) = 0 \quad \text{if } \dim T \geq 2;$

(2) $(\partial T)\phi = T \lfloor d\phi + (-1)^k \partial(T \lfloor \phi);$

(3) $\partial T = -\sum_{j=1}^{N} (D_{x_j} T) \lfloor dx_j \quad \text{if } \dim T \geq 1;$

(4) $T = \sum_{J \in \bigwedge(N,M)} [T \lfloor dx_J] \wedge e_J;$

(5) $D_{x_j}(T \lfloor \phi) = (D_{x_j} T) \lfloor \phi + T \lfloor (\partial \phi / \partial x_j);$

(6) $D_{x_j}(T \wedge \xi) = (D_{x_j} T) \wedge \xi + T \wedge (\partial \xi / \partial x_j);$

(7) $(T \wedge \xi) \lfloor \phi = T \wedge (\xi \lfloor \phi) \quad \text{if } \dim T = 0 \text{ and } k \leq p;$

(8) $\partial(T \wedge \xi) = -T \wedge \operatorname{div} \xi - \sum_{j=1}^{N} (D_{x_j} T) \wedge (\xi \lfloor dx_j) \quad \text{if } \dim T = 0 \leq p.$

In the above, the partial derivatives $\partial \phi / \partial x_j$ of the form ϕ and $\partial \xi / \partial x_j$ of the vector field ξ are obtained by differentiating the coefficient functions and we say that $\dim T = M$ *if $T \in \mathcal{D}_M(U)$.*

One can easily verify the above proposition by linearity.

2.1.1 Currents Representable by Integration

We want figure out what kinds of currents could be represented by integration. Let $U \in \mathbb{R}^N$ be an open set, we start from 0-currents:

Lemma 2.1.1. *Let $T \in \mathcal{D}_0(U)$, if for each open set $W \subset\subset U$ there exists a positive real number $M < \infty$ such that*

$$|T(\phi)| \leq M ||\phi||_\infty \tag{2.1}$$

holds for all $\phi \in C_c^\infty(U)$, then there exists a total variation measure μ_T, such that

$$T(\phi) = \int_U \phi d\mu_T. \tag{2.2}$$

Proof. Since $C_c^\infty(U)$ is dense in $C_c(U)$, by Hahn-Banach Theorem. T can be extended to a continuous functional in $C_c(U)$. Then by Theorem 1.1.1, the lemma holds. $\square$

Next, we endow $\mathcal{D}_M(U)$ with the mass-norm:

Definition 2.1.8. Let $T \in \mathcal{D}_M(U)$. We define the mass of T on the open set U by

$$M(T) = \sup_{\substack{\omega \in \mathcal{D}^M(U) \\ |\omega| \leq 1}} T(\omega).$$

If $W \subseteq U$ is an open subset, then we have the local mass given by

$$M_W(T) = \sup_{\substack{|\omega| \leq 1, \omega \in \mathcal{D}^M(W) \\ \operatorname{supp} \omega \subset W}} T(\omega).$$

Since $(\mathcal{D}^M, |\cdot|)$ is a normed space, its dual space $\{T \in \mathcal{D}_M(U) : M(T) < \infty\}$ is a Banach space. Now, we can prove the representation theorem for M-currents:

Theorem 2.1.1 (Representation theorem)**.** *If $T \in \mathcal{D}_M(U)$ and $M_W(T) < \infty$ for all open $W \subset\subset U$, then there exists a Radon measure μ_T on U and a μ_T-measurable M-vector field $\vec{T} : U \to \bigwedge_M(\mathbb{R}^N)$ such that $|\vec{T}| = 1$ μ_T-almost everywhere such that*

$$T(\omega) = \int_U \langle \omega(x), \vec{T}(x) \rangle d\mu_T(x)$$

for all $\omega \in \mathcal{D}_M(U)$. Moreover, the measure μ_T, which we call the total variation measure associated with T, is characterized by the identity

$$\mu_T(W) = \sup_{\substack{|\omega| \leq 1, \omega \in \mathcal{D}^M(W) \\ \operatorname{supp} \omega \subset W}} T(\omega) = M_W(T)$$

in particular, $\mu_T(U) = M(T)$.

Proof. If $M_W(T) < \infty$ for all open $W \subset\subset U$, then, for each sequence $J \in \bigwedge(N, M)$ the 0-dimensional current $T \lfloor dx_J$ satisfies the condition (2.1) and thus defines a total variation measure μ_{j_J} and function f_J as in (2.2). Using the identity

$$T = \sum_{J \in \bigwedge(N,M)} [T \lfloor dx_J] \wedge e_J$$

together the total variation measures μ_{f_J} and functions $f_J e_J$ and normalizing the resulting function, we obtain the Radon measure μ_T and the μ_T-measurable vector field $\vec{T}$. $\square$

The total variation measure μ_T will also be denoted by $\|T\|$ and the vector field $\vec{T}$ is called the orientation function.

Definition 2.1.9 (Restrictions of currents)**.** If $T \in \mathcal{D}_M(U)$, $M(T) < \infty$, and $A \subseteq \mathbb{R}^N$ is Borel, then the restriction of T to A is the m-current $T\lfloor A \in \mathcal{D}_M(U)$,

$$(T\lfloor A)(\omega) = \int_A \langle \vec{T}(x), \omega(x)\rangle d\mu_T(x), \quad \omega \in \mathcal{D}^M(U),$$

where $\vec{T}$ and μ_T are as in the theorem above. Similarly, if g is a μ_T-integrable function, we also define $T\lfloor g \in \mathcal{D}_M(U)$, the interior multiplication by g, by

$$(T\lfloor g)(\omega) = \int_U g(x)\langle \vec{T}(x), \omega(x)\rangle d\mu_T(x), \quad \omega \in \mathcal{D}^M(U).$$

2.1.2 Plateau's Problem for Normal Currents

First, we say something about the topology induced by the mass.

Definition 2.1.10. A sequence $\{T_k\} \subset \mathcal{D}_M(U)$ is said to converge weakly to $T \in \mathcal{D}_M(U)$ if

$$T_k(\omega) \to T(\omega) \quad \text{for every } \omega \in \mathcal{D}^M(U), \quad \text{as } k \to +\infty.$$

We write $T_k \rightharpoonup T$.

Notice that if T, T_k have finite mass, then this is just the weak-$*$ convergence in the dual space $(\{T \in \mathcal{D}_M(U) : M(T) < \infty\}, M(\cdot))$.

A simple but important property is following:

Theorem 2.1.2. *(Lower semicontinuity of mass). If a sequence $\{T_k\} \subset \mathcal{D}_M(U)$ converges weakly to $T \in \mathcal{D}_M(U)$, then*

$$M(T) \leq \liminf_{k\to\infty} M(T_k).$$

Proof. For every $\omega \in D^M(U)$ with $|\omega| \leq 1$ we have

$$T(\omega) = \lim_{k\to\infty} T_k(\omega) \leq \liminf_{k\to\infty} M(T_k)$$

and hence $M(T) \leq \liminf_{k\to\infty} M(T_k)$. □

We can now solve Plateau's problem in a very weak sense.

Definition 2.1.11 (Normal currents)**.** Let $T \in \mathcal{D}_M(U)$, we define $N(T) = M(T) + M(\partial T)$, the space $\{T \in \mathcal{D}_M(U) : N(T) < \infty\}$ is denoted by $N_M(U)$ and elements in this space are called normal M-currents in U.

Theorem 2.1.3 (Plateau's problem for normal currents)**.** *Let $S \in N_M(U)$, then there exists $T \in N_M(U)$ such that $\partial T = \partial S$ and*

$$M(T) = \inf\{M(S') : S' \in N_M(U), \quad \partial S' = \partial S\}.$$

Proof. Let $\{T_k\} \subset N_M(U)$ be a mass minimizing sequence with $\partial T_k = \partial S$ for all $k \in \mathbb{N}$. Thus

$$M(T_k) \to L := \inf\{M(S') : S' \in N_M(U), \partial S' = \partial S\}.$$

By the Banach-Alaoglu Theorem 1.1.3, there exists a subsequence (T_{k_j}) and $T \in N_M(U)$ such that

$$T_{k_j} \rightharpoonup T.$$

Since $T_{k_j} \rightharpoonup T$, it follows that $\partial T_{k_j} \rightharpoonup \partial T$ and hence $\partial T = \partial S$, in particular, $T \in N_M(U)$. By the lower semi-continuity of mass, we have

$$M(T) \leq \liminf_{j \to \infty} M(T_{k_j}) = L.$$

$\square$

Thus T is a mass minimizing current with boundary ∂S. This theorem is not satisfying because normal M-currents are in general very far from M-dimensional submanifolds.

2.1.3 Association with Oriented Submanifold

Firstly, we fix some notation. For $S \geq 0$, we denote the S-dimensional Hausdorff measure by $\mathcal{H}^S$. Let $N \in \mathbb{N}$, if the N-dimensional Lebesgue measure is denoted by $\mathcal{L}^N$, then we have that $\mathcal{H}^N = \mathcal{L}^N$.

Not every normal M-current is associated with M-dimensional submanifold, see the following example:

Example 2.1.1. The 1-current on $\mathbb{R}^2$ given by

$$T(\omega) := \int_{[0,1]^2} \langle \omega, e_1 \rangle d\mathcal{L}^2(x)$$

Satisfies $M(T) = 1$ and $M(\partial T) = 2$ since

$$\partial T(f) = T(df) = \int_{[0,1]^2} \frac{\partial f}{\partial x}(x,y) dx dy = \int_0^1 \left[f(1,y) - f(0,y) \right] dy,$$

So $T \in N_1(\mathbb{R}^2)$, but this current is not associated with any 1-dimensional submanifold.

Actually, we can construct a class of currents that are associated with oriented submanifolds of $\mathbb{R}^N$. Suppose that S is a C^1 oriented M-dimensional submanifold. Here S being oriented means that for each point $x \in S$ there is a set of M orthonormal tangent vectors $\xi_1(x), \xi_2(x), ..., \xi_M(x)$ such that

$$\vec{S}(x) = \xi_1(x) \wedge \xi_2(x) \wedge ... \wedge \xi_M(x)$$

defines a continuous vector field $\vec{S} : S \to \bigwedge_M(\mathbb{R}^N)$. We define the current $[|S|] \in \mathcal{D}_M(\mathbb{R}^N)$ by setting

$$[|S|](\omega) = \int_S \langle \omega, \vec{S} \rangle d\mathcal{H}^M.$$

As a special case of this definition, we can take S to be a Lebesgue measurable subset of $\mathbb{R}^N$ and define

$$[|S|](\omega) = \int_S \langle \omega, e_1 \wedge e_2 \wedge ... \wedge e_N \rangle d\mathcal{L}^N,$$

for $\omega \in \mathcal{D}^N(\mathbb{R}^N)$.

In case S is an oriented submanifold with oriented boundary, the classical Stokes's theorem tells us that

$$[|S|](d\omega) = [|\partial_o S|](\omega),$$

where $\partial_o S$ is the oriented boundary of S. By the definition of the boundary of a current we have

$$[|S|](d\omega) = (\partial[|S|])(\omega).$$

Thus, we have $[|\partial_o S|] = \partial[|S|]$. The definition of boundary of a current is consistent with the classical definition of oriented boundary. We also observe that the mass generalizes the area of a submanifold:

$$M([|S|]) = \mathcal{H}^M(S).$$

2.2 Constancy Theorem

Treat $\mathcal{L}^N$ as the 0-current that gives the value $\int_U \phi d\mathcal{L}^N$ when applied to $\phi \in \mathcal{D}^0(\mathbb{R}^N)$. If ξ is an M-vector field with $\mathcal{L}^N$-measurable coefficients, satisfying

$$\int_K ||\xi|| d\mathcal{L}^N < \infty$$

for each compact subset $K \subseteq \mathbb{R}^N$, then there is a corresponding current $\mathcal{L}^N \wedge \xi \in \mathcal{D}_M(\mathbb{R}^N)$ given by

$$(\mathcal{L}^N \wedge \xi)(\psi) = \int_{\mathbb{R}^N} \langle \psi, \xi \rangle d\mathcal{L}^N \quad \text{for all } \psi \in \mathcal{D}^M(\mathbb{R}^N).$$

If $\phi \in \mathcal{E}^k(U)$, with $k \leq M$, $(\mathcal{L}^N \wedge \xi)\lfloor\phi \in \mathcal{D}_{M-k}(U)$ is given by

$$[(\mathcal{L}^N \wedge \xi)\lfloor\phi](\psi) = \int_{\mathbb{R}^N} \langle \phi \wedge \psi, \xi \rangle d\mathcal{L}^N$$

for $\psi \in \mathcal{D}^{M-k}(\mathbb{R}^N)$. We can also write this as $(\mathcal{L}^N \wedge \xi)\lfloor\phi = \mathcal{L}^N \wedge (\xi\lfloor\phi)$, where we define the interior product $\xi\lfloor\phi$ by requiring that $\langle \psi, \xi\lfloor\phi \rangle = \langle \phi \wedge \psi, \xi \rangle$.

If ξ has C^1 coefficients, then (using the fact that when $\mathcal{L}^N$ is treated as a current, all its partial derivatives vanish) we have

$$D_{x_j}(\mathcal{L}^N \wedge \xi) = \mathcal{L}^N \wedge (\partial\xi/\partial x_j)$$

and

$$\partial(\mathcal{L}^N \wedge \xi) = -\sum_{j=1}^N [D_{x_j}\mathcal{L}^N \wedge \xi]\lfloor dx_j = -\mathcal{L}^N \wedge \sum_{j=1}^N (\partial\xi/\partial x_j)\lfloor dx_j.$$

In case $M = 1$, in which case ξ is a 1-vector field, we see that

$$\sum_{j=1}^N (\partial\xi/\partial x_j)\lfloor dx_j = \operatorname{div}\xi. \tag{2.3}$$

Letting (2.3) define the divergence of an M-vector field for all $1 \leq M \leq N$, we have

$$\partial(\mathcal{L}^N \wedge \xi) = -\mathcal{L}^N \wedge \operatorname{div}\xi.$$

Let ξ be an M-vector field on U. We define the differential form $D_M\xi$ by setting

$$D_M\xi = \xi \rfloor (dx_1 \wedge \cdots \wedge dx_N).$$

The differential form $D_M\xi$ has degree $N - M$. Also, with each differential form ϕ of degree M on U we associate the $(N - M)$-vector field

$$\mathbf{D}^M\phi = (\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_N) \lfloor \phi.$$

If $\phi \in \mathcal{D}^{N-M}$ and $\psi \in \mathcal{D}^M$, then we see that

$$\begin{aligned}\int (\mathcal{L}^N \wedge \mathbf{D}^{N-M}\phi)(\psi) &= \int \langle \psi, \mathbf{D}^{N-M}\phi \rangle dL^N \\ &= \int \langle \phi \wedge \psi, \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_N \rangle dL^N.\end{aligned}$$

Define $\mathbf{E}^N \in \mathcal{D}_N(\mathbb{R}^N)$ by

$$\mathbf{E}^N = \mathcal{L}^N \wedge e_1 \wedge \cdots \wedge e_N$$

so, if $\phi \in \mathcal{D}^N(\mathbb{R}^N)$, then

$$\mathbf{E}^N(\phi) = \int \langle \phi(x), e_1 \wedge e_2 \wedge \cdots \wedge e_N \rangle d\mathcal{L}^N(x).$$

We see that

$$D_{x_j}\mathbf{E}^N = 0 \quad \text{for each } j = 1, \ldots, N \quad \text{and} \quad \partial \mathbf{E}^N = 0.$$

We also see that for any Lebesgue measurable set $A \subseteq \mathbb{R}^N$,

$$\mathbf{E}^N \lfloor A = [|A|].$$

If $T \in \mathcal{D}_N(U)$ and $j \in \{1, \ldots, N\}$, then, using the formula

$$\partial T = -\sum_{l=1}^{N} (D_{x_l}T) \lfloor dx_l$$

and the fact that $\bigwedge^{N+1} \mathbb{R}^N = 0$, we can calculate that

$$(\partial T) \wedge e_j = (-1)^N D_{x_j} T. \tag{2.4}$$

Thus the vanishing of the boundary of an N-dimensional current is equivalent to the vanishing of its partial derivatives. Accordingly, we expect that an N-dimensional current with vanishing boundary should be given by integration. That intuition is confirmed by the next proposition.

Theorem 2.2.1 (Constancy Theorem)**.** *If $T \in \mathcal{D}_N(U)$ with $\partial T = 0$ and if U is a connected open set, then there is a real number c such that*

$$T = c(\mathbf{E}^N \lfloor U) = c[|U|].$$

In order to prove the constancy theorem, we will need to introduce the notion of smoothing currents. In what follows, we will use mollifiers in a standard manner.

Definition 2.2.1. Let $T \in \mathcal{D}'(\mathbb{R}^N)$, we define a new current $T_\sigma \in \mathcal{D}'(\mathbb{R}^N)$ by

$$T_\sigma(\omega) = T(\varphi_\sigma * \omega).$$

Here the convolution of φ_σ with ω is defined by convolution of φ_σ with the coefficient functions of ω, The process of forming T_σ from T is called smoothing.

Lemma 2.2.1. *The smoothing has the following properties:*

1. *T_σ weakly converges to T as $\sigma \downarrow 0$.*
2. *$D_{x_j} T_\sigma = (D_{x_j} T)_\sigma$ for $j = 1, 2, ..., N$.*
3. *If $M = N$, then for each $\sigma > 0$, there exists a real-valued function F_σ such that*

$$T_\sigma(\omega) = \int_{\mathbb{R}^N} F_\sigma \langle \omega, e_1 \wedge \cdots \wedge e_N \rangle d\mathcal{L}^N \quad \forall \omega \in \mathcal{D}^N(\mathbb{R}^N).$$

Proof.
1. For $\omega \in \mathcal{D}^M(\mathbb{R}^N)$, $\varphi_\sigma * \omega$ converges to ω in the topology of $\mathcal{D}^M(\mathbb{R}^N)$, so $T_\sigma \rightharpoonup T$.
2. Fix $j \in \{1, ..., N\}$ and $\omega \in \mathcal{D}^M(\mathbb{R}^N)$. We have

$$\varphi_\sigma * (\partial\omega/\partial x_j) = \partial(\varphi_\sigma * \omega)/\partial x_j.$$

Then we can compute

$$\begin{aligned}(D_{x_j} T_\sigma)(\omega) &= -T_\sigma(\partial\omega/\partial x_j) = -T[\varphi_\sigma * (\partial\omega/\partial x_j)] \\ &= -T[\partial(\varphi_\sigma * \omega)/\partial x_j] = D_{x_j} T(\varphi_\sigma * \omega) = (D_{x_j} T)_\sigma(\omega).\end{aligned}$$

3. Define the function $F_\sigma(z) = T[\varphi_\sigma(x - z) dx_1 \wedge \cdots dx_N]$.
Let $\omega = g(x) dx_1 \wedge \cdots dx_N$, then

$$T_\sigma(\omega) = T[\int_{\mathbb{R}^N} g(z)\varphi_\sigma(x - z) d\mathcal{L}^N dx_1 \wedge \cdots dx_N].$$

Denote $I = \int_{\mathbb{R}^N} g(z)\varphi_\sigma(x - z) d\mathcal{L}^N$. Since the support of g is compact, there exists a family of open balls $\{A_k\}_{k=1}^p$ with the same radius r_p such that $\operatorname{supp} g \subset \bigcup_{k=1}^p A_k$. Let z_k be the center of A_k and denote

$$S_p = \sum_{k=1}^p g(z)\varphi_\sigma(x - z_k) d\mathcal{L}^N(A_k).$$

Then we have $\lim_{r_p \to 0} S_p = I$. By linearity, we also have

$$\begin{aligned}T(S_p dx_1 \wedge \cdots dx_N) &= \sum_{k=1}^p T[\varphi_\sigma(x - z_k) dx_1 \wedge \cdots dx_N] g(z_k) \mathcal{L}^N(A_k) \\ &= F_\sigma(z_k) g(z_k) \mathcal{L}^N(A_k) \\ &= \sum_{k=1}^p F_\sigma(z_k) \langle \omega(z_k), e_1 \wedge \cdots \wedge e_N \rangle \mathcal{L}^N(A_k).\end{aligned}$$

Passing to the limit, the result follows. $\square$

Proof of the constancy theorem. Without loss of generality, assume $U = \mathbb{R}^N$. We need to show that

$$T(\omega) = c\int_{\mathbb{R}^N} \langle \omega, e_1 \wedge \cdots \wedge e_N \rangle d\mathcal{L}^N \quad \forall \omega \in \mathcal{D}^N(\mathbb{R}^N).$$

From (2.4) we see $D_{x_j}T = 0$. By Lemma 2.2.1, we have that for each $\sigma > 0$, there exists a function F_σ such that

$$T_\sigma(\omega) = \int_{\mathbb{R}^N} F_\sigma \langle \omega, e_1 \wedge \cdots \wedge e_N \rangle d\mathcal{L}^N \quad \forall \omega \in \mathcal{D}^N(\mathbb{R}^N).$$

and

$$[D_{x_j}T]_\sigma = 0 = D_{x_j}T_\sigma.$$

Let $\omega = \frac{\partial F_\sigma}{\partial x_j} dx_1 \wedge \cdots dx_N$, then

$$0 = D_{x_i}T_\sigma(\omega) = -\int_{\mathbb{R}^N} \left(\frac{\partial F_\sigma}{\partial x_j}\right)^2 d\mathcal{L}^N,$$

thus F_σ must be a constant. Selecting a subsequence $\sigma_i \downarrow 0^+$, we complete the proof. $\square$

We also have two following generalization of the constancy theorem

Proposition 2.2.1. *Let $U \subset \mathbb{R}^N$ be a bounded open set and $T \in N_M(U)$. Then there exists a function $f \in BV(U)$ such that $T = [|U|]\lfloor f$.*

Proof. By Lemma 2.2.1, the smoothing T_σ of T can be written as

$$T_\sigma = [|U|]\lfloor f_\sigma.$$

By the definition of smoothing, we have that for all σ,

$$||f_\sigma||_{L^1(U)} \leq M(T_\sigma) \leq M(T) < +\infty$$

and

$$\int_U |Df_\sigma| \leq M(\partial T_\sigma) \leq M(\partial T) < +\infty.$$

So, $f_\sigma \in BV(U)$ and then by Theorem 1.1.4, there exists a subsequence σ_i and a function $f \in BV(U)$ such that $f_{\sigma_i} \to f$ in $L^1(U)$ as $\sigma_i \downarrow 0$, thus $T_{\sigma_i} \to T = [|U|]\lfloor f$. $\square$

Proposition 2.2.2. *If V is an M-dimensional plane, $T \in \mathcal{D}_M(\mathbb{R}^N)$, $\partial T = 0$ and $\operatorname{spt} T \subseteq V$, then there is a real number c, such that*

$$T = c[|V|],$$

i.e. $T(\omega) = c\int_V \langle \omega, \vec{V} \rangle d\mathcal{H}^M$, where $\vec{V} = v_1 \wedge ... \wedge v_M$ is an orthonormal vector parallel to V.

Proof. Without loss of generality, let

$$V = \{(x_1, ..., x_N) : x_{M+1} = x_{M+2} = ... = x_N = 0\}$$

and choose an index $1 \leq i_1 < i_2 < ... < i_M \leq N$.
1) Assume $i_M > M$, let ϕ be an arbitrary smooth function with compact support, $\omega = (-1)^{M-1}\phi(x)x_{i_M}dx_{i_1} \wedge ... \wedge dx_{i_{M-1}}$, then

$$d\omega = \phi(x)dx_{i_1} \wedge ... \wedge x_M + \sum_{j \notin \{i_1,...,i_M\}} \frac{\partial \phi}{\partial x_j} x_{i_M} dx_j \wedge dx_{i_1} \wedge ... \wedge dx_{M-1}.$$

Since $\operatorname{spt} T \subseteq V$, so

$$T(d\omega) = T(\phi(x)dx_{i_1} \wedge ... \wedge dx_{i_M}) = \partial T(\omega) = 0$$

and thus

$$T\lfloor dx_{i_1} \wedge ... \wedge dx_{i_M} = 0$$

for every $i_M > M$.
2) By the identity:

$$T = \sum_{J \in \bigwedge(N,M)} [T\lfloor dx_J] \wedge e_J$$

and from 1), we know that the only nonzero term is

$$T = (T\lfloor dx_1 \wedge ... \wedge dx_M) \wedge e_1 \wedge ... \wedge e_M.$$

We let $\widetilde{T} = (T\lfloor dx_1 \wedge ... \wedge dx_M) \wedge e_1 \wedge ... \wedge e_N \in \mathcal{D}_N(\mathbb{R}^N)$, and we use the constancy theorem to finish the proof: we have to check that $\partial\widetilde{T} = 0$.

Let $\omega_j = (-1)^{j-1}\phi dx_1 \wedge ... \wedge dx_{j-1} \wedge dx_{j+1} \wedge ... \wedge dx_N$.

If $j \leq M$, then

$$\begin{aligned}\partial\widetilde{T}(\omega_j) &= \widetilde{T}(\frac{\partial \phi}{\partial x_j}dx_1 \wedge ... \wedge dx_N) = T(\frac{\partial \phi}{\partial x_j}dx_1 \wedge ... \wedge dx_M) \\ &= \partial T[(-1)^{j-1}\phi dx_1 \wedge ... \wedge dx_{j-1} \wedge dx_{j+1} \wedge ... \wedge dx_M] = 0.\end{aligned}$$

If $j > M$, then

$$\begin{aligned}\partial\widetilde{T}(\omega_j) &= \widetilde{T}(\frac{\partial \phi}{\partial x_j}dx_1 \wedge ... \wedge dx_N) = T(\frac{\partial \phi}{\partial x_j}dx_1 \wedge ... \wedge dx_M) \\ &= \partial T[(\int_{-\infty}^{x_1} \frac{\partial \phi(t, x_2, ..., x_N)}{\partial x_j} dt) dx_2 \wedge ... \wedge dx_M] \\ &= 0.\end{aligned}$$

Finally, by the Constancy Theorem 2.2.2, we have $T = c[|V|]$. □

2.3 Further Constructions

2.3.1 Product of Currents

Definition 2.3.1. Suppose $U_1 \subseteq \mathbb{R}^{N_1}$, $T_1 \in \mathcal{D}_{M_1}(U_1)$, and $U_2 \subseteq \mathbb{R}^{N_2}$, $T_2 \in \mathcal{D}_{M_2}(U_2)$. We define $T_1 \times T_2 \in \mathcal{D}_{M_1+M_2}(U_1 \times U_2)$ as follows:

(1) We denote the basis covectors in $\mathbb{R}^{N_1}$ by dx_α and the basis covectors in $\mathbb{R}^{N_2}$ by dy_β.

(2) If $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{M_1} \leq N_1$, $1 \leq \beta_1 < \beta_2 < \cdots < \beta_{M_2} \leq N_2$, and $g \in \mathcal{D}(U_1 \times U_2, \mathbb{R})$, then set

$$\begin{aligned}
&[T_1 \times T_2](g\ dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_{M_1}} \wedge dy_{\beta_1} \wedge \cdots \wedge dy_{\beta_{M_2}}) \\
&\quad = T_1(T_2[g(x,y)dy_{\beta_1} \wedge \cdots \wedge dy_{\beta_{M_2}}]dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_{M_1}})).
\end{aligned}$$

(3) If $\omega \in \mathcal{D}^{M_1'}(U_1)$, $\omega_2 \in \mathcal{D}^{M_2'}(U_2)$ with $M_1 + M_2 = M_1' + M_2'$ but $M_1' \neq M_1$ and $M_2' \neq M_2$, then $[T_1 \times T_2](\omega_1 \wedge \omega_2) = 0$.

(4) Extend $T_1 \times T_2$ to $\mathcal{D}^{M_1+M_2}(U_1 \times U_2)$ by linearity.

Now it is immediate that

$$\partial(T_1 \times T_2) = (\partial T_1) \times T_2 + (-1)^{M_1} T_1 \times \partial T_2.$$

In case either $M_1 = 0$ or $M_2 = 0$ then the last formula is still valid, provided the corresponding terms are interpreted to be zero.

In the special case that $T \in \mathcal{D}_M(U)$ with $U \subseteq \mathbb{R}^N$ and $[|(0,1)|]$ is the 1-current in $\mathbb{R}^1$, then the equation above becomes

$$\begin{aligned}
\partial([|(0,1)|] \times T) &= (\delta_1 - \delta_0) \times T - [|(0,1)|] \times \partial T \\
&= \delta_1 \times T - \delta_0 \times T - [|(0,1)|] \times \partial T,
\end{aligned}$$

where δ_p denotes the 0-current that is given by a point mass at p.

2.3.2 The Pushforward

Definition 2.3.2. Let $U \subseteq \mathbb{R}^{N_1}$ be open sets and $V \subset \mathbb{R}^{N_2}$, $f : U \to V$ be a smooth map such that $f|_{\operatorname{spt} T}$ is proper. Let $\omega \in \mathcal{D}^M(U)$, and $f^*\omega$ be its pull-back. The pushforward $f_\# T$ of $T \in \mathcal{D}_M(U)$ is defined by

$$f_\# T(\omega) = T(\zeta \cdot f^*\omega),$$

where $\zeta \in C_c^\infty(U)$ and equals to 1 in a neighborhood of $\operatorname{spt} T \cap \operatorname{supp} f^*\omega$. The definition is independent of ζ.

Here, we require $f|_{\operatorname{spt} T}$ to be proper, so that $\operatorname{supp} f_\# T$ is compact. Observe that

$$\partial f_\# T(\omega) = f_\# T(d\omega) = T(\zeta \cdot f^* d\omega) = T(\zeta \cdot d f^*\omega) = f_\# \partial T(\omega),$$

so we have $\partial f_\# T = f_\# \partial T$.

Definition 2.3.3. Let U and V be open sets as above. For a linear mapping $L : \mathbb{R}^{N_1} \to \mathbb{R}^{N_2}$, the linear map

$$\bigwedge\nolimits_M L : \bigwedge\nolimits_M(\mathbb{R}^{N_1}) \to \bigwedge\nolimits_M(\mathbb{R}^{N_2})$$

is defined by

$$\bigwedge\nolimits_M L(e_{i_1} \wedge \cdots \wedge e_{i_M}) = Le_{i_1} \wedge \cdots \wedge Le_{i_M}$$

for every $(i_1, \ldots, i_M) \in \bigwedge(N_1, M)$.

If $f : U \to V$ is smooth, v is an M-covector, we see that

$$\langle f^*\omega(x), v \rangle = \langle \omega(f(x)), \bigwedge\nolimits_M df_x(v) \rangle,$$

so the pushforward can be identified as

$$f_{\#}T(\omega) = \int_U \langle \omega(f(x)), \bigwedge_M df_x \vec{T}(x) \rangle d\mu_T.$$

The next result is about vanishing of currents on sets that project to measure 0 in all coordinate directions.

Lemma 2.3.1. *Let $\alpha = (i_1, ..., i_M) \in \bigwedge(N, M)$ be a multi index, and let $\mathbf{p}_\alpha$ be the orthogonal projection:*

$$\mathbf{p}_\alpha : \mathbb{R}^N \to \mathbb{R}^M$$

such that $\mathbf{p}_\alpha(x_1, ..., x_N) = (x_{i_1}, ..., x_{i_M})$. Assume $U \subseteq \mathbb{R}^N$ open, and let $E \subset U$ be closed and such that $\mathcal{L}^M(\mathbf{p}_\alpha E) = 0$. Then for each $T \in \mathcal{D}_M(U)$ with $M_W(T) + M_W(\partial T) < +\infty, \forall W \subset\subset U$, we have

$$T\lfloor E = 0.$$

Proof. Let $\omega \in \mathcal{D}^M(U)$. Write

$$\omega = \sum_{\alpha \in \bigwedge N(N,M)} \omega_\alpha dx_\alpha$$

with $\omega_\alpha \in C_c^\infty(U)$. Thus

$$T(\omega) = \sum_\alpha T(\omega_\alpha dx_\alpha) = \sum_\alpha (T\lfloor\omega_\alpha) dx_\alpha = \sum_\alpha (T\lfloor\omega_\alpha) \mathbf{p}_\alpha^* dy.$$

Here $dy = dy_1 \wedge \cdots \wedge dy_M$ in the standard coordinates on $\mathbb{R}^M$. So we have

$$T(\omega) = \sum_\alpha \mathbf{p}_{\alpha\#}(T\omega_\alpha)(dy). \tag{2.5}$$

Since $\operatorname{spt} T\omega_\alpha \subseteq \operatorname{supp}\ \omega_\alpha$ is compact in U, so (2.5) makes sense.

Next we will show $M(\partial \mathbf{p}_{\alpha\#} T\lfloor\omega_\alpha) < \infty$, It is enough to show $M(\partial T\lfloor\omega_\alpha) < \infty$. For any $\tau \in \mathcal{D}^{N-1}(U)$, we have

$$\begin{aligned}
\partial(T\lfloor\omega_\alpha)(\tau) &= (T\lfloor\omega_\alpha)(d\tau) \\
&= T(\omega_\alpha d\tau) \\
&= T(d(\omega_\alpha \tau)) - T(d\omega_\alpha \wedge \tau) \\
&= \partial T(\omega_\alpha \tau) - T(d\omega_\alpha \wedge \tau),
\end{aligned}$$

thus

$$M_W(\partial(T\lfloor\omega_\alpha)) \leq M_W(\partial T) \cdot \sup|\omega_\alpha| + M_W(T) \cdot \sup|d\omega_\alpha| < +\infty.$$

By Proposition 2.2.1, there exists $\theta_\alpha \in BV(p_\alpha(U))$ such that

$$p_{\alpha\#}(T\lfloor\omega_\alpha) = [p_\alpha(U)]\lfloor\theta_\alpha.$$

It follows that $p_{\alpha\#}(T\lfloor\omega_\alpha)\lfloor p_\alpha(E) = 0$ since $\mathcal{L}^M(p_\alpha(E)) = 0$. Assuming without loss of generality that E is closed, we now see that

$$\begin{aligned} M(p_\alpha(T\lfloor\omega_\alpha)) &\leq M(p_{\alpha\#}(T\lfloor\omega_\alpha)\lfloor(\mathbb{R}^N \setminus p_\alpha(E))) & (2.6)\\ &= M(p_{\alpha\#}(T\lfloor\omega_\alpha)\lfloor(\mathbb{R}^N \setminus p_\alpha^{-1}p_\alpha(E))) & (2.7)\\ &\leq M((T\lfloor\omega_\alpha)\lfloor(\mathbb{R}^N \setminus p_\alpha^{-1}p_\alpha E)) & (2.8)\\ &\leq M_W(T\lfloor(\mathbb{R}^N \setminus p_\alpha^{-1}E) \cdot |\omega_\alpha| & (2.9)\\ &\leq M_W(T\lfloor(\mathbb{R}^N \setminus E)) \cdot |\omega_\alpha|, & (2.10) \end{aligned}$$

for any open set W such that $\operatorname{supp}\omega \subseteq W \subseteq U$. Now we combine (2.5) and (2.10) to obtain

$$M_W(T) \leq cM_W(T\lfloor(\mathbb{R}^N \setminus E)).$$

Also, we have

$$M_W(T\lfloor E) \leq cM_W(T\lfloor(\mathbb{R}^N \setminus E)). \tag{2.11}$$

If K is any compact subset of E, then we can choose sets $\{W_q\}$ such that

$$W_q \subset\subset U, \quad W_{q+1} \subseteq W_q, \quad \bigcap_{q=1}^{\infty} W_q = K.$$

By (2.11), with $W = W_q$, we conclude that $M(T\lfloor K) = 0$. Since K was arbitrary, we see that $M(T\lfloor E) = 0$.

□

2.3.3 The Homotopy Formula

Next we introduce the homotopy formula for currents, Let $U \subseteq \mathbb{R}^{N_1}$, $V \subseteq \mathbb{R}^{N2}$ and $f, g : U \to V$ be smooth mappings, and let h be a smooth homotopy from g to h, i.e. $h : [0,1] \times U \to V$, s.t. $h(0,x) = f(x)$ and $h(1,x) = g(x)$. For $T \in \mathcal{D}_M(U)$, if $h|_{[0,1]\times \operatorname{spt} T}$ is proper then $h_\#([|(0,1)|] \times T)$ is well-defined and we have

$$\begin{aligned} \partial h_\#([|(0,1)|] \times T) &= h_\#\partial([|(0,1)|] \times T)\\ &= h_\#(\delta_1 \times T - \delta_0 \times T - h_\#([|(0,1)|] \times \partial T))\\ &= g_\# T - f_\# T - h_\#([|(0,1)|] \times \partial T). \end{aligned}$$

Then, the *Homotopy Formula* is the following:

$$g_\# T - f_\# T = \partial h_\#([|(0,1)|] \times T) + h_\#([|(0,1)|] \times \partial T) \tag{2.12}$$

Remark 2.3.1. If we consider the linear homotopy

$$h(t,x) = tg(x) + (1-t)f(x)$$

then, for $\omega \in \mathcal{D}^M(V)$ we have:

$$
\begin{aligned}
h_\#([|(0,1)|] \times T)(\omega) &= \int_0^1 \int_U \langle h^\#(\omega), e_1 \wedge \vec{T} \rangle d\mu_T d\mathcal{L}^1 \\
&= \int_0^1 \int_U \langle \omega(h(t,x)), \bigwedge\nolimits_{M+1} dh_{t,x}(e_1 \wedge \vec{T}) \rangle d\mu_T d\mathcal{L}^1 \\
&= \int_0^1 \int_U \langle \omega(h(t,x)), \bigwedge\nolimits_{M+1} [g(x) - f(x) \wedge (tdg + (1-t)df)](e_1 \wedge \vec{T}) \rangle d\mu_T d\mathcal{L}^1 \\
&\leq \sup_{\operatorname{spt} T} |f-g| \sup_{\operatorname{spt} T} (||Df|| + ||Dg||)^M M(T).
\end{aligned}
$$

So, we have

$$
M[h_\#([|(0,1)|] \times T)] \leq \sup_{\operatorname{spt} T} |f-g| \sup_{\operatorname{spt} T} (||Df|| + ||Dg||)^M M(T). \tag{2.13}
$$

Applications of the Homotopy Formula

The next lemma shows that the homotopy formula can be used to define $f_\# T$ in case f is only Lipschitz, provided that $f|\operatorname{spt} T$ is proper and both $M_W(T), M_W(\partial T)$ are finite for all $W \subset\subset U$.

Lemma 2.3.2. *Let $T \in \mathcal{D}_M(U)$ be such that $\forall W \subset\subset U$ $M_W(T) + M_W(\partial T) < +\infty$, let $f : U \to V$ Lipschitz and assume $f|_{\operatorname{spt} T}$ is proper. Then for each $\omega \in \mathcal{D}^M(U)$, the following limit exists:*

$$
f_\# T(\omega) := \lim_{\sigma \to 0^+} f_{\sigma\#} T(\omega).
$$

Proof. Let $\sigma, \tau > 0$ and h be the affine homotopy from f_τ to f_σ. Then by the homotopy formula and (2.13), for each $\omega \in \mathcal{D}^M(U)$ we have

$$
\begin{aligned}
|f_{\sigma\#} T(\omega) - f_{\tau\#} T(\omega)| &= |h_\#([|(0,1)|] \times T)(d\omega) + h_\#([|(0,1)|] \times \partial T)(\omega)| \\
&\leq ||d\omega|| \sup_{\operatorname{spt} T} |f_\sigma - f_\tau| \sup_{\operatorname{spt} T} (||Df_\sigma|| + ||Df_\tau||)^M M(T) \\
&+ ||\omega|| \sup_{\operatorname{spt} T} |f_\sigma - f_\tau| \sup_{\operatorname{spt} T} (||Df_\sigma|| + ||Df_\tau||)^M M(\partial T) \\
&\to 0
\end{aligned}
$$

as $|\sigma - \tau| \to 0$. Then the result follows. □

We also have

$$
\operatorname{spt} f_\# T \subseteq f(\operatorname{spt} T)
$$

and

$$
M(f_\# T) \leq (ess \sup |Df|^M M_{f^{-1}(W)}(T))
$$

for all $W \subset\subset U$.

Now we need the notion of a cone over a current $T \in \mathcal{D}_M(U)$. We first start from the special case that $T = [|S|]$, where S is a submanifold of the sphere $S^{N-1} \subset \mathbb{R}^N$. In this case, the cone over T is $[|C_S|]$, where

$$
C_S = \{\lambda x : x \in S, 0 \leq \lambda \leq 1\}.
$$

Then, let

- U be star-shaped with respect to the point 0 (i.e., $tx \in U$, for each $x \in U$ and each $0 \leq t \leq 1$);
- $\operatorname{spt} T$ be compact;
- $h : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be defined by $h(t,x) = tx$.

The cone over T, denoted by $\delta_0 \divideontimes T$ is given by

$$\delta_0 \divideontimes T = h_{\#}([|(0,1)|] \times T).$$

It follows that $\delta_0 \divideontimes T \in \mathcal{D}_{M+1}(U)$ and by the homotopy formula,

$$\partial(\delta_0 \divideontimes T) = T - \delta_0 \divideontimes \partial T.$$

Also, if $\operatorname{spt} T \subseteq \{x : |x| = r\}$ holds, then by

$$\begin{aligned} h_{\#}([[(0,1)]] \times T)(\omega) &= \int_0^1 \int \langle \omega(h(t,x)), \bigwedge\nolimits_{M+1} dh_{t,x}(e_1 \wedge \vec{T}) \rangle d\mu_T(x) d\mathcal{L}^1(t) \\ &= \int_0^1 \int t^M \langle \omega(tx), x \wedge \vec{T}(x) \rangle d\mu_T(x) d\mathcal{L}^1(t) \end{aligned}$$

we have

$$M(\delta_0 \divideontimes T) \leq \frac{r}{M+1} M(T).$$

We can also define the cone over T with vertex p, which we denote by $\delta_p \divideontimes T$. In this case, we have

$$\partial(\delta_p \divideontimes T) = T - \delta_p \divideontimes \partial T \tag{2.14}$$

and, if $\operatorname{spt} T \subseteq \{x : |x - p| = r\}$ holds,

$$M(\delta_p \divideontimes T) \leq \frac{r}{M+1} M(T).$$

Chapter 3

Plateau's Problem for Integral Currents

3.1 Integral Currents

As already observed, general currents of finite mass have very little in common with oriented submanifolds. In this section, we will introduce a subclass of currents which are much closer to submanifolds called Integral Currents. Before that, we need some preliminary tools.

Lipschitz Functions and Rectifiable Sets

Definition 3.1.1. Let X and Y be metric spaces with metrics dist_X and dist_Y, respectively. A function $f : X \to Y$ is said to be *Lipschitz* of order 1, or simply *Lipschitz*, if there exists $M < \infty$ such that

$$\mathrm{dist}_Y[f(x_1), f(x_2)] \leq M \, \mathrm{dist}_X[x_1, x_2]$$

holds, for all $x_1, x_2 \in X$. The least choice of M that makes the above inequality true is called the Lipschitz constant for f and is denoted by $\mathrm{Lip}(f)$.

Definition 3.1.2. Let M be an integer with $1 \leq M \leq N$. A set $S \subseteq \mathbb{R}^N$ is said to be *countably M-rectifiable* if

$$S \subseteq S_0 \cup \bigcup_{j=1}^{\infty} F_j(\mathbb{R}^M),$$

where $\mathcal{H}^M(S_0) = 0$ and $F_j : \mathbb{R}^M \to \mathbb{R}^N$ are Lipschitz functions.

Tangent Spaces and Approximate Tangent Spaces

Definition 3.1.3. An M-dimensional C^1 submanifold of $\mathbb{R}^N$ is a set $S \subset \mathbb{R}^N$ for which each point has an open neighborhood $V \subset \mathbb{R}^N$ such that there exists a one-to-one C^1 map $\phi : U \to \mathbb{R}^N$ where $U \subset \mathbb{R}^M$ is open, such that

1. $D\phi$ is of rank M at all points of U,

2. $\phi(U) = V \cap S$.

Definition 3.1.4 (Tangent Spaces)**.** Suppose that S is an M-dimensional C^1 submanifold of $\mathbb{R}^N$. Let x be a point of S and let ϕ be as above. Then the range of $D\phi(u)$, $u \in U$, is called the tangent space to S at $x = \phi(u)$ and is denoted by $T_x S$.

Definition 3.1.5. Let $S \subset \mathbb{R}^N$ be $\mathcal{H}^M$-measurable with $\mathcal{H}^M(S \cap K) < \infty$ for every compact K. We say that an M-dimensional linear subspace W of $\mathbb{R}^N$ is the approximate tangent space to S at $x \in \mathbb{R}^N$ if

$$\lim_{\lambda \to 0^+} \frac{1}{\lambda^M} \int_{y \in \lambda^{-1}(S-x)} f(y) d\mathcal{H}^M(y) = \int_W f(y) d\mathcal{H}^M(y)$$

for all $f \in C_c(\mathbb{R}^N)$. Here,

$$y \in \lambda^{-1}(S - x) \iff y = \lambda^{-1}(z - x) \text{ for some } z \in S.$$

When the approximate tangent space to S at x exists, we will also denote it by $T_x S$. Here the dimension M should always be understood to be the Hausdorff dimension of S.

If S is an M-dimensional C^1 submanifold of $\mathbb{R}^N$, then the approximate tangent space coincides with the usual tangent space.

Theorem 3.1.1. *If S is $\mathcal{H}^M$-measurable and countably M-rectifiable and if $\mathcal{H}^M(S \cap K) < \infty$ holds for every compact $K \subset \mathbb{R}^N$, then $T_x S$ exists for $\mathcal{H}^M$-almost every $x \in S$.*

One can find the proof in Theorem 5.4.6 of [1].

Area and Co-area Formula

Definition 3.1.6. Let $S \subset \mathbb{R}^N$ be $\mathcal{H}^M$-measurable and countably M-rectifiable with $\mathcal{H}^M(S \cap K) < \infty$ for every compact K and $f : \mathbb{R}^N \to \mathbb{R}$ be a Lipschitz function.

(1) The approximate tangential gradient of f is defined by

$$\nabla^S f(x) = \sum_{j=1}^{M} \partial_{v_j} f(x) v_j, \quad \mathcal{H}^M\text{-a.e. } x \in S$$

where $(v_1, \ldots, v_M)$ is an orthonormal basis of $T_x S$ and $\partial_{v_j} f(x)$ denotes the directional derivative of f in the direction v_j. Note that we can also write

$$S = S_0 \cup \bigcup_{j=1}^{\infty} S_j,$$

where $\mathcal{H}^M(S_0) = 0$ and $S_j \subset S_j$, with S_j an M-dimensional C^1-submanifold of $\mathbb{R}^N$. Then $\nabla^S f(x) = \nabla^{S_j} f(x)$ whenever $x \in S_j$ and $f|_{S_j}$ is differentiable at x (which holds $\mathcal{H}^M$-a.e. in S_j by Rademacher's theorem).

(2) Having defined $\nabla^S f(x)$, we can define the linear map $d^S f_x : T_x^S S \to \mathbb{R}$ by

$$d^S f_x(v) = \langle v, \nabla^S f(x) \rangle,$$

at all points where $T_x S$ and $\nabla^S f(x)$ exist. Above $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^N$.

(3) If $f = (f_1, ..., f_K) : \mathbb{R}^N \to \mathbb{R}^K$ is Lipschitz, we define the linear map $d^S f_x : T_x S \to \mathbb{R}^K$ by

$$d^S f_x(v) = \sum_{j=1}^{K} \langle v, \nabla^S f_j(x) \rangle e_j,$$

where $e_1, ..., e_K$ is the standard basis of $\mathbb{R}^K$

(4) If $K \geq M$, we define the approximate Jacobian of f, denoted by $J^S f(x)$ for $\mathcal{H}^M$ a.e. $x \in S$ by

$$J_M^S f(x) = \sqrt{det[(d^S f_x)^t (d^S f_x)]}.$$

5) If $K < M$, we can define

$$J_K^S f(x) = \sqrt{det[(d^S f_x)(d^S f_x)^t]}.$$

Theorem 3.1.2 (Area Formula)**.** *If $K \geq M$, f, S as above, then*

$$\int_A J_M^S f(x) d\mathcal{H}^M(x) = \int_{\mathbb{R}^K} \mathcal{H}^0(A \cap f^{-1}(y)) d\mathcal{H}^M(y),$$

for every $\mathcal{H}^M$-measurable set $A \subset S$.

Theorem 3.1.3 (Co-area Formula)**.** *If $K < M$, f, S as above, then*

$$\int_A J_K^S f(x) d\mathcal{H}^M(x) = \int_{\mathbb{R}^K} \mathcal{H}^{M-K}(A \cap f^{-1}(y)) d\mathcal{H}^K(y),$$

for every $\mathcal{H}^M$-measurable set $A \subset S$.

Theorem 3.1.4. *If $K < M$ and f, S as above, then $J_K^S f$ exists $\mathcal{H}^M$-almost everywhere in S and*

$$\int_S g J_K^S f d\mathcal{H}^M = \int_{\mathbb{R}^K} \int_{S \cap f^{-1}(y)} g d\mathcal{H}^{M-K} d\mathcal{H}^\nu(y),$$

holds for every $\mathcal{H}^M$-measurable function g.

One can find more details in [2], [8] and [5].

3.1.1 Integer-Multiplicity Currents

We first introduce the Integer-Multiplicity currents. Integral currents are just the Normal currents which have Integer-Multiplicity.

Definition 3.1.7 (Integer-Multiplicity current)**.** Let M be an integer, $1 \leq M \leq N$, $T \in \mathcal{D}_M(U)$ for $U \subseteq \mathbb{R}^N$ open. T is an Integer-Multiplicity (rectifiable) M-current if $\exists S, \theta, \xi$ such that

$$T(\omega) = \int_S \langle \omega(x), \xi(x) \rangle \theta(x) d\mathcal{H}^M(x)$$

$\forall \omega \in \mathcal{D}^M(U)$, where

1. S is a $\mathcal{H}^M$-measurable and M-rectifiable subset of U with $\mathcal{H}^M(S \cap K) < +\infty, \forall K \subseteq U$ compact;

2. θ is a locally $\mathcal{H}^M$-integrable, nonnegative, integer-valued function;

3. $\xi : S \to \bigwedge_M \mathbb{R}^N$ is a $\mathcal{H}^M$-measurable function such that for $\mathcal{H}^M$-almost every point $x \in S$, $\xi(x)$ is a simple unit M-vector in T_xS. ($\xi(x)$ is simple if $\xi(x) = \tau_1 \wedge ... \wedge \tau_M$; we may choose $\{\tau_i\}$ to be an orthonormal basis of T_xS.)

The function θ is called the multiplicity of T and ξ is the orientation of T, we can write T as $T = \tau(S, \theta, \xi)$.

For $M = 0$, we have the following definition:

Definition 3.1.8. T $\in \mathcal{D}_0(U)$ is an Integer-multiplicity 0-current if $\exists S \subseteq U, \theta : S \to \mathbb{Z}$, such that for every $K \subseteq U$ compact, $S \cap K$ is finite, and

$$T(\omega) = \sum_{x \in S \cap \operatorname{supp} \omega} \theta(x)\omega(x) \quad \forall \omega \in \mathcal{D}_0(U).$$

In this case, we write $T = \tau(S, \theta, sign(\theta))$.

We also introduce the notation

$$\mathcal{I}_M(U) := \{T \in \mathcal{D}_M(U) : T \text{ is Integer-multiplicity}\}$$

and

$$I_M(U) := \mathcal{I}_M(U) \cap N_M(U).$$

Elements of $I_M(U)$ are called Integral M-currents.

Proposition 3.1.1. *For Integer-multiplicity currents have the following properties:*

1. *If $T_1, T_2 \in \mathcal{I}_M(U)$ and $p_1, p_2 \in \mathbb{N}$, then $p_1T_1 + p_2T_2 \in \mathcal{I}_M(U)$.*
2. *If $T_1 = \tau(V_1, \theta_1, \xi_1) \in \mathcal{I}_M(U)$ and $T_2 = \tau(V_2, \theta_2, \xi_2) \in \mathcal{I}_K(V)$, then*
$$T_1 \times T_2 = (V_1 \times V_2, \theta_1\theta_2, \xi_1 \wedge \xi_2) \in \mathcal{I}_{M+K}(U \times V).$$
3. *If $f : U \to V$ is Lipschitz, $T = (S, \theta, \xi) \in \mathcal{I}_M(U)$, and $f|_{\operatorname{spt} T}$ is proper, and $f_\# T \in \mathcal{D}_M(V)$ is defined by*
$$f_\# T(\omega) = \int_S \langle \omega(f(x)), \bigwedge_M d_x f \xi(x) \rangle \theta(x) d\mathcal{H}^M(x), \quad \forall \omega \in \mathcal{D}^M(V),$$
then we have $f_\# T \in \mathcal{I}_M(V)$.

Proof. 1. and 2. are easy, now we prove 3. Note that

$$\left| \bigwedge_M d_x f \xi(x) \right| = J_M^S f(x).$$

We get from the area formula that

$$f_\# T(\omega) = \int_{fS} \langle \omega(y), \sum_{x \in f^{-1}(y) \cap S_+} \theta(x) \frac{\bigwedge_M d_x f \xi(x)}{|\bigwedge_M d_x f \xi(x)|} \rangle d\mathcal{H}^M(y), \tag{3.1}$$

where $S_+ = \{x \in S : J_M^S f(x) > 0\}$. Notice that fS is M-rectifiable, and therefore the approximate tangent space $T_y fS$ exists at $\mathcal{H}^M$-a.e. $y \in fS$. Hence at points $y \in fS$ where $T_y fS$ exists and for which $T_x S$ and $d_x f$ exist for all $x \in f^{-1}(y) \cap S_+$, we have

$$\frac{\bigwedge_M d_x f \xi(x)}{|\bigwedge_M d_x f \xi(x)|} = \pm \tau_1 \wedge \cdots \wedge \tau_m,$$

where $\tau_1, \ldots, \tau_m$ is an orthonormal basis of $T_y fS$. Hence we obtain from (3.1)

$$f_\# T(\omega) = \int_{fS} \langle \omega(y), \eta(y) \rangle N(y) d\mathcal{H}^M(y),$$

where $\eta(y)$ is an orientation of $T_y fS$ and $N(y)$ is a positive integer satisfying

$$\sum_{x \in f^{-1}(y) \cap S_+} \theta(x) \frac{\bigwedge_M d_x f \xi(x)}{|\bigwedge_M d_x f \xi(x)|} = N(y)\eta(y).$$

So $f_\# T \in \mathcal{I}_M(V)$. □

3.1.2 The Slicing

Our goal in this section is to define the concept of the "slice" of an Integer-Multiplicity current. Roughly speaking, we slice a current by intersecting it with the level set of a Lipschitz function. Let's start from the following lemma, which is a special case of Theorem 3.1.4 and the Co-area Formula.

Lemma 3.1.1. *Let $S \subset \mathbb{R}^N$ be M-rectifiable and $f : \mathbb{R}^N \to \mathbb{R}$ Lipschitz. Then for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$:*

1. *$S_t := f^{-1}(t) \cap S$ is $(M-1)$-rectifiable and*
2. *for $\mathcal{H}^{M-1}$-a.e. $x \in S_t$, the tangent spaces $T_x S_t$ and $T_x S$ exist, $T_x S_t \subset T_x S$, and $T_x S = \{y + \lambda \nabla^S f(x) : y \in T_x S_t, \lambda \in \mathbb{R}\}$.*
3. *For every nonnegative $\mathcal{H}^M$-measurable function $g : S \to \mathbb{R}$, we have (co-area formula)*
$$\int_{-\infty}^{\infty} \int_{S_t} g \; d\mathcal{H}^{M-1} dt = \int_S |\nabla^S f| g \; d\mathcal{H}^M.$$

Replacing g by $g \cdot \chi_{\{x : f(x) \le t\}}$. Then 3 becomes

$$\int_{S \cap \{x : f(x) < t\}} |\nabla^S f| d\mathcal{H}^M = \int_{-\infty}^{t} \int_{S_u} d\mathcal{H}^{M-1} d\mathcal{L}^1(u).$$

Hence the left-hand side is an absolutely continuous function of t and we have

$$\frac{d}{dt} \int_{S \cap \{x : f(x) < t\}} |\nabla^S f| d\mathcal{H}^M = \int_{S_t} d\mathcal{H}^{M-1} \quad \text{for a.e. } t \in \mathbb{R}.$$

Let $T = \tau(S, \theta, \xi)$ be an Integer-Multiplicity current in U, with U an open set in $\mathbb{R}^{M+K}$. Let f be a Lipschitz function on U and let

$$\theta_+(x) = \begin{cases} 0 & \text{if } \nabla^S f(x) = 0, \\ \theta(x) & \text{if } \nabla^S f(x) \neq 0. \end{cases}$$

For $\mathcal{L}^1$-almost every $t \in \mathbb{R}$ with $T \lfloor S_t$, $T_x \lfloor S_t$ existing for $\mathcal{H}^{M-1}$-almost every $x \in S_t$, and such that 3. of Lemma 3.1.1 holds, we define $\xi_t(x)$ by

$$\xi_t(x) = \xi(x) \lfloor \left(\frac{\nabla^S f(x)}{|\nabla^S f(x)|} \right), \tag{3.2}$$

where $\frac{\nabla^S f(x)}{|\nabla^S f(x)|}$ is regarded as a 1-form. We observe that $\xi_t(x)$ has the following properties:

- $\xi_t(x)$ is simple;
- $\xi_t(x)$ lies in $\bigwedge_{M-1}(T_x \lfloor S_t) \subseteq \bigwedge_{M-1}(T_x \lfloor S)$;
- $\xi_t(x)$ has unit length for $\mathcal{H}^{M-1}$-almost every $x \in S_t$.

Now, we can define the slice of a current as follows.

Definition 3.1.9. Assume that $S \subset \mathbb{R}^N$ be M-rectifiable, let $T = \tau(S, \theta, \xi) \in \mathcal{I}_M(U)$ and $f : \mathbb{R}^N \to \mathbb{R}$ be Lipschitz. For $\mathcal{L}^1$-almost every $t \in \mathbb{R}$, we know that $T_x S$, $T_x S_t$ exist and 3 of Lemma 3.1.1 holds for $\mathcal{H}^{M-1}$-almost every $x \in S_t$. Then we can define the Integer-Multiplicity current $\langle T, f, t \rangle \in \mathcal{I}_{M-1}(S_t)$ by

$$\langle T, f, t \rangle = \tau(S_t, \theta_t, \xi_t),$$

where $\xi_t(x)$ is as in (3.2) and

$$\theta_t = \theta_+ \big|_{S_t}.$$

We call $\langle T, f, t \rangle$ the *slice* of the current T by the function f at t.

Lemma 3.1.2. *Let $S \subset \mathbb{R}^N$ be an M-rectifiable set, $T = \tau(S, \theta, \xi) \in \mathcal{I}_M(U)$ and $f : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz. Then the slices have the following properties:*

1. *For each open set $W \subseteq U$,*

$$\begin{aligned} \int_{\mathbb{R}} M_W(\langle T, f, t \rangle) d\mathcal{L}^1(t) &= \int_{S \cap W} |\nabla^S f| \theta d\mathcal{H}^M \\ &\leq \left(\operatorname*{ess\,sup}_{S \cap W} |\nabla^S f| \right) M_W(T). \end{aligned}$$

2. *If $M_W(\partial T) < \infty$ for all $W \subset\subset U$, then for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, we have*

$$\langle T, f, t \rangle = \partial(T \lfloor \{x : f(x) < t\}) - (\partial T) \lfloor \{x : f(x) < t\}.$$

3. If $M_W(T) + M_W(\partial T) < \infty$ for all $W \subset\subset U$, then for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, we have

$$M_W\langle T, f, t\rangle \leq \operatorname*{ess\,sup}_{S\cap W}|Df| \liminf_{h\to 0^+} \frac{1}{h} M_W(T\lfloor\{x : t < f(x) < t+h\}) \tag{3.3}$$

$$M_W\langle T, f, t\rangle \leq \operatorname*{ess\,sup}_{S\cap W}|Df| \liminf_{h\to 0^+} \frac{1}{h} M_W(T\lfloor\{x : t-h < f(x) < t\}) \tag{3.4}$$

and

$$\int_a^b M_W(\langle T, f, t\rangle)dt \leq \operatorname*{ess\,sup}_{S\cap W}|Df| M_W(T\lfloor\{x : a < f(x) < b\}) \tag{3.5}$$

4. If ∂T is of Integer-Multiplicity in $\mathcal{D}_{M-1}(U)$, then for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, we have

$$\langle \partial T, f, t\rangle = -\partial\langle T, f, t\rangle.$$

Proof. 1. Follows from 3 of Lemma 3.1.1.

2. Since $S \subset \mathbb{R}^N$ is M-rectifiable, so we can write

$$S = \bigcup_{j=0}^{\infty} S_j,$$

with $S_i \cap S_j = \emptyset$ when $i \neq j$, $\mathcal{H}^M(S_0) = 0$ and each $S_j \subseteq V_j$ with V_j embedded C^1 submanifold of $\mathbb{R}^{M+K}$. For $h : \mathbb{R}^{M+K} \to \mathbb{R}$ Lipschitz map, let h_σ be its mollification. Then as $\sigma \to 0$, we have

$$v \cdot \nabla^S h_\sigma \text{ converges to } v \cdot \nabla^S h \tag{3.6}$$

for any fixed, bounded $\mathcal{H}^M$-measurable function $v : \mathbb{R}^{M+K} \to \mathbb{R}^{M+K}$; that is, $\nabla^S h_\sigma$ converges to $\nabla^S h$ weakly in $L^2(\mu_T)$. To check 2., one need only check that (3.6) holds with the C^1 submanifolds V_j replacing S_j and with v vanishing on $\mathbb{R}^{M+K} \setminus S_j$; one approximates v by a smooth function and uses the fact that the h_σ converge uniformly to h.

Now let $\varepsilon > 0$ and let γ be the unique piecewise linear, continuous function satisfying

$$\gamma(s) = \begin{cases} 1 & \text{if } s < t - \varepsilon \\ 0 & \text{if } s > t. \end{cases}$$

Then γ is Lipschitz and let $h = \gamma \circ f$. For $\omega \in \mathcal{D}^M(U)$, we have

$$\begin{aligned} \partial T(h_\sigma \omega) &= T(d(h_\sigma \omega)) \\ &= T(dh_\sigma \wedge \omega) + T(h_\sigma d\omega). \end{aligned}$$

Now, applying the integral representation of ∂T, we see that

$$\begin{aligned}(\partial T\lfloor h)(\omega) &= \int_U \langle h\omega, \partial\vec{T}\rangle d\mu_{\partial T}\\ &= \lim_{\sigma\to 0^+}\int_U \langle h_\sigma\omega, \partial\vec{T}\rangle d\mu_{\partial T}\\ &= \lim_{\sigma\to 0^+}\partial T(h_\sigma\omega)\\ &= \lim_{\sigma\to 0^+} T(dh_\sigma\wedge\omega) + (T\lfloor h)(d\omega).\end{aligned}$$

Since $\xi(x)$ orients T_xS, let λ^T be the orthogonal projection of $\bigwedge^q(\mathbb{R}^{M+K})$ onto $\bigwedge^q(T_xS)$. we have

$$\begin{aligned}\langle dh_\sigma\wedge\omega, \xi(x)\rangle &= \langle (dh_\sigma(x))^T\wedge\omega^T, \xi(x)\rangle\\ &= \langle (dh_\sigma(x))^T\wedge\omega, \xi(x)\rangle.\end{aligned}$$

Then

$$\begin{aligned}T(dh_\sigma\wedge\omega) &= \int_S \langle (dh_\sigma(x))^T\wedge\omega, \xi(x)\rangle\theta d\mathcal{H}^M\\ &= \int_S \langle \omega, \xi(x)\lfloor\nabla^S h_\sigma(x)\rangle\theta d\mathcal{H}^M.\end{aligned}$$

Thus letting $\sigma\to 0^+$ and using (3.6), we have

$$\lim_{\sigma\to 0^+} T(dh_\sigma\wedge\omega) = \int_S \langle\omega, \xi(x)\lfloor\nabla^S h(x)\rangle\theta d\mathcal{H}^M. \tag{3.7}$$

By the definition of $\nabla^S h$ and the chain rule for Lipschitz functions, we have

$$\nabla^S h = \gamma'(f)\nabla^S f \quad \text{for } \mathcal{H}^M\text{-almost every point of } S. \tag{3.8}$$

Here we assume $\gamma'(f) = 0$ when $f = t$ or $f = t-\varepsilon$ for which γ is not differentiable. Notice also that

$$\nabla^S h(x) = \nabla^S f(x) = 0$$

for $\mathcal{H}^M$-almost every point in $\{x\in S : f(x) = c\}$, c is a constant. Now we have

$$\begin{aligned}(\partial T\lfloor h)(\omega) &= \int_S \langle\omega, \xi(x)\lfloor\nabla^S h(x)\rangle\theta d\mathcal{H}^M + (T\lfloor h)(d\omega)\\ &= \frac{1}{\varepsilon}\int_{S\cap\{t-\varepsilon<f<t\}} \langle\omega, \xi\lfloor\nabla^S f\rangle\theta d\mathcal{H}^M + (T\lfloor h)(d\omega)\\ &= A + B\end{aligned}$$

Let $\varepsilon \to 0$, consider A, and choose $g = \langle \omega, \xi \lfloor \frac{\nabla^S f}{|\nabla^S f|} \rangle |\nabla^S f| \theta$. Then

$$\begin{aligned}
\lim_{\varepsilon \to 0} A &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{S \cap \{t - \varepsilon < f < t\}} \langle \omega, \xi \lfloor \frac{\nabla^S f}{|\nabla^S f|} \rangle |\nabla^S f| \theta d\mathcal{H}^M \\
&= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big(\int_{S \cap \{f < t\}} g |\nabla^S f| d\mathcal{H}^M - \int_{S \cap \{f < t - \varepsilon\}} g |\nabla^S f| d\mathcal{H}^M \Big) \\
&= \frac{d}{dt} \int_{S \cap \{f < t\}} g |\nabla^S f| d\mathcal{H}^M \\
&= \int_{S_t} g d\mathcal{H}^{M-1} \\
&= \langle T, f, t \rangle (\omega)
\end{aligned}$$

Consider B:

$$\lim_{\varepsilon \to 0} B = \lim_{\varepsilon \to 0} \partial (T \lfloor h)(\omega) = \partial (T \lfloor \{x : f(x) < t\}),$$

and

$$\lim_{\varepsilon \to 0} (\partial T \lfloor h)(\omega) = (\partial T) \lfloor \{x : f(x) < t\}.$$

Since $\mathcal{D}^M(U)$ is separable, then by considering a countable dense set of $\omega \in \mathcal{D}^M(U)$, we see that the previous computation is applicable with this choice of g except on a set F of points having measure 0, with F independent of ω. That completes the proof of 2.

3. For (3.3), we approximate the characteristic function $\chi_{\{x : f(x) > t\}}$ by a sequence of C^∞ functions $\{g_h\}$ such that $g_h(x) = 0$ if $f(x) < t$, $g_h(x) = 1$ if $f(x) > t + h$,

$$Dg_h \leq \frac{\lambda Df}{h},$$

where $\lambda > 1$ but close to 1. Using 2 and Proposition 2.1.1, we have

$$\begin{aligned}
M_W((\partial T) \lfloor g_h - \partial (T \lfloor g_h)) &= M_W(T \lfloor Dg_h) \\
&\leq \operatorname*{ess\,sup}_{S \cap W} |Df| \cdot \frac{1}{h} \cdot M_W(T \lfloor \{x : t < f(x) < t + h\}).
\end{aligned}$$

Letting $h \to 0^+$, we get (3.3). The proof of (3.4) is similar.

For (3.5), we just need to integrate (3.3).

4. Since $\partial^2 = 0$, so

$$\begin{aligned}
\langle \partial T, f, t \rangle &= \partial [\partial T \lfloor \{x : f < t\}] \\
&= \partial [\partial [T \lfloor \{x : f < t\}] - \langle T, f, t \rangle] \\
&= -\partial \langle T, f, t \rangle
\end{aligned}$$

then 4 follows. □

Remark 3.1.1. The right-hand side of the equation in part 2 of Lemma 3.1.2 makes sense when T and ∂T are representable by integration, without the necessity of assuming that T is an Integer-Multiplicity current. Thus we may consider slicing for an arbitrary current $T \in \mathcal{D}_M(U)$ that together with its boundary has locally finite mass in U. So suppose that $M_W(T) + M_W(\partial T) < \infty$ for all $W \subset\subset U$. Initially, we define two types of slices by

$$\langle T, f, t_- \rangle := \partial[T\lfloor\{x : f(x) < t\}] - (\partial T)\lfloor\{x : f(x) < t\}$$

and

$$\langle T, f, t_+ \rangle := -\partial[T\lfloor\{x : f(x) > t\}] + (\partial T)\lfloor\{x : f(x) > t\}.$$

Claim. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a Lipschitz function. We have the following result*

(1) For only countably many values of t does it holds that

$$M[T\lfloor\{x : f(x) = t\}] + M[(\partial T)\lfloor\{x : f(x) = t\}] > 0. \tag{3.9}$$

Thus, for all other values of t, we have

$$\langle T, f, t_- \rangle - \langle T, f, t_+ \rangle = \partial[T\lfloor\{x : f(x) \neq t\}] - (\partial T)\lfloor\{x : f(x) \neq t\} = 0.$$

Then we could denote the common value of $\langle T, f, t_+ \rangle$ and $\langle T, f, t_- \rangle$ by $\langle T, f, t \rangle$.

(2) Moreover, we have

$$\operatorname{spt}\langle T, f, t \rangle \subset \operatorname{spt} T \cap \{x : f(x) = t\rangle. \tag{3.10}$$

(3) 3 of Lemma 3.1.2 is also valid for $\langle T, f, t \rangle$.

Proof of the Claim.

(1) Let $\{W_i\}_{i=1}^{\infty} \subset\subset U$ such that $U = \bigcup_{i=1}^{\infty} W_i$, and let

$$\mathcal{A}_{W_i} := \{t \in \mathbb{R} : M_W[T\lfloor\{x : f(x) = t\} > 0\};$$
$$\mathcal{A}_{W_i}^k := \{t \in \mathbb{R} : M_W[T\lfloor\{x : f(x) = t\} > \frac{1}{k}\}.$$

Then we have $\mathcal{A}_{W_i} = \bigcup_{k=i}^{\infty} \mathcal{A}_{W_i}^k$. Since $\{x : f(x) = t_1\} \cap \{x : f(x) = t_2\} = \emptyset$ for $t_1 \neq t_2$, we get

$$\begin{aligned}
\infty > M_{W_i}(T) > M_{W_i}(T\lfloor\{x : f(x) = t, t \in \mathcal{A}_{W_i}^k\}) \\
= \sum_{t \in \mathcal{A}_{W_i}^k} (T\lfloor\{x : f(x) = t\}) \\
\geq \mathcal{H}^0(\mathcal{A}_{W_i}^k) \cdot \frac{1}{k};
\end{aligned}$$

thus $\mathcal{A}_{W_i}^k$ is a finite set, and the result follows.

(2) First consider the case that f is C^1 and let γ be any smooth, increasing function from $\mathbb{R}$ to $\mathbb{R}^+$. We have

$$\begin{aligned}
\partial(T\lfloor\gamma \circ f)(\omega) - ((\partial T)\lfloor\gamma \circ f)(\omega) &= (T\lfloor\gamma \circ f)(d\omega) - ((\partial T)\lfloor\gamma \circ f)(\omega) \\
&= T(\gamma \circ f \; d\omega) - T(d(\gamma \circ f\omega)) \\
&= -T(\gamma'(f) df \wedge \omega).
\end{aligned}$$

Now let $\varepsilon > 0$ be arbitrary and select γ piece-wise linear such that

$$\gamma(t) = \begin{cases} 0 & \text{for } t < a, \\ 1 & \text{for } t > b. \end{cases}$$

We also suppose that $0 \leq \gamma'(t) \leq [1+\varepsilon]/[b-a]$ for $a < t < b$. Then

$$\{\partial(T\lfloor \gamma \circ f) - ((\partial T)\lfloor \gamma \circ f)\} \rightharpoonup \langle T, f, a_+ \rangle \quad \text{as } b \downarrow a.$$

Hence (3.10) follows because $\operatorname{supp} \gamma' \subset [a,b]$. For a general Lipschitz function f, we just approximate f by f_σ, where f_σ is a mollifier, and let $\sigma \downarrow 0$ to obtain the conclusion.

(3) Using similar argument as in the proof of Lemma 3.1.2, the result follows. □

We conclude this section with a discussion about slicing a current by a general Lipschitz function.

Definition 3.1.10. Let $T \in \mathcal{I}_M(\mathbb{R}^{M+K})$, $F : \mathbb{R}^{M+K} \to \mathbb{R}^L$ be a Lipschitz function where $2 \leq L \leq M$. Then the slice of T by F at $(t_1, \ldots, t_L)$ is defined by

$$\langle T, F, (t_1, \ldots, t_L) \rangle = \langle \langle \ldots \langle \langle T, F_1, t_1 \rangle, F_2, t_2 \rangle, \ldots \rangle, F_L, t_L \rangle$$

where $F_1, F_2, \ldots, F_L$ are the components of F.

Next we will see the slicing of an Integer-Multiplicity current by the orthogonal projection onto a coordinate M-plane.

Let

$$\begin{aligned} \mathbf{p} : \mathbb{R}^{M+K} &\to \mathbb{R}^M \\ (x_1, \ldots, x_{M+K}) &\mapsto (x_1, \ldots, x_M). \end{aligned}$$

be the orthogonal projection and $T = \tau(S, \theta, \xi) \in \mathcal{I}_M(\mathbb{R}^{M+K})$ be an Integer-Multiplicity current. Proceeding in a manner similar to Lemma 3.1.1, we define S_+ to be the set of $x \in S$ for which $T_x S$ and $D_S\mathbf{p}(x)$ exist and for which rank $D_S\mathbf{p}(x) = M$. Then for $\mathcal{L}^M$-almost every $t = (t_1, \ldots, t_M)$ we have

$$\langle T, \mathbf{p}, t \rangle = \sum_{x \in \mathbf{p}^{-1}(t) \cap S_+} \sigma(x)\theta(x)\delta_x, \tag{3.11}$$

where $\sigma(x) = \operatorname{sgn}(a)$ when $\bigwedge_M d_x\mathbf{p}\, \xi(x) = a\, dx_1 \wedge \cdots \wedge dx_M$.

The next proposition is an application of (3.11).

Proposition 3.1.2. *Let $T \in \mathcal{I}_M$ be an Integer-Multiplicity current and $\mathbf{p} : \mathbb{R}^{M+K} \to \mathbb{R}^M$ be the projection as above.*

(1) If $h : \mathbb{R}^M \to \mathbb{R}^K$, $A \subseteq \mathbb{R}^M$ is $\mathcal{L}^M$-measurable, and $H : \mathbb{R}^M \to \mathbb{R}^{M+K}$ is given by $H(t) = (t, h(t))$, then

$$\langle H_\#[|A|], \mathbf{p}, t \rangle = \delta_{H(t)}.$$

(2) For continuous $\phi : \mathbb{R}^{M+K} \to \mathbb{R}$ and $\psi : \mathbb{R}^M \to \mathbb{R}$ and if at least one of the two functions is compactly supported, then

$$\int \psi(t) \langle T, \mathbf{p}, t \rangle(\phi)\, d\mathcal{L}^M(t) = [T\lfloor(\psi \circ \mathbf{p})\, dx_1 \wedge \cdots \wedge dx_M](\phi).$$

3.2 The Deformation Theorem

The deformation theorem is one of the fundamental results of the theory of currents and provides a useful approximation of a normal current T by a polyhedral chain P lying on a certain M-skeleton such that the error is of the form $T - P = \partial R + S$. The main error term is ∂R, where R is the $(M+1)$-dimensional surface through which T is deformed to P. The other error term S arises in moving ∂T into the skeleton.

There are both scaled and unscaled versions of this result. The scaled version of the theorem is obtained by applying homotheties to the unscaled version, so we will concentrate on the unscaled version.

Some Notation

First we need some notation that will be particular to this treatment:

- For $1 \le M$, $K \in \mathbb{Z}$, we will consider currents in $N_M(\mathbb{R}^{M+K})$;
- $C = [0,1] \times [0,1] \times \cdots \times [0,1]$ is the standard unit cube in $\mathbb{R}^{M+K}$;
- $\mathbb{Z}^{M+K} = \{(z_1, z_2, \ldots, z_{M+K}) : z_j \in \mathbb{Z}\}$ is the integer lattice in $\mathbb{R}^{M+K}$;
- For $j = 0, 1, \ldots, M+K$, we will use $\mathcal{L}_j$ to denote the collection of all the j-dimensional faces in the cubes.
- Let $t_z : \mathbb{R}^{M+K} \to \mathbb{R}^{M+K}$ denote the translation by $z \in \mathbb{R}^{M+K}$, so that
$$t_z(x) = x + z.$$
Then the translation of the cube $t_z(C)$ is
$$t_z(C) = [z_1, z_1+1] \times [z_2, z_2+1] \times \cdots \times [z_{M+K}, z_{M+K}+1],$$
where $z = (z_1, z_2, \ldots, z_{M+K}) \in \mathbb{Z}^{M+K}$ ranges over the integer lattice.

Each M-dimensional face $F \in \mathcal{L}_M$ corresponds (once we make a choice of orientation) to an Integer-Multiplicity current $[|F|]$. The precise statement of the theorem is as follows.

Unscaled Deformation Theorem

Theorem 3.2.1 (Unscaled Deformation Theorem). *Suppose that $T \in N_M(\mathbb{R}^{M+K})$ is an M-dimensional normal current. Then we have*
$$T - P = \partial R + S,$$
where $P \in \mathcal{D}_M(\mathbb{R}^{M+K})$, $R \in \mathcal{D}_{M+1}(\mathbb{R}^{M+K})$, and $S \in \mathcal{D}_M(\mathbb{R}^{M+K})$ are such that
$$P = \sum_{F \in \mathcal{L}_M} p_F [|F|], \text{ where } p_F \in \mathbb{R} \text{ for } F \in \mathcal{L}_M, \tag{3.12}$$
$$M(P) \le c\, M(T), \quad M(\partial P) \le c\, M(\partial T), \tag{3.13}$$

$$M(R) \le c\, M(T), \quad M(S) \le c\, M(\partial T). \tag{3.14}$$

The constant c depends on M and K. Further,

$$\operatorname{spt} P \cup \operatorname{spt} R \subset \left\{ x : \operatorname{dist}(x, \operatorname{spt} T) < 2\sqrt{M+K} \right\},$$

$$\operatorname{spt} \partial P \cup \operatorname{spt} S \subset \left\{ x : \operatorname{dist}(x, \operatorname{spt} \partial T) < 2\sqrt{M+K} \right\}.$$

Moreover, if T is an integral current, then P and R can be chosen to be integral currents. Also, in this case, the numbers p_F in (3.12) are integers. If in addition ∂T is an integral current, then S can be chosen to be an Integral current.

Proof: Unscaled Version

The proof of the unscaled deformation theorem is based on a retraction to deform the current T onto the M-skeleton L_M. The first step is to construct the retraction. For this construction, we introduce additional notation.

- For $j = 0, 1, \ldots, M+K$, set
$$L_j = \bigcup_{F \in \mathcal{L}_j} F.$$
Thus L_j is the j-skeleton of the cubical decomposition
$$\bigcup_{z \in \mathbb{Z}^{M+K}} (z + C)$$
of $\mathbb{R}^{M+K}$;
- for $j = 0, 1, \ldots, M+K$, set
$$\widetilde{L}_j = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} \right) + L_j.$$

Clearly we have

$$\mathbb{R}^{M+K} = L_{M+K} \supseteq L_{M+K-1} \supseteq L_{M+K-2} \supseteq \cdots \supseteq L_0,$$

and similar results hold for the $\widetilde{L}_j$.

Observe that

$$\widetilde{L}_0 \cap L_{M+K-1} = \emptyset,\ \widetilde{L}_1 \cap L_{M+K-2} = \emptyset, \cdots, \widetilde{L}_{K-1} \cap L_M = \emptyset;$$

these identities hold because

- a point in $L_{M+K-j-1}$ must have $j+1$ integral coordinate values,
- a point in $\widetilde{L}_j$ must have $M+K-j$ coordinate values that are multiples of $1/2$.

Similarly we see that, for any face $F \in \mathcal{L}_{M+K-j}$, the center of F is the point of intersection of F and $\widetilde{L}_j$. Thus the nearest-point-retraction

$$\xi_j : L_{M+K-j} \setminus L_{M+K-j-1} \to \widetilde{L}_j$$

is well-defined. We define the retraction

$$\psi_j : L_{M+K-j} \setminus \widetilde{L}_j \to L_{M+K-j-1}$$

by requiring that

- $\psi_j(x) = x$, if $x \in L_{M+K-j-1}$;
- the line segment connecting $\psi_j(x)$ and $\xi_j(x)$ contains x if $x \in L_{M+K-j} \setminus [\widetilde{L}_j \cup L_{M+K-j-1}]$.

In fact, ψ_j radially projects the points in $F \in \mathcal{L}_{M+K-j}$ from the center of F onto the relative boundary of F, so of course ψ_j cannot be defined at the center of F and still be continuous.

We also define the retraction

$$\psi : \mathbb{R}^{M+K} \setminus \widetilde{L}_{K-1} \to L_M$$

by

$$\psi = \psi_{K-1} \circ \psi_{K-2} \circ \cdots \circ \psi_0.$$

Let $A_0 = \{x = (x_1, ..., x_{M+K}) : 0 < x_1 < ... < x_{M+K} < \frac{1}{2}\}$ and $x \in A_0$, and consider $\psi_0|_{A_0}$. The line segment that connect x and $(\frac{1}{2}, ..., \frac{1}{2})$ is denoted by l_x and

$$l_x = \left\{ y : y = (1-t)(\frac{1}{2}, ..., \frac{1}{2}) + t(x_1, ..., x_{M+K}), t \in \mathbb{R} \right\}.$$

By definition, $\psi_0(x) \in F \cap l_x$ for $F \in \mathcal{L}_{M+K-1}$, so $\psi_0(x)$ has a coordinate x_i that equals to 0. We find $\psi_0(x)$ by finding $t_{\min}$ such that

$$t_{\min} = \min \left\{ t : (1-t)\frac{1}{2} + tx_i = 0, 1 \leq i \leq M+K \right\}.$$

Then $t_{\min} = \frac{1}{2x_1 - 1}$ and

$$\psi_0(x) = \frac{1}{2x_1 - 1}(0, x_2 - x_1, ..., x_{M+K} - x_1) \quad \forall x \in A_0.$$

Similarly,

$$\psi_1 \circ \psi_0(x) = \frac{1}{1 - 2(x_2 - x_1)} \frac{1}{1 - 2x_1}(0, 0, x_3 - x_2, ..., x_{M+K} - x_2),$$

and proceeding in this way, for $x_0 = 0$ we get

$$\psi(x) = \prod_{j=0}^{K-1} \frac{1}{1 - 2(x_{j+1} - x_j)}(0, ..., 0, x_{K+1} - x_K, ..., x_{M+K} - x_K) \in L_M. \tag{3.15}$$

Example 3.2.1. For M = 1 and K = 2, consider a curve in the unit cube, then ψ_0 maps it onto the faces of the cube by radially projecting from the center of the cube, then ψ_1 maps that projected curve onto the edges of the cube by radially projecting from the centers of the faces.

Next, we want to estimate the norm of $D\psi$, which is a crucial pint to prove the theorem. We need the following lemma.

Lemma 3.2.1. *If* $0 \le a_0 \le a_1 \le \cdots \le a_{j+1} < 1/2$, *then*

$$\prod_{i=0}^{j}(1+2a_i-2a_{i+1})^{-1} \le \frac{1}{1-2a_{j+1}}.$$

Proof. We prove the lemma by induction. For $j=0$ and $j=1$, the results are obvious. Assume that the inequality holds for $j=k$, we check the result for $j=k+1$, by using the result for $j=1$, we have

$$\begin{aligned}\prod_{i=0}^{k+1}(1+2a_i-2a_{i+1})^{-1} &\le (1-2a_{k+1})^{-1}(1+2a_{k+1}-2a_{k+2})^{-1}\\ &\le \frac{1}{1-2a_{k+2}},\end{aligned}$$

then we finish the proof. □

Lemma 3.2.2. *There is a constant* $C = C(M,K)$ *such that*

$$1 \le |D\psi(x)| \le \frac{C}{\rho}$$

holds for $\mathcal{L}^{M+K} - a.e.\ x \in \mathbb{R}^{M+K}\backslash\widetilde{L}_{K-1}$, *where* $\rho = \operatorname{dist}(x,\widetilde{L}_{K-1})$.

Proof. If θ is the composition of reflections through the planes $\{x : e_j \cdot x = \frac{k}{2}\}, k \in \mathbb{Z}$, translation $t_z, z \in \mathbb{Z}^{M+K}$, and permutations of coordinates, then $|D\theta| = 1$ and $\theta\circ\psi\circ\theta = \psi$, so it is sufficient to consider the case that $x \in A_0$. Let

$$\psi(x) = (\psi^1(x), ..., \psi^{M+K}(x))$$

and using Lemma 3.2.1 to compute the absolute value of the partial derivative of $\psi^i(x) = (x_i - x_K)\prod_{j=0}^{K-1}\frac{1}{1-2(x_{j+1}-x_j)}$, we get

$$\begin{aligned}\left|\frac{\partial\psi^i}{\partial x_j}\right| &= \prod_{j=0}^{K-1}\frac{1}{1-2(x_{j+1}-x_j)}(x_i-x_K)\frac{-4(x_{j+1}+x_{j-1})}{(1-2(x_j-x_{j-1}))(1-2(x_{j+1}-x_j))}\\ &\le \frac{2}{1-2x_{M+K}}\end{aligned}$$

for $i \ge K+1, j \ne i,k$, and

$$\left|\frac{\partial\psi^i}{\partial x_i}(x)\right| \le \frac{4}{1-2x_{M+K}}, \quad \left|\frac{\partial\psi^i}{\partial x_K}(x)\right| \le \frac{4}{1-2x_{M+K}}$$

The nearest point of $\widetilde{L}_{K-1}$ t0 x is $(x_1, ..., x_{K-1}, 1/2, ..., 1/2)$, so

$$\rho = \frac{1}{2}\left(\sum_{j=K}^{M+K}(1-2x_j)^2\right)^{1/2} \ge \frac{1}{2}x_{M+K},$$

thus $|D\psi| \leq \frac{C}{\rho}$, and for each ψ^i, we have $\left|\frac{\partial \psi^i}{\partial x_i}\right| \geq 1$, so

$$1 \leq |D\psi| \leq \frac{C}{\rho}$$

Remark: Here the norm of $D\psi$ is the *Hilbert-Schmidt* norm. $\square$

Proof of the unscaled deformation theorem. We divide the proof into four steps.

Step 1. We claim that

$$\int_{\widetilde{C}} |D\psi(x)|^M \, d\mathcal{L}^{M+K}(x) < \infty,$$

where $\widetilde{C} = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \times \cdots \times [-\frac{1}{2}, \frac{1}{2}]$.

Using the estimate in Lemma 3.2.2, we see that

$$\int_{\widetilde{C}} |D\psi(x)|^M \, d\mathcal{L}^{M+K}(x) \leq \int_{\widetilde{C}} \rho^{-M} \, d\mathcal{L}^{M+K} = \int_{\widetilde{C}} \widetilde{\rho}^{-M} \, d\mathcal{L}^{M+K},$$

where $\widetilde{\rho}(x)$ is the distance from a point in $\mathbb{R}^{M+K}$ to the union of the $(K-1)$-dimensional coordinate planes. Since $\widetilde{\rho}(x)$ is the minimum of the distances from x to each of the individual $(K-1)$-dimensional coordinate planes, if we write $x = (x', x'') \in \mathbb{R}^{M+K}$, where $x' \in \mathbb{R}^{M+1}$ and $x'' \in \mathbb{R}^{K-1}$, then $\widetilde{\rho}(x) \leq \widetilde{\rho}(x', 0) = |x'|$, so it will suffice to estimate $\int_{\widetilde{C}} |x'|^{-M} \, d\mathcal{L}^{M+K}(x)$. Let

$$B_1 = \{x' \in \mathbb{R}^{M+1} : |x'| \leq \frac{1}{2}\sqrt{M+1}\},\ B_2 = \{x'' \in \mathbb{R}^{K-1} : |x''| \leq \frac{1}{2}\sqrt{K-1}\}.$$

We have $\widetilde{C} \subset B_1 \times B_2$, and then

$$\begin{aligned}
\int_{\widetilde{C}} |x'|^{-M} \, d\mathcal{L}^{M+K}(x) &\leq \int_{B_1} \int_{B_2} |x'|^{-M} \, d\mathcal{L}^{M+1}(x') \, d\mathcal{L}^{K-1}(x'') \\
&= \mathcal{L}^{K-1}(B_2) \cdot \int_0^{\frac{1}{2}\sqrt{M+1}} \int_{\mathbb{R}^{M+1} \cap \{\xi : |\xi| = r\}} r^{-M} \, d\mathcal{H}^M(\xi) \, dr \\
&= \mathcal{L}^{K-1}(B_2) \cdot \mathcal{H}^M(\mathbb{R}^{M+1} \cap \{\xi : |\xi| = 1\}) \frac{1}{2}\sqrt{M+1} < \infty.
\end{aligned}$$

Step 2. There exists a point $a \in \widetilde{C}$ such that

$$\int |D\psi(x)|^M d||t_{a\#}T||(x) \leq cM(T),$$

$$\int |D\psi(x)|^M d||t_{a\#}\partial T||(x) < cM(\partial T).$$

Above, $c = c(M, K)$ is a constant and $||t_{a\#}T||$ denotes the total variation measure $\mu_{t_{a\#}T}$ of the current $t_{a\#}T$.

Set

$$c = 4 \int_{\widetilde{C}} |D\psi(x)|^M d\mathcal{L}^{M+K}(x).$$

By the symmetry in the construction of ψ we have, $\forall x \in \mathbb{R}^{M+K}$,

$$\int_{\widetilde{C}} |D\psi(x+a)|^M d\mathcal{L}^{M+K}(a) = \int_{\widetilde{C}} |D\psi(a)|^M d\mathcal{L}^{M+K}(a) = c/4.$$

By Fubini theorem, we have

$$\begin{aligned}\frac{c}{4}M(T) &= \int\int_{\widetilde{C}} |D\psi(x+a)|^M d\mathcal{L}^{M+K}(a) d||T||(x) \\ &= \int_{\widetilde{C}}\int |D\psi(x+a)|^M d||T||(x) d\mathcal{L}^{M+K}(a).\end{aligned}$$

Set

$$\begin{aligned} G_1 &= \left\{ a \in \widetilde{C} : \int_{\widetilde{C}} |D\psi(x+a)|^M d||T||(x) \le cM(T) \right\}, \\ H_1 &= \widetilde{C} \setminus G_1 = \left\{ a \in \widetilde{C} : \int_{\widetilde{C}} |D\psi(x+a)|^M d||T||(x) > cM(T) \right\}. \end{aligned}$$

We have

$$\begin{aligned} \int_{\widetilde{C}} \int |D\psi(x+a)|^M d||T||(x) d\mathcal{L}^{M+K}(a) \\ &\ge \int_{H_1} \int |D\psi(x+a)|^M d||T||(x) d\mathcal{L}^{M+K}(a) \\ &\ge cM(T)\mathcal{L}^{M+K}(H_1), \end{aligned}$$

so if $\mathcal{L}^{M+K}(H_1) \ge 1/3$ held, then we would have $(c/4)M(T) \ge (c/3)M(T)$. That is a contradiction. So we have $\mathcal{L}^{M+K}(H_1) < 1/3$ and $\mathcal{L}^{M+K}(G_1) \ge 2/3$.

Also set

$$G_2 = \left\{ a \in \widetilde{C} : \int_{\widetilde{C}} |D\psi(x+a)|^M d||\partial T||(x) \le cM(\partial T) \right\}.$$

Similarly, we have $\mathcal{L}^{M+K}(G_2) \ge 2/3$. So $\mathcal{L}^{M+K}(G_1 \cap C_2) > 0$, and there exists $a \in G_1 \cap C_2$. Finally, again by the symmetry in the construction of ψ, we observe that

$$\int |D\psi(x)|^M d||t_{a\#}T||(x) = \int |D\psi(x+a)|^M d||T||(x)$$

and

$$\int |D\psi(x)|^M d||t_{a\#}\partial T||(x) = \int_{\widetilde{C}} |D\psi(x+a)|^M d||\partial T||(x).$$

Then the result follows.

Step 3. Now we fix an $a \in \widetilde{C}$ as in Step 2 above and write $\widetilde{T} = t_{a\#}T$. Applying the homotopy formula (2.12) we have

$$T = \widetilde{T} + \partial h_{\#}([|(0,1)|] \times T) + h_{\#}([|(0,1)|] \times \partial T), \tag{3.16}$$

where h is the affine homotopy

$$h(t,x) = tx + (1-t)\psi(x)$$

between the identity map and t_a. Then by (2.13) we have the following estimates

$$\begin{aligned} M[h_\#([|(0,1)|] \times T)] &\leq |a| M(T), \\ M[h_\#([|(0,1)|] \times \partial T)] &\leq |a| M(\partial T). \end{aligned}$$

Let $k(t,x) = tx + (1-t)\psi(x)$ be another homotopy, again by the homotopy formula, we have

$$\widetilde{T} = \psi_\# \widetilde{T} + \partial k_\#([|(0,1)|] \times \widetilde{T}) + k_\#([|(0,1)|] \times \partial \widetilde{T}). \tag{3.17}$$

Since $|D\psi(x) - x| < 1/2\sqrt{M+K}$, we also have the following estimates.

$$\begin{aligned} M[k_\#([|(0,1)|] \times \widetilde{T})] &\leq 1/2\sqrt{M+K} \int |D\psi(x)|^M d||\widetilde{T}||(x) \\ &\leq 1/2\sqrt{M+K} \cdot cM(T); \\ M[k_\#([|(0,1)|] \times \partial\widetilde{T})] &\leq 1/2\sqrt{M+K} \int |D\psi(x)|^{M-1} d||\partial\widetilde{T}||(x) \\ &\leq 1/2\sqrt{M+K} \int |D\psi(x)|^{M} d||\partial\widetilde{T}||(x) \\ &\leq 1/2\sqrt{M+K} \cdot cM(\partial T); \\ M(\psi_\# \widetilde{T}) &\leq \int |D\psi(x)|^M d||\widetilde{T}||(x) \leq cM(T); \\ M(\psi_\# \partial\widetilde{T}) &\leq \int |D\psi(x)|^{M-1} d||\partial\widetilde{T}||(x) \\ &\leq \int |D\psi(x)|^{M} d||\partial\widetilde{T}||(x) \leq cM(\partial T). \end{aligned}$$

Combining (3.16) and (3.17), we have

$$\begin{aligned} T - \psi_\# \widetilde{T} = \partial & \left[h_\# \left([|(0,1)|] \times T\right) + k_\# \left([|(0,1)|] \times \widetilde{T} \right) \right] \\ & + h_\# \left([|(0,1)|] \times \partial T\right) + k_\# \left([|(0,1)|] \times \partial\widetilde{T} \right). \end{aligned}$$

We set

$$R = h_\# \left([|(0,1)|] \times T\right) + k_\# \left([|(0,1)|] \times \widetilde{T} \right)$$

and

$$S_1 = h_\# \left([|(0,1)|] \times \partial T\right) + k_\# \left([|(0,1)|] \times \partial\widetilde{T} \right).$$

Note that R is of Integer-Multiplicity if T is, and S_1 is of Integer-Multiplicity if ∂T is. Also we have

$$\operatorname{spt} R \subset \left\{ x : \operatorname{dist}(x, \operatorname{spt} T) < 2\sqrt{M+K} \right\}, \operatorname{spt} S_1 \subset \left\{ x : \operatorname{dist}(x, \operatorname{spt} \partial T) < 2\sqrt{M+K} \right\}.$$

Step 4. Let $Q = \psi_{\#}\widetilde{T}$, then $\operatorname{spt} Q \subset L_M$ and $T - Q = \partial R - S_1$. We will show that Q can be used to construct

$$P = \sum_{F \in \mathcal{L}_M} p_F[|F|].$$

as in (3.12). Let $F \in \mathcal{L}_M$ and $\mathring{F}$ be the interior of F. Suppose that $F \subset \mathbb{R}^M \times \{0\} \subset \mathbb{R}^{M+K}$ and let $\mathbf{p}$ be orthogonal projection onto $\mathbb{R}^M \times \{0\}$, then $\mathbf{p} \circ \psi = \psi$ in a neighborhood of any point $y \in \mathring{F}$. Thus we have

$$\mathbf{p}_{\#}(Q \lfloor F) = Q \lfloor \mathring{F}.$$

Identifying $\mathbb{R}^M \times \{0\}$ with $\mathbb{R}^M$ and applying Proposition 2.2.1, we get that there exists $\theta_F \in BV(\mathbb{R}^M)$ such that

$$M(Q \lfloor \mathring{F}) = \int_{\mathring{F}} |\theta_F| d\mathcal{L}^M(x) \tag{3.18}$$

and

$$M((\partial Q) \lfloor \mathring{F}) = \int_{\mathring{F}} |D\theta_F| \tag{3.19}$$

holds, and such that

$$(Q \lfloor \mathring{F})(\omega) = \int_{\mathring{F}} \langle \omega(x), e_1 \wedge e_2 \wedge \cdots \wedge e_M \rangle \, \theta_F(x) \, d\mathcal{L}^M(x) \tag{3.20}$$

holds for all $\omega \in \mathcal{D}^M(\mathbb{R}^M)$.

In addition, by (3.20), we have

$$(Q \lfloor \mathring{F} - \beta[|F|])(\omega) = \int_{\mathring{F}} (\theta_F - \beta) \langle \omega(x), e_1 \wedge \cdots \wedge e_M \rangle \, d\mathcal{L}^M(x), \tag{3.21}$$

for some constant β. Thus we have

$$M(Q \lfloor \mathring{F} - \beta[|F|]) = \int_{\mathring{F}} |\theta_F - \beta| \, d\mathcal{L}^M(x), \tag{3.22}$$

$$M(\partial(Q \lfloor \mathring{F} - \beta[|F|])) = \int_{\mathbb{R}^M} |D(\chi_{\mathring{F}}(\theta_F - \beta))|. \tag{3.23}$$

Now, since $\mathcal{L}^M(F) = 1$, we can take $\beta = \beta_F$ such that

$$\min \Big\{ \mathcal{L}^M\{x \in \mathring{F} : \theta_F(x) \geq \beta\}, \mathcal{L}^M\{x \in \mathring{F} : \theta_F(x) \leq \beta\} \Big\} \geq \frac{1}{2}.$$

Also we may take $\beta_F \in \mathbb{Z}$ whenever θ_F is integer-valued.

Then by Theorem 1.1.7, Theorem 1.1.8, (3.18), (3.19), (3.22), and (3.23), we have

$$M(Q \lfloor \mathring{F} - \beta[|F|]) \leq c \int_{\mathring{F}} |D\theta_F| = c \, M(\partial Q \lfloor \mathring{F}), \tag{3.24}$$

$$M(\partial(Q \lfloor \mathring{F} - \beta[|F|])) \leq c \int_{\mathring{F}} |D\theta_F| = c \, M(\partial Q \lfloor \mathring{F}), \tag{3.25}$$

for some constant c. Also we have $Q\lfloor\partial\mathring{F} = 0$, so

$$M\left(Q - Q\lfloor \bigcup_{F\in\mathcal{L}_M} \mathring{F}\right) = 0. \tag{3.26}$$

Now, summing over $F \in \mathcal{L}_M$ and using (3.24), (3.25), and (3.26), with

$$P := \sum_{F\in\mathcal{L}_M} \beta_F[|F|],$$

we see that

$$M(Q-P) \leq cM(\partial Q), \tag{3.27}$$
$$M(\partial Q - \partial P) \leq cM(\partial Q). \tag{3.28}$$

Our choice of β_F also tells us that

$$2\int_{\mathring{F}} |\beta_F| d\mathcal{L}^M \geq 2\int_{\{x\in\mathring{F}:\theta_F(x)\geq\beta\}} |\beta_F| d\mathcal{L}^M \geq |\beta_F|. \tag{3.29}$$

Thus, again using (3.22), and since $M(P) = \sum_{F\in\mathcal{L}_M} |\beta_F|$, we see that

$$M(P) \leq cM(Q). \tag{3.30}$$

We also know, from (3.28) above (and the triangle inequality), that

$$M(\partial P) \leq cM(\partial Q).$$

Finally, by setting $S := S_1 + (Q-P)$ and $p_F := \beta_F$, we obtain

$$T - P = \partial R + S, \tag{3.31}$$

and the deformation theorem follows. □

Scaled Deformation Theorem

Let the map η_t be defined by

$$\begin{aligned} \eta_t : \mathbb{R}^{M+K} &\longrightarrow \mathbb{R}^{M+K} \\ x &\longmapsto tx \qquad \forall t\in\mathbb{R}. \end{aligned}$$

The scaled deformation theorem is the following.

Theorem 3.2.2 (Scaled Deformation Theorem)**.** *Fix $\rho > 0$. Let $T \in N_M(\mathbb{R}^{M+K})$ be an M-dimensional normal current, then we have*

$$T - P = \partial R + S,$$

where $P \in \mathcal{D}_M(\mathbb{R}^{M+K})$, $R \in \mathcal{D}_{M+1}(\mathbb{R}^{M+K})$, *and* $S \in \mathcal{D}_M(\mathbb{R}^{M+K})$ *are such that*

$$P = \sum_{F \in \mathcal{L}_M} p_F \eta_{\rho\#}[|F|], \tag{3.32}$$

with $p_F \in \mathbb{R}$ *for* $F \in \mathcal{L}_M$, *and*

$$M(P) \le c\, M(T), \qquad M(\partial P) \le c\, M(\partial T), \tag{3.33}$$

$$M(R) \le c\,\rho\, M(T), \qquad M(S) \le c\,\rho\, M(\partial T). \tag{3.34}$$

The constant c *depends only on* M *and* K. *Further,*

$$\operatorname{spt} P \cup \operatorname{spt} R \subset \{x : \operatorname{dist}(x, \operatorname{spt} T) < 2\sqrt{M+K}\rho\},$$
$$\operatorname{spt} \partial P \cup \operatorname{spt} S \subset \{x : \operatorname{dist}(x, \operatorname{spt} \partial T) < 2\sqrt{M+K}\rho\}.$$

Moreover, if T *is an integral current, then so are* P *and* R, $p_F \in \mathbb{Z}$. *If* ∂T *is an integral current, then so is* S.

Proof. Applying the Unscaled Deformation Theorem 3.2.1 to $\eta_{1/\rho\#}T$ and then applying $\eta_{\rho\#}$ to P, R and S, the result follows. □

Some Applications

Theorem 3.2.3 (Isoperimetric Inequality). *Let* $M \le 2$. *Suppose that* $T \in \mathcal{I}_{M-1}(\mathbb{R}^{M+K})$, $\operatorname{spt} T$ *is compact and* $\partial T = 0$. *Then there is a compactly supported* $T \in \mathcal{I}_M(\mathbb{R}^{R+K})$ *such that* $\partial R = T$ *and*

$$[M(R)]^{M-1/M} \le cM(T),$$

where $c = c(M, K)$ *is a constant.*

Example 3.2.2. Let $T \in \mathcal{D}_1(\mathbb{R}^2)$ be a current given by integration on a simple, closed curve γ in $\mathbb{R}^2$. Then $M(T)$ is the length of γ. Let the current $R \in \mathcal{D}_2(\mathbb{R}^2)$ be the region in the plane whose boundary is T . The conclusion of the theorem is that the square root of the area of R is bounded by a constant times the mass of T: this is the classical isoperimetric inequality.

Proof. For $T = 0$, the result is trivial. We consider the case that $T \ne 0$. Let $P, S \in \mathcal{I}_{M-1}(\mathbb{R}^{M+K})$ and $R \in \mathcal{I}_M(\mathbb{R}^{M+K})$, also for each $\rho > 0$ let $\eta_\rho(x) = \rho x$. Then by Theorem 3.2.2 we have

$$T - P = \partial R + S.$$

But $\partial T = \partial\partial R = 0$, so $M(S) = 0$, and

$$M(\eta_{\rho\#}[|F|]) = \mathcal{H}^{M-1}(\rho F) = \rho^{M-1}.$$

So, $M(P) = N(\rho)\rho^{M-1}$ for some $N(\rho) \in \mathbb{N}$. Now, choose $\rho = [2cM(T)]^{1/(M-1)}$, then

$$M(P) = N(\rho)\rho^{M-1} = 2N(\rho)cM(T) \le cM(T);$$

thus $2N(\rho) \le 1$, so $N(\rho) = 0$ and $P = 0$. Then $T = \partial R$ for some $R \in \mathcal{I}_M(\mathbb{R}^{M+K})$ and

$$M(R) \le c\rho\, M(T) = 2^{1/(M-1)} c^{(M/(M-1)} [M(T)]^{M/(M-1)}.$$

□

Theorem 3.2.4 (Weak Polyhedral Approximation). *Let $T \in \mathcal{I}_M(U)$ be am Integer-Multiplicity current with $M_W(\partial T) < \infty$ for all $W \subset\subset U$. Then there is a sequence $\{P_l\}$ of currents of the form*

$$P_l = \sum_{F \in \mathcal{L}_M} p_F^{(l)} \eta_{\rho_l \#}[|F|], \quad p_F^{(l)} \in \mathbb{Z}, \tag{3.35}$$

such that P_l and ∂P_l converge weakly to T and ∂T, respectively, in U as $\rho_l \downarrow 0$.

Proof. First consider the case $U = \mathbb{R}^{M+K}$ and $T \in \mathcal{I}_M(\mathbb{R}^{M+K})$. For any sequence $\rho_l \to 0$, by Theorem 3.2.2 we have

$$T - P_l = \partial R_l + S_l$$

for some R_l, S_l such that

$$M(R_l) \le c\,\rho_l M(T) \to 0, \ M(S_l) \le c\,\rho_l M(\partial T) \to 0,$$

and

$$M(P_l) \le cM(T), \, M(\partial P_l) \le cM(\partial T).$$

Thus we have $P_l(\omega) \to T(\omega)$ for all $\omega \in \mathcal{D}^M(\mathbb{R}^{M+K})$, also $\partial P_l = 0$ if $\partial T = 0$.

For the general case, let ϕ be a Lipschitz function on $\mathbb{R}^{M+K}$ such that $\phi > 0$ in U and $\phi = 0$ on $\mathbb{R}^{M+K} \setminus U$. Also assume that $\{x : \phi(x) > \lambda\} \subset\subset U$ for all $\lambda > 0$. Letting $T_\lambda = T\lfloor\{x : \phi(x) > \lambda\}$, then for $\mathcal{L}^1$-almost every $\lambda > 0$, Lemma 3.1.2 implies that T_λ is such that $M(\partial T_\lambda) < \infty$. Since $\operatorname{spt} T_\lambda \subset\subset U$, we can use the above argument to approximate T_λ for any such λ. Then, for a suitable sequence $\lambda_j \to 0$, the required approximation is an immediate consequence. □

3.3 The Compactness Theorem

In this section, we will prove the compactness theorem for Integer-Multiplicity currents with finite local mass and finite boundary local mass. The compactness theorem for integral currents will be a simple corollary.

Theorem 3.3.1 (Compactness Theorem for Integer-Multiplicity Currents). *Let $U \subset \mathbb{R}^{M+K}$ be an open set. Let $\{T_j\} \subset \mathcal{D}_M(U)$ be a sequence of Integer-Multiplicity currents such that*

$$\sup_{j\ge1}\,[M_W(T_j) + M_W(\partial T_j)] < \infty \quad \forall W \subset\subset U.$$

Then there is an Integer-Multiplicity current $T \in \mathcal{D}_M(U)$ and a subsequence $\{T_{j'}\}$ such that $T_j \to T$ weakly in U.

The proof of this theorem is complicate, but the idea is to use induction, We first start from integer multiplicity 0-currents.

3.3.1 Integer-Multiplicity 0-Currents

We first fix some notation.

1. Let $\mathcal{R}_0(\mathbb{R}^{M+K})$ denote the space of finite-mass Integer-Multiplicity 0-currents in $\mathbb{R}^{M+K}$, notice that $\mathcal{R}_0(\mathbb{R}^{M+K}) = I_0(\mathbb{R}^{M+K})$.
2. A nonzero current T in $\mathcal{R}_0(\mathbb{R}^{M+K})$ can be written

$$T = \sum_{j=1}^{\alpha} c_j \delta_{p_j}, \tag{3.36}$$

where α is a positive integer, $p_j \in \mathbb{R}^{M+K}$ for each $1 \leq j \leq \alpha$, $p_i \neq p_j$ for $1 \leq i \neq j \leq \alpha$, δ_{p_j} is the Dirac mass at p_j, and $c_j \in \mathbb{Z} \setminus \{0\}$ for each $1 \leq j \leq \alpha$.

Next, we prove the compactness theorem for $\mathcal{R}_0(\mathbb{R}^{M+K})$.

Proof. Suppose that $T_j \in \mathcal{R}_0(\mathbb{R}^{M+K})$, $j = 1, 2, \ldots$, and that

$$L = \sup_{j \geq 1} M(T_j) < \infty, \quad L \in \mathbb{Z}^+.$$

By the Banach-Alaoglu theorem there is a $T \in \mathcal{D}_0(\mathbb{R}^{M+k})$ such that a subsequence of the T_j converges weakly to T, still denoting the subsequence by T_j. What we must prove is that $T \in \mathcal{R}_0(\mathbb{R}^{M+K})$.

Let $\mathbb{B}(x, r)$ denote the standard open ball in $\mathbb{R}^{M+k}$ centered in x with radius r. Choose $0 < m < \infty$ large enough such that $T \lfloor \mathbb{B}(0, m) \neq 0$. We can write each $T_j \lfloor \overline{\mathbb{B}}(0, m) \in \mathcal{R}_0(\mathbb{R}^{M+K})$ as

$$T_j \lfloor \overline{\mathbb{B}}(0, m) = \sum_{i=1}^{L} c_i^{(j)} \delta_{p_i^{(j)}},$$

where

$$c_i^{(j)} \in \mathbb{Z}, \quad -L \leq c_i^{(j)} \leq L, \quad p_i^{(j)} \in \overline{\mathbb{B}}(0, m).$$

We now allow $c_i^{(j)} = 0$ because it is possible that $M[T_j \lfloor \overline{\mathbb{B}}(0, m)] < L$ holds.

By the Bolzano-Weierstrass theorem, we can pass to a subsequence (without changing the notation) so that for $j = 1, 2, \ldots, L$,

$$p_i^{(j)} \to p_i \in \overline{\mathbb{B}}(0, m) \text{ as } j \to \infty$$

and

$$c_i^{(j)} \to c_i \in \mathbb{Z}.$$

If $\phi \in \mathcal{D}^0(\mathbb{R}^{M+K})$ with supp $\phi \subseteq \mathbb{B}(0, m)$ then we have

$$T_j(\phi) = T_j \lfloor \overline{\mathbb{B}}(0, m)(\phi) \to \sum_{i=1}^{L} c_i \phi(p_i)$$

and we have $T_j(\phi) \to T(\phi)$ because T_j converges weakly to T. Thus we can write

$$T \lfloor \mathbb{B}(0, m) = \sum_{i=1}^{\alpha} c_i \delta_{p_i},$$

where $\alpha \leq L$ is a positive integer, $p_i \in \mathbb{B}(0, m)$ each $1 \leq i \leq \alpha$, $p_j \neq p_h$ for $1 \leq j \neq h \leq \alpha$, and $c_i \in \mathbb{Z} \backslash \{0\}$, for each $1 \leq i \leq \alpha$. Since $M(T) < \infty$, then for m large enough, we can write $T \lfloor \mathbb{B}(0, m) = T$, so $T \in \mathcal{R}_0(\mathbb{R}^{M+K})$. □

Definition 3.3.1. We have the following definitions:

1. Form (3.36), we see that if $T \in \mathcal{R}_0(\mathbb{R}^{M+K})$ and $\phi \in \mathcal{D}^0(\mathbb{R}^{M+K})$ then

$$T(\phi) = \sum_{i=1}^{\alpha} c_i \phi(p_i).$$

   Now we also define $T(\phi)$ in the same way for $\phi \in C(\mathbb{R}^{M+K})$ is only continuous.

2. Endow $\mathcal{R}_0(\mathbb{R}^{M+K})$ with the metric d_0 defined by

$$d_0(T_1, T_2) = \sup\{(T_1 - T_2)(\phi) : \phi \text{ is Lipschitz}, ||\phi||_\infty \le 1, ||\phi'||_\infty \le 1\}.$$

3. We let $\mathcal{F}^{M+K}$ denote the space of nonempty finite subsets of $\mathbb{R}^{M+K}$ metrized by the Hausdorff distance. The Hausdorff distance between A and B, denoted by $\mathrm{HD}(A, B)$, is defined by

$$\mathrm{HD}(A, B) = \max\left\{\sup_{a\in A} \mathrm{dist}(a, B), \sup_{b\in B} \mathrm{dist}(A, b)\right\}$$

   for $A, B \in \mathcal{F}^{M+K}$.

4. Define

$$\varrho : \mathcal{R}_0(\mathbb{R}^{M+K}) \to \overline{\mathbb{R}}$$

   by

$$\varrho(T) = \inf\{|p - q| : p, q \in \operatorname{spt} T, p \ne q\}.$$

   Note that if either $T = 0$ or $\mathcal{H}^0(\operatorname{spt} T) = 1$, then $\varrho(T) = +\infty$.

Lemma 3.3.1. *If $T_j \in \mathcal{R}_0(\mathbb{R}^{M+K})$ and $T_j \to T \in \mathcal{R}_0(\mathbb{R}^{M+K})$ weakly as $j \to \infty$, then*

$$\mathcal{H}^0(\operatorname{spt} T) \le \liminf_{j\to\infty} \mathcal{H}^0(\operatorname{spt} T_j).$$

If additionally

$$\mathcal{H}^0(\operatorname{spt} T) = \mathcal{H}^0(\operatorname{spt} T_j),\ \ j = 1, 2, \ldots,$$

then

$$\varrho(T) = \lim_{j\to\infty} \varrho(T_j).$$

Proof. For each $p \in \operatorname{spt} T$ we can find $\phi_p \in \mathcal{D}^0(\mathbb{R}^{M+K})$ for which $\phi_p(p) = 1$, $\phi_p(x) < 1$ for $x \ne p$, and $\phi_p(q) = 0$ for $q \in \operatorname{spt} T$ with $q \ne p$. The existence of such a function ϕ_p implies that p is a limit point of any set of the form $\bigcup_{i\ge I} \operatorname{spt} T_{j_i}$, and the result follows. □

Lemma 3.3.2. *If $T, \widetilde{T} \in \mathcal{R}_0(\mathbb{R}^{M+K})$ satisfy $0 < M(T) = M(\widetilde{T})$, then it holds that*

$$\min\left\{1, \frac{1}{3}\varrho(T), HD(\operatorname{spt} T, \operatorname{spt} \widetilde{T})\right\} \le d_0(T, \widetilde{T}).$$

Proof. Write $T = \sum_{j=1}^{\alpha} c_j \delta_{p_j}$ and $\widetilde{T} = \sum_{q\in \operatorname{spt} \widetilde{T}} \gamma_q \delta_q$. Set

$$r = \min\left\{1, \frac{1}{3}\varrho(T)\right\}.$$

and assume that $d_0(T, \widetilde{T}) < r$.

Because $M(T) = M(\widetilde{T})$ holds, we have

$$\sum_{j=1}^{\alpha} |c_j| = \sum_{q \in \operatorname{spt} \widetilde{T}} |\gamma_q|. \tag{3.37}$$

For $j = 1, 2, \ldots, \alpha$, define ϕ_j by setting

$$\phi_j(x) = \begin{cases} \operatorname{sgn}(c_j) \cdot [r - |x - p_j|] & \text{if } |x - p_j| < r, \\ 0 & \text{if } |x - p_j| \geq r. \end{cases}$$

Since $|\phi_j| \leq r \leq 1$ and $|\phi_j'| \leq 1$ hold, we have $(T - \widetilde{T})(\phi_j) \leq d_0(T, \widetilde{T})$.

If there were $1 \leq j \leq \alpha$ for which $\operatorname{spt} \widetilde{T} \cap \mathbb{B}(p_j, r) = \emptyset$ held, then we would have

$$d_0(T, \widetilde{T}) \geq (T - \widetilde{T})(\phi_j) = T(\phi_j) = r|c_j| \geq r,$$

contradicting the assumption that $d_0(T, \widetilde{T}) < r$. We conclude that

$$\operatorname{spt} \widetilde{T} \cap \mathbb{B}(p_j, r) \neq \emptyset, \text{ for } j = 1, 2, \ldots, \alpha. \tag{3.38}$$

Now define $\phi = \sum_{j=1}^{\alpha} \phi_j$. Since the ϕ_j have disjoint support, we see that $|\phi| \leq r \leq 1$ and $|\phi'| \leq 1$ hold. Setting

$$A_j = \operatorname{spt} \widetilde{T} \cap \mathbb{B}(p_j, r), \quad B = \operatorname{spt} \widetilde{T} \setminus \bigcup_{j=1}^{\alpha} A_j,$$

and using (3.37), we have

$$\begin{aligned}
d_0(T, \widetilde{T}) &\geq (T - \widetilde{T})(\phi) = T(\phi) - \widetilde{T}(\phi) && (3.39)\\
&= r \sum_{j=1}^{\alpha} |c_j| - \sum_{j=1}^{\alpha} \sum_{q \in A_j} \operatorname{sgn}(c_j)[r - |q - p_j|]\gamma_q && (3.40)\\
&= r \sum_{q \in \operatorname{spt}(\widetilde{T})} |\gamma_q| - \sum_{j=1}^{\alpha} \sum_{q \in A_j} \operatorname{sgn}(c_j)[r - |q - p_j|]\gamma_q && (3.41)\\
&= \sum_{q \in B} r|\gamma_q| + \sum_{j=1}^{\alpha} \sum_{q \in A_j} \left(r|\gamma_q| - \operatorname{sgn}(c_j)[r - |q - p_j|]\gamma_q\right). && (3.42)
\end{aligned}$$

Note that each $(r|\gamma_q| - \operatorname{sgn}(c_j)[r - |q - p_j|]\gamma_q)$ is nonnegative.

If there existed $q \in B$, then we would have

$$d_0(T, \widetilde{T}) \geq r|\gamma_q| \geq r,$$

contradicting the assumption that $d_0(T, \widetilde{T}) < r$. We conclude that

$$\operatorname{spt}(\widetilde{T}) \subseteq \bigcup_{j=1}^{\alpha} \mathbb{B}(p_j, r). \tag{3.43}$$

Now we consider $q_* \in \operatorname{spt}(\widetilde{T})$ and $1 \leq j_* \leq \alpha$ such that $q_* \in A_{j_*}$. Looking only at the summand in (3.42) that corresponds to j_* and q_*, we see that

$$d_0(T, \widetilde{T}) \geq r|\gamma_{q_*}| - \operatorname{sgn}(c_{j_*})[r - |q_* - p_{j_*}|]|\gamma_{q_*}| \tag{3.44}$$

holds.

In assessing the significance of (3.44) there are two cases to be considered according to the sign of $c_{j_*}\gamma_{q_*}$.

Case 1. In case $\operatorname{sgn}(c_{j_*}\gamma_{q_*}) = -1$ holds, we have

$$\operatorname{sgn}(c_{j_*})\gamma_{q_*} = \operatorname{sgn}(c_{j_*})\operatorname{sgn}(\gamma_{q_*})|\gamma_{q_*}| = \operatorname{sgn}(c_{j_*}\gamma_{q_*})|\gamma_{q_*}| = -|\gamma_{q_*}|.$$

The fact that $\operatorname{sgn}(c_{j_*})\gamma_{q_*} = -|\gamma_{q_*}|$ holds implies

$$\begin{aligned} d_0(T, \widetilde{T}) &\geq r|\gamma_q| - \operatorname{sgn}(c_j)[r - |q - p_j|]|\gamma_q| \\ &= (r + r - |q_* - p_{j_*}|)|\gamma_{q_*}| \geq r, \end{aligned}$$

and this last inequality contradicts the assumption that $d_0(T, \widetilde{T}) < r$.

Case 2. We see that $\operatorname{sgn}(c_{j_*}\gamma_{q_*}) = +1$, thus $\operatorname{sgn}(c_{j_*})\gamma_{q_*} = |\gamma_{q_*}|$, then

$$d_0(T, \widetilde{T}) \geq (r - r + |q - p_{j_*}|)|\gamma_{q_*}| \geq |q_* - p_{j_*}|.$$

By (3.43), for $q_* \in \operatorname{spt}(\widetilde{T})$, there exists j_* such that $q_* \in A_{j_*}$. Similarly, by (3.38), for $1 \leq j_* \leq \alpha$, there exists $q_* \in \operatorname{spt}(\widetilde{T})$ such that $q_* \in A_{j_*}$. Thus we conclude that $d_0(T, \widetilde{T}) \geq \operatorname{HD}[\operatorname{spt} T, \operatorname{spt} \widetilde{T}]$. □

Theorem 3.3.2.

1. *If $A \subseteq \mathbb{R}^M$ and $f : A \to \mathcal{F}^{M+K}$ is a Lipschitz function, then*

$$\bigcup_{x \in A} f(x) \tag{3.45}$$

is a countably M-rectifiable subset of $\mathbb{R}^{M+K}$.

2. *If $A \subseteq \mathbb{R}^M$ and $g : A \to \mathcal{R}_0(\mathbb{R}^{M+K})$ is a Lipschitz function, then*

$$\bigcup_{x \in A} \operatorname{spt}[g(x)] \tag{3.46}$$

is a countably M-rectifiable subset of $\mathbb{R}^{M+K}$.

Proof.

1. Let $\operatorname{Lip}(f) = m$ be a Lipschitz bound for f, then $\operatorname{Lip}(f(x)/m) = 1$. Thus, without loss of generality, we may suppose that $\operatorname{Lip}(f) = 1$.

In this proof, we will need to consider open balls in both $\mathbb{R}^M$ and in $\mathbb{R}^{M+K}$. Accordingly, we will use the notation $\mathbb{B}^M(x, r)$ for the open ball in $\mathbb{R}^M$ and $\mathbb{B}^{M+K}(x, r)$ for the open ball in $\mathbb{R}^{M+K}$.

For $\ell = 1, 2, \ldots$, set $A_\ell = \{x \in A : \mathcal{H}^0[f(x)] = \ell\}$. Note that $\bigcup_{x \in A_1} f(x)$ is the image of the Lipschitz function $u : A_1 \to \mathbb{R}^{M+K}$ defined by requiring $f(x) = \{u(x)\}$.

Now consider $\ell \geq 2$ and $x \in A_\ell$. Write $f(x) = \{p_1, p_2, \ldots, p_\ell\}$ and set $r(x) = \min_{i \neq j} |p_i - p_j|$.

If $z \in A_\ell \cap \mathbb{B}^M(x, \frac{r(x)}{4})$, then for each $i = 1, 2, \ldots, \ell$ there is a unique $q \in f(z) \cap \mathbb{B}^{M+K}(p_i, \frac{r(x)}{4})$ and we define $u_i(z) = q$.

The functions $u_1, u_2, \ldots, u_\ell$ are Lipschitz because, for

$$z_1, z_2 \in A_\ell \cap \mathbb{B}^M\left(x, \frac{r(x)}{4}\right),$$

we have

$$\mathrm{HD}[f(z_1), f(z_2)] = \max\{|u_i(z_1) - u_i(z_2)| : i = 1, 2, \ldots, \ell\}.$$

Since

$$\bigcup_{z \in A_\ell \cap \mathbb{B}^M(x, \frac{r(x)}{4})} f(z) = \bigcup_{i=1}^{\ell} \{u_i(z) : z \in A_\ell \cap \mathbb{B}^M(x, \frac{r(x)}{4})\},$$

we see that $\bigcup_{z \in A_\ell \cap \mathbb{B}^M(x, \frac{r(x)}{4})} f(z)$ is a countably M-rectifiable subset of $\mathbb{R}^{M+K}$.

As a subspace of a second countable space, A_ℓ is second countable, so it has the Lindelöf property; that is, every open cover has a countable subcover. Thus there is a countable cover of A_ℓ by sets of the form $A_\ell \cap \mathbb{B}^M(x, \frac{r(x)}{4})$, $x \in A_\ell$. We conclude that $\bigcup_{z \in A_\ell} f(z)$ is a countably M-rectifiable subset of $\mathbb{R}^{M+K}$ and hence $\bigcup_{\ell=1}^{\infty} \bigcup_{z \in A_\ell} f(z)$ is also countably M-rectifiable.

2. Without loss of generality, also suppose that $\mathrm{Lip}(g) = 1$. For i and j positive integers, set

$$A_{i,j} = \{x \in A : M[g(x)] = j \text{ and } 2^{-i} < r_{g(x)}\},$$

where

$$r_{g(x)} = \min\left\{1, \frac{1}{3}\varrho[g(x)]\right\}.$$

Fix $x \in A_{i,j}$. For $z_1, z_2 \in A_{i,j} \cap \mathbb{B}(x, 2^{-i-1})$, we have

$$M[g(z_1)] = M[g(z_2)] = j \text{ and } d_0[g(z_1), g(z_2)] < 2^{-i} < r_{g(z_1)}.$$

So, by Lemma 3.3.2, $\mathrm{HD}[\mathrm{spt}(g(z_1)), \mathrm{spt}(g(z_2))] \leq d_0[g(z_1), g(z_2)]$ holds. Thus

$$f : A_{i,j} \cap \mathbb{B}(x, 2^{-i-1}) \to \mathcal{F}^{M+K}$$

defined by $f(z) = \mathrm{spt}[g(z)]$ is Lipschitz. By part 1. we conclude that

$$\bigcup_{z \in A_{i,j} \cap \mathbb{B}(x, 2^{-i-1})} \mathrm{spt}[g(z)]$$

is a countably M-rectifiable subset of $\mathbb{R}^{M+K}$. As in the proof of 1., we observe that $A_{i,j}$ has the Lindelöf property, and so the result follows. □

Rectifiability Criterion

The next theorem provides a criterion for guaranteeing that a current is an Integer-Multiplicity rectifiable current. Later we shall use this criterion to complete the proof of the compactness theorem.

Before stating this theorem, we first need some tools.

Definition 3.3.2. Let μ be a measure on $\mathbb{R}^N$. Fix a point $p \in \mathbb{R}^N$ and fix $0 \leq m < \infty$.

1. The m-dimensional upper density of μ at p is denoted by $\Theta^{*m}(\mu, p)$ and is defined by setting
$$\Theta^{*m}(\mu, p) = \limsup_{r \downarrow 0} \frac{\mu[\overline{\mathbb{B}}(p, r)]}{\Omega_m r^m}.$$

2. The m-dimensional lower density of μ at p is denoted by $\Theta_*^m(\mu, p)$ and is defined by setting
$$\Theta_*^m(\mu, p) = \liminf_{r \downarrow 0} \frac{\mu[\overline{\mathbb{B}}(p, r)]}{\Omega_m r^m}.$$

3. If $\Theta_*^m(\mu, p) = \Theta^{*m}(\mu, p)$, we call their common value the m-dimensional density of μ at p and denote it by $\Theta^m(\mu, p)$.

Here Ω_m is the the m-dimensional volume of the unit ball in Euclidean m-space.

Theorem 3.3.3. *Fix $t > 0$. If μ is a Borel regular measure on $\mathbb{R}^N$ and $A \subseteq C \subseteq \mathbb{R}^N$, then*
$$t \leq \Theta^{*M}(\mu \lfloor C, x), \text{ for all } x \in A, \text{ implies } t \cdot \mathcal{S}^M(A) \leq \mu(C).$$
Here $\mathcal{S}^M$ is the M-dimensional spherical measure.

For the proof, see Theorem 4.3.7 in [1].

Theorem 3.3.4 (Rectifiability Criterion). *If $T \in \mathcal{D}_M(\mathbb{R}^{M+K})$ satisfies the following conditions:*

1. *$M(T) + M(\partial T) < \infty$,*

2. *$\|T\| = \mathcal{H}^M \lfloor \theta$, where θ is integer valued and nonnegative,*

3. *$\{x : \theta(x) > 0\}$ is a countably M-rectifiable set,*

then T is an Integer-Multiplicity rectifiable current.

Proof. Set $S = \{x : \theta(x) > 0\}$. We need to show that for $\mathcal{H}^M$-almost every point in S, $\vec{T}(x) = v_1 \wedge \cdots \wedge v_M$, where $v_1, \ldots, v_M$ is an orthonormal system parallel to $T_x S$.

Of course, $\mathcal{H}^M$-almost every point x of S is a Lebesgue point of θ and is a point where $\vec{T}(x)$ and $T_x S$ both exist. By Theorem 3.3.3, we see that $\Theta^{*M}(\|\partial T\|, x) < \infty$ holds for $\mathcal{H}^M$-almost every $x \in S$. Hence $\Theta^{M-1}(\|\partial T\|, x) = 0$ also holds for $\mathcal{H}^M$-almost every $x \in S$.

Without loss of generality, suppose that $x = 0$. Let $\eta_r : \mathbb{R}^{M+K} \to \mathbb{R}^{M+K}$ given by $\eta_r(z) = r^{-1}z$ be a rescaling. Consider a sequence $r_i \downarrow 0$, passing to a subsequence but without changing notation, we have that
$$\eta_{r_i \#} T \rightharpoonup R \quad \text{and} \quad \eta_{r_i \#} \partial T \rightharpoonup \partial R, \quad \text{as } r_i \downarrow 0$$

for some $R \in \mathcal{D}_M(\mathbb{R}^{M+K})$, that is

$$\begin{aligned}\lim_{i\to+\infty} \eta_{r_i\#}T(\omega) &= \lim_{i\to+\infty} \int_S \langle \bigwedge\nolimits_M d_x\eta_{r_i}\vec{T}(\eta_{r_i}(x)), \omega(\eta_{r_i}(x))\rangle \theta(x) d\mathcal{H}^M(x) \\ &= \int_{T_0S} \langle \vec{T}(0), \omega(y)\rangle \theta(0) d\mathcal{H}^M(y) \\ &= R(\omega) = \int_{T_0S} \langle \vec{R}, \omega\rangle d\mu_R\end{aligned}$$

for all $\omega \in \mathcal{D}_M(\mathbb{R}^{M+K})$, and

$$\begin{aligned}\eta_{r_i\#}\partial T(\omega') &= \int_{\mathbb{R}^{M+K}} \langle \bigwedge\nolimits_{M-1} d_x\eta_{r_i}\partial\vec{T}, \omega'(\eta_{r_i}(x))\rangle d||\partial T|| \\ &\leq \frac{1}{r^{M-1}} \cdot ||\partial T||[\mathbb{B}(0, r_i \cdot r')] \to 0 \quad \text{as } i \to +\infty\end{aligned}$$

for all $\omega' \in \mathcal{D}^{M-1}(\mathbb{B}(0, r'))$. Then we have $\vec{R}(0) = \vec{T}(0)$, $\partial R = 0$, and $\operatorname{spt} R \subseteq T_0S$. By Proposition 2.2.2, we have $\vec{R}(x) = v_1 \wedge \cdots \wedge v_M = \vec{T}(0)$, where $v_1, \ldots, v_M$ is an orthonormal system parallel to T_0S. □

3.3.2 MBV Functions

In this subsection, we introduce a class of metric-space-valued functions of bounded variation.

Definition 3.3.3.

1. A function $u : \mathbb{R}^M \to \mathcal{R}_0(\mathbb{R}^{M+K})$ can be written as

$$u(y) = \sum_{i=1}^{\infty} c_i(y)\delta_{p_i(y)} \tag{3.47}$$

where for any $y \in \mathbb{R}^M$, $p_i(y) \in \mathbb{R}^{M+K}$ and only finitely many $c_i(y)$ are nonzero.

2. If u is as in (3.47) and $\phi : \mathbb{R}^{M+K} \to \mathbb{R}$, then we define $u\Diamond\phi : \mathbb{R}^M \to \mathbb{R}$ by setting

$$(u\Diamond\phi)(y) = \sum_{i=1}^{\infty} c_i(y)\phi[p_i(y)] \tag{3.48}$$

for $y \in \mathbb{R}^M$; thus the value of $(u\Diamond\phi)(y)$ is the result of applying the 0-current $u(y)$ to the function ϕ.

3. A Borel function $u : \mathbb{R}^M \to \mathcal{R}_0(\mathbb{R}^{M+K})$ is a metric space valued function of bounded variation (MBV) if for every bounded Lipschitz function $\phi : \mathbb{R}^{M+K} \to \mathbb{R}$, the function $u\Diamond\phi$ is locally BV in the traditional sense (see for instance Section 3.6 in [10]).

4. If $u : \mathbb{R}^M \to \mathcal{R}_0(\mathbb{R}^{M+K})$ is MBV, then we denote the total variation measure of u by V_u and define it by

$$
\begin{aligned}
(\mathrm{V}_u)(A) &= \sup\left\{\int_A |D(u\Diamond\phi)| : \phi : \mathbb{R}^{M+K} \to \mathbb{R}, |\phi| \le 1, |d\phi| \le 1\right\} \\
&= \sup\left\{\int_A (u\Diamond\phi)\operatorname{div} g \, d\mathcal{L}^M : \operatorname{supp} g \subseteq A, |g| \le 1, |\phi| \le 1, |d\phi| \le 1\right\},
\end{aligned}
$$

for $A \subseteq \mathbb{R}^M$ open.

For us the most important example of an MBV function will be provided by slicing a current. That is the content of the next proposition.

Proposition 3.3.1. *Let* $\mathbf{p} : \mathbb{R}^{M+K} = \mathbb{R}^M \times \mathbb{R}^K \to \mathbb{R}^M$ *be the projection onto the first factor. If* $T \in I_M(\mathbb{R}^{M+K})$ *is an integral current, then* $u : \mathbb{R}^M \to \mathcal{R}_0(\mathbb{R}^{M+K})$ *defined by*

$$u(x) = \langle T, \mathbf{p}, x\rangle, \quad x \in \mathbb{R}^M,$$

is MBV and

$$\mathrm{V}_u(A) \le M\left[\|\partial T\|(A) + \|T\|(A)\right]$$

holds for each open set $A \subseteq \mathbb{R}^M$.

Proof. Fix an open set $A \subseteq \mathbb{R}^M$. Let $g = (g_q, \cdots, g_M) \in C^1(\mathbb{R}^M, \mathbb{R}^M)$ satisfy $|g| \le 1$ and $\operatorname{supp} g \subseteq A$. Let $\phi : \mathbb{R}^{M+K} \to \mathbb{R}$ be such that $|\phi| \le 1$ and $|d\phi| \le 1$. Pick i with $1 \le i \le M$ and set

$$d\widehat{x_i} = dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_M.$$

Using (2) of Proposition 3.1.2 and

$$(u\Diamond\phi)\partial_{x_i} g_i = D_{x_i} g_i \langle T, \mathbf{p}, x\rangle(\phi)$$

we have

$$
\begin{aligned}
\left|\int D_{x_i} g_i \langle T, \mathbf{p}, x\rangle(\phi) d\mathcal{L}^M(x)\right| &= \left|\left(T\lfloor[(D_{x_i} g_i)\circ \mathbf{p}]\, dx_1 \wedge \cdots \wedge dx_M\right)(\phi)\right| \\
&= \left|T\left(\phi\left[(D_{x_i} g_i)\circ \mathbf{p}\right] dx_1 \wedge \cdots \wedge dx_M\right)\right| \\
&= \left|T\left[\phi d(g_i \circ \mathbf{p}) \wedge d\widehat{x_i}\right]\right| \\
&= \left|(\partial T)[\phi(g_i \circ \mathbf{p}) d\widehat{x_i}] - T[(g_i \circ \mathbf{p}) d\phi \wedge d\widehat{x_i}]\right| \\
&\le \|\partial T\|(A) + \|T\|(A),
\end{aligned}
$$

so

$$\left|\int_A (u\Diamond\phi)\operatorname{div} g \, d\mathcal{L}^M\right| = \left|\int \langle T, \mathbf{p}, x\rangle \phi \operatorname{div}(g) d\mathcal{L}^n(x)\right| \le M\left[\|\partial T\|(A) + \|T\|(A)\right].$$

Then the result follows. □

Theorem 3.3.5. *Let* $\mathbf{p} : \mathbb{R}^{M+K} = \mathbb{R}^M \times \mathbb{R}^K \to \mathbb{R}^M$ *be the projection onto the first factor and fix* $0 < L < \infty$. *If for* $\ell = 1, 2, \ldots$, *we have that* $T_\ell \in I_M(\mathbb{R}^{M+K})$ *is an integral current with* $M(T_\ell) + M(\partial T_\ell) \le L$ *and if* $T_\ell \rightharpoonup T$ *weakly, then for* $\mathcal{L}^M$*-almost* *every $x \in \mathbb{R}^M$, it holds that $\langle T, \mathbf{p}, x\rangle$ is an Integer-Multiplicity current. Furthermore, the function $u : \mathbb{R}^M \to \mathcal{R}_0(\mathbb{R}^{M+K})$ defined by*

$$u(x) = \langle T, \mathbf{p}, x\rangle$$

is MBV, and

$$V_u(A) \leq ML$$

holds for each open set $A \subseteq \mathbb{R}^M$.

Proof. Since $T_\ell \to T$ weakly, so $\langle T_\ell, \mathbf{p}, x\rangle \to \langle T, \mathbf{p}, x\rangle$ weakly for $\mathcal{L}^M$-almost every $x \in \mathbb{R}^M$, then by the same argument as in the previous proof, and passing to the limit, the result follows. □

Definition 3.3.4. For a measure μ on $\mathbb{R}^M$, we define the maximal function for μ, denoted by $\mathcal{M}_\mu$, by

$$\mathcal{M}_\mu(x) = \sup_{r>0} \frac{1}{\Omega_M r^M} \mu\left[\overline{\mathbb{B}}(x,r)\right].$$

Lemma 3.3.3. *If v is a real-valued BV function and 0 is a Lebesgue point for f, then we have*

$$\begin{aligned}&\frac{1}{\Omega_M r^M}\int_{\mathbb{B}(0,r)} \frac{|v(x)-v(0)|}{|x|} d\mathcal{L}^M(x)\\ &\qquad \leq \int_0^1 \frac{1}{\Omega_M (\tau r)^M}\int_{\mathbb{B}(0,\tau r)} |Dv(x)| d\mathcal{L}^M(x) d\mathcal{L}^1(\tau) \leq \mathcal{M}_{|Dv|}(0).\end{aligned}$$

Proof. For a function $v \in C^\infty(\mathbb{R}^M)$, we have

$$\begin{aligned}|v(x)-v(0)| &= \left|\int_0^1 \frac{d}{d\tau} v(\tau x) d\mathcal{L}^1(\tau)\right|\\ &= \left|\int_0^1 \langle Dv(\tau x), x\rangle d\mathcal{L}^1(\tau)\right| \leq |x| \int_0^1 |Dv(\tau x)| d\mathcal{L}^1(\tau).\end{aligned}$$

So

$$\begin{aligned}&\frac{1}{\Omega_M r^M}\int_{\mathbb{B}(0,r)} \frac{|v(x)-v(0)|}{|x|} d\mathcal{L}^M(x)\\ &\qquad\qquad \leq \int_{\mathbb{B}(0,r)}\int_0^1 \frac{1}{\Omega_M r^M} |Dv(\tau x)| d\mathcal{L}^1(\tau) d\mathcal{L}^M(x)\\ &\qquad\qquad = \int_0^1 \int_{\mathbb{B}(0,r)} \frac{1}{\Omega_M r^M} |Dv(\tau x)| d\mathcal{L}^M(x) d\mathcal{L}^1(\tau)\\ &\qquad\qquad = \int_0^1 \frac{1}{\Omega_M (\tau r)^M}\int_{\mathbb{B}(0,\tau r)} |Dv(x)| d\mathcal{L}^M(x) d\mathcal{L}^1(\tau),\end{aligned}$$

then by a smoothing argument in Theorem 1.1.5, the result follows. □

Theorem 3.3.6. *If $v : \mathbb{R}^M \to \mathbb{R}$ is a BV function and y and z are Lebesgue points for v, then*

$$|v(y)-v(z)| \leq \left[\mathcal{M}_{|Dv|}(y) + \mathcal{M}_{|Dv|}(z)\right] |y-z|.$$

Proof. Suppose that $y \neq z$. Let p be the mid point of the segment connecting y and z and set $r = \frac{|y-z|}{2}$. For $x \in \overline{\mathbb{B}}(p,r)$ we have

$$\begin{aligned}\frac{|v(y)-v(z)|}{|y-z|} &\leq \frac{|v(y)-v(x)|}{|y-z|} + \frac{|v(x)-v(z)|}{|y-z|},\\ |x-y| &\leq |y-z|,\\ |x-z| &\leq |y-z|,\end{aligned}$$

so

$$\begin{aligned}\frac{|v(y)-v(z)|}{|y-z|} &\leq \frac{|v(y)-v(x)|}{|y-z|} + \frac{|v(x)-v(z)|}{|y-z|}\\ &\leq \frac{|v(y)-v(x)|}{|y-x|} + \frac{|v(x)-v(z)|}{|x-z|}.\end{aligned}$$

As a result,

$$\begin{aligned}&\frac{|v(y)-v(z)|}{|y-z|} = \frac{1}{\Omega_M r^M}\int_{\overline{\mathbb{B}}(p,r)} \frac{|v(y)-v(z)|}{|y-z|}dx\\ &\leq \frac{1}{\Omega_M r^M}\int_{\overline{\mathbb{B}}(p,r)} \frac{|v(y)-v(x)|}{|y-x|}dx + \frac{1}{\Omega_M r^M}\int_{\overline{\mathbb{B}}(p,r)} \frac{|v(x)-v(z)|}{|x-z|}dx\\ &\leq \mathcal{M}_{|Dv|}(y) + \mathcal{M}_{|Dv|}(z).\end{aligned}$$

□

Lemma 3.3.4. *If $u : \mathbb{R}^M \to \mathcal{R}_0(\mathbb{R}^{M+K})$ is an MBV function, then there is a set E with $\mathcal{L}^M(E) = 0$ such that, for $y, z \in \mathbb{R}^M \setminus E$, it holds that*

$$d_0[u(y), u(z)] \leq [\mathcal{M}_{\mathrm{V}_u}(y) + \mathcal{M}_{\mathrm{V}_u}(z)]\,|y-z|.$$

Proof. Let ϕ_i, $i = 1, 2, \ldots$, be a dense set in $\mathcal{D}^0(\mathbb{R}^M)$ and let E_i be the set of non-Lebesgue points for $u \diamond \phi_i$. Then we set $E = \bigcup_{i=1}^{\infty} E_i$ and the result follows from Theorem 3.3.6. □

Lemma 3.3.5. *For each $\lambda > 0$, it holds that*

$$\mathcal{L}^M\{x : \mathcal{M}_\mu(x) > \lambda\} \leq \frac{B_M}{\lambda}\mu(\mathbb{R}^M),$$

where B_M is the constant for $\mathbb{R}^M$ from the Besicovitch covering theorem which is stated as follows.

Theorem 3.3.7 (Besicovitch's Covering Theorem)**.**
Let $\mathbb{R}^M$ be the M-dimensional Euclidean space, then there exists a constant B_M, depending only on the dimension M, with the following property:
If $\mathcal{F}$ is any collection of nondegenerate closed balls in $\mathbb{R}^M$ with

$$\sup\{\mathrm{diam}(\mathbb{B}) : \mathbb{B} \in \mathcal{F}\} < \infty$$

and if A is the set of centers of balls in $\mathcal{F}$, then there exist B_M countable collections

$\mathcal{G}_1, \ldots, \mathcal{G}_{B_M}$ *of disjoint balls in* $\mathcal{F}$ *such that*

$$A \subseteq \bigcup_{i=1}^{B_M} \bigcup_{\mathbb{B} \in \mathcal{G}_i} \mathbb{B}.$$

The proof of this theorem is in section 1.5.2 of [8].

Proof of Lemma 3.3.5. Set

$$L = \{x : \mathcal{M}_\mu(x) > \lambda\}.$$

For each $x \in L$, choose a ball $\overline{\mathbb{B}}(x, r_x)$ such that

$$\frac{1}{\Omega_M r^M} \mu[\overline{\mathbb{B}}(x, r_x)] > \lambda.$$

Since $L \subseteq \bigcup\limits_{x \in L} \overline{\mathbb{B}}(x, r_x)$, we can apply Theorem 3.3.7 to find families $\mathcal{G}_1, \ldots, \mathcal{G}_{B_M}$ of pairwise-disjoint balls $\overline{\mathbb{B}}(x, r_x), x \in L$, such that $L \subseteq \bigcup_{i=1}^{B_M} \bigcup_{B \in \mathcal{G}_i} B$. Then we have

$$\begin{aligned} \mathcal{L}^M(L) \leq \mathcal{L}^M \left(\bigcup_{i=1}^{B_M} \bigcup_{B \in \mathcal{G}_i} B \right) &\leq \sum_{i=1}^{B_M} \sum_{B \in \mathcal{G}_i} \Omega_M (\frac{\operatorname{diam}(B)}{2})^M \\ &< \frac{1}{\lambda} \sum_{i=1}^{B_M} \sum_{B \in \mathcal{G}_i} \mu(B) \leq \frac{B_M}{\lambda} \mu(\mathbb{R}^M). \end{aligned}$$

□

We also aobserve that if we apply Lemma 3.3.5 to the measure V_u for some MBV function u, since V_u is finite, then $\mathcal{M}_{\mathrm{V}_u}(y) < +\infty$ for $\mathcal{L}^M$-a.e. y.

Theorem 3.3.8. *If* $u : \mathbb{R}^M \to \mathcal{R}_0(\mathbb{R}^{M+K})$ *is an* MBV *function, then there is a set* E_1 *with* $\mathcal{L}^M(E_1) = 0$ *such that*

$$M = \bigcup_{y \in \mathbb{R}^M \setminus E_1} \operatorname{spt}[u(y)]$$

is a countably M*-rectifiable subset of* $\mathbb{R}^{M+K}$.

The idea of the proof is to consider points lying over the set $\{\mathcal{M}_{\mathrm{V}_u} < \frac{1}{i}\}$ for each i.

Proof. Let $A_i = \{y \in \mathbb{R}^M : \mathcal{M}_{\mathrm{V}_u}(y) < \frac{1}{i}\}$, we apply Lemma 3.3.5 to write $\mathbb{R}^M$ as the union of sets A_i. By Lemma 3.3.4, there is a set $E_i \subseteq A_i$ of measure zero such that u is Lipschitz on $A_i \setminus E_i$. So we can apply Theorem 3.3.2 to see that $\bigcup_{y \in A_i \setminus E_i} \operatorname{spt}[u(y)]$ is countably M-rectifiable. □

Lemma 3.3.6 (Slicing Lemma)**.** *Let* $U \subseteq \mathbb{R}^{M+K}$ *be an open set and* $\{T_i\} \subset \mathcal{I}_M(U)$ *Suppose that* $f : U \to \mathbb{R}$ *is Lipschitz. If* T_i *converges weakly to* $T \in \mathcal{D}_M(U)$ *and*

$$\sup \left(M_W(T_i) + M_W(\partial T_i) \right) < \infty$$

for every open set $W \subset\subset U$*, then, for* $\mathcal{L}^1$*-almost every* r*, there is a subsequence* i_j *such that*

$$\langle T_{i_j}, f, r \rangle \text{ converges weakly to } \langle T, f, r \rangle \tag{3.49}$$

and

$$\sup\left(M_W[\langle T_{i_j}, f, r\rangle] + M_W[\partial\langle T_{i_j}, f, r\rangle]\right) < \infty$$

holds for $W \subset\subset U$.

If additionally $W_0 \subset\subset U$ *is such that*

$$\lim_{i\to\infty}\left(M_{W_0}(T_i) + M_{W_0}(\partial T_i)\right) = 0,$$

then the subsequence can be chosen so that

$$\lim_{i\to\infty}\left(M_{W_0}[\langle T_{i_j}, f, r\rangle] + M_{W_0}[\partial\langle T_{i_j}, f, r\rangle]\right) = 0.$$

Proof. Passing to a subsequence for which $\|T_{i_j}\| + \|\partial T_{i_j}\|$ converges weakly to a Radon measure μ, we see that (3.49) holds, except possibly for the at most countably many r for which $\mu\{x : f(x) = r\}$ has positive measure.

The remaining conclusions follow by passing to additional subsequences and using (3.5) in Lemma 3.1.2 and the fact that $\partial\langle T_i, f, r\rangle = -\langle \partial T_i, f, r\rangle$. □

Lemma 3.3.7 (Density Lemma). *Suppose that* $T \in \mathcal{D}_M(U)$. *For* $\mathbb{B}(x,r) \subseteq U$, *set*

$$\lambda(x,r) = \inf\{M(S) : \partial S = \partial[T\lfloor\mathbb{B}(x,r)], S \in \mathcal{D}_M(U)\}.$$

(1) If $M_W(T) + M_W(\partial T) < \infty$ *holds for every* $W \subset\subset U$, *then*

$$\lim_{r\downarrow 0}\frac{\lambda(x,r)}{\|T\|(\mathbb{B}(x,r))} = 1 \tag{3.50}$$

holds for $\|T\|$*-almost every* $x \in U$.

(2) If

1. $\partial T = 0$,
2. $\partial[T\lfloor\mathbb{B}(x,r)]$ *is Integer-Multiplicity for every* $x \in U$ *and almost every* $r > 0$,
3. $M_W(T) + M_W(\partial T) < \infty$ *holds for every* $W \subset\subset U$,

then there exists $\delta > 0$ *such that*

$$\Theta_*^M(\|T\|, x) > \delta$$

holds for $\|T\|$*-almost every* $x \in U$.

Proof.

(1) We argue by contradiction. Since $\lambda(x,r) \le \|T\|(\mathbb{B}(x,r))$ is true by definition, we suppose that there is an $\varepsilon > 0$ and $E \subseteq U$ with $\|T\|(E) > 0$ such that for each $x \in E$ there exist arbitrarily small $r > 0$ such that

$$\lambda(x,r) < (1-\varepsilon)\|T\|(\mathbb{B}(x,r)).$$

We may assume that $E \subseteq W$ for an open $W \subset\subset U$.

Consider $\rho > 0$. Cover $\|T\|$-almost all of E by disjoint balls $B_i = \mathbb{B}(x_i, r_i)$, where $x_i \in E$ and $r_i < \rho$. For each i, let $S_i \in \mathcal{D}_M(U)$ satisfy

$$\partial S_i = \partial[T\lfloor\mathbb{B}(x_i,r_i)], \quad M(S_i) < (1-\varepsilon)M[T\lfloor\mathbb{B}(x_i,r_i)].$$

Set

$$T_\rho = T - \sum_i T\lfloor B_i + \sum_i S_i.$$

For any $\omega \in \mathcal{D}^M(U)$, using (2.14) we get

$$\begin{aligned}
(T - T_\rho)(\omega) &= \sum_i (T\lfloor B_i - S_i)(\omega) \\
&= \sum_i [\partial(\delta_{x_i} \divideontimes (T\lfloor B_i - S_i)) + \delta_{x_i} \divideontimes \partial(T\lfloor B_i - S_i)](\omega) \\
&= \sum_i (\delta_{x_i} \divideontimes (T\lfloor B_i - S_i))(d\omega) + 0 \\
&\leq \sum_i M(\delta_{x_i} \divideontimes (T\lfloor B_i - S_i)) \cdot \sup |d\omega| \\
&\leq \rho \sum_i M(T\lfloor B_i - S_i) \cdot \sup |d\omega| \\
&\leq 2\rho \sum_i M(T\lfloor B_i) \cdot \sup |d\omega| \\
&\leq 2\rho M(T) \cdot \sup |d\omega|.
\end{aligned}$$

Thus we see that T_ρ converges weakly to T as ρ decreases to zero. By the lower semi-continuity of mass, we have

$$M_W(T) \leq \liminf_{\rho \downarrow 0} M_W(T_\rho).$$

On the other hand, we have

$$\begin{aligned}
M_W(T_\rho) &\leq M_W\left(T - \sum_i T\lfloor B_i\right) + \sum_i M_W(S_i) \\
&\leq M_W\left(T - \sum_i T\lfloor B_i\right) + (1-\varepsilon) \sum_i M_W(T\lfloor B_i) \\
&\leq M_W(T) - \varepsilon \sum_i M_W(T\lfloor B_i) \\
&\leq M_W(T) - \varepsilon \|T\|(E),
\end{aligned}$$

a contradiction. (2) Let x be a point at which (3.50) holds. Set $f(r) = M(T\lfloor \mathbb{B}(x,r))$. For sufficiently small r we have

$$f(r) < 2\lambda(x,r). \tag{3.51}$$

To be specific, we suppose that (3.51) holds for $0 < r < R$.

Let $g(y) = |y - x|$ then $\mathbb{B}(x,r) = \{y : g(y) < r\}$, thus we have

$$\begin{aligned}
\langle T, g, r\rangle &= \partial[T\lfloor\{y : g(y) < r\}] - (\partial T)\lfloor\{y : g(y) < r\} \\
&= \partial[T\lfloor\{y : g(y) < r\}] - 0,
\end{aligned}$$

and by (3.3) we have

$$M(\langle T, g, r\rangle) = M[\partial(T\lfloor \mathbb{B}(x,r))] \leq f'(r).$$

holds for $\mathcal{L}^1$-almost every r. Applying the isoperimetric inequality, we have

$$\lambda(x, r)^{(M-1)/M} \leq c_0 f'(r),$$

where c_0 is a constant depending only on the dimensions M and K. So, by (3.51), we have

$$[f(r)]^{(M-1)/M} \leq c_1 f'(r) \quad (0 < r < R),$$

where c_1 is another constant. Thus we have

$$\frac{d}{dr}[f(r)]^{1/M} = (1/M) f'(r)[f(r)]^{(1-M)/M} \geq 1/c_1.$$

Since f is a nondecreasing function, we have

$$[f(\rho)]^{1/M} \geq \int_0^\rho \frac{d}{dr}[f(r)]^{1/M} d\mathcal{L}^1(r) \geq \int_0^\rho 1/c_1 d\mathcal{L}^1(r) = \rho/c_1.$$

We conclude that $f(r) \geq (r/c_1)^M$ holds for $0 < r < R$. □

3.3.3 The Proof of The Compactness Theorem

Now, we can start to prove the Compactness Theorem 3.3.1.

Theorem. *Let $U \subset \mathbb{R}^{M+K}$ be an open set. Let $\{T_j\} \subset \mathcal{I}_M(U)$ be a sequence of Integer-Multiplicity currents such that*

$$\sup_{j \geq 1} [M_W(T_j) + M_W(\partial T_j)] < \infty \quad \forall W \subset\subset U.$$

Then there is an Integer-Multiplicity current $T \in \mathcal{D}_M(U)$ and a subsequence $\{T_{j'}\}$ such that $T_j \to T$ weakly in U.

Proof. Assume $\{T_j\} \subset \mathcal{D}_M(U)$ is a sequence of Integer-Multiplicity currents such that

$$\sup_{j \geq 1} [M_W(T_j) + M_W(\partial T_j)] < \infty \quad \forall W \subset\subset U.$$

Then by Banach-Alaoglu Theorem 1.1.3 and passing to a subsequence if necessary, but without changing notation, there exists $T \in \mathcal{D}_M(U)$ such that $T_j \rightharpoonup T$ and $\partial T_j \rightharpoonup \partial T$ in U. Next, we need to show that T is an Integer-Multiplicity current.

First we show that it is enough to consider the case $U = \mathbb{R}^{M+K}$. Assume the Compactness Theorem is valid for $U = \mathbb{R}^{M+K}$ and $\operatorname{spt} T_j \subset K$ for some fixed compact set K. Then there exists a point $a \in \operatorname{supp} T_j$ for all j.

Consider the function $f(x) = |x - a|$, by Lemma 3.3.6, there exists a subsequence of T_j, still denoted by T_j such that for $\mathcal{L}^1 - a.e.\ r$, $\langle T_j, f, r\rangle \to \langle T, f, r\rangle$ weakly in K, that is

$$[\partial(T_j \lfloor \mathbb{B}(a, r)) - (\partial T_j)\lfloor \mathbb{B}(a, r)] \rightharpoonup [\partial(T \lfloor \mathbb{B}(a, r)) - (\partial T)\lfloor \mathbb{B}(a, r)]$$

and $M_W[\partial(T_j \lfloor \mathbb{B}(a, r)) - (\partial T_j)\lfloor \mathbb{B}(a, r)] < \infty$.

Then $M_W[T_j \lfloor \mathbb{B}(a, r)] + M_W[\partial(T_j \lfloor \mathbb{B}(a, r))] < \infty$, so we have

$$T_j \lfloor \mathbb{B}(a, r) \rightharpoonup T \lfloor \mathbb{B}(a, r).$$

Then the Compactness Theorem is valid for arbitrary open set $U \subset \mathbb{R}^{M+N}$.

Without loss of generality, assume $U = \mathbb{R}^{M+N}$, we use the induction to complete the proof.

1) For $M = 0$, the Compactness Theorem is already shown in Section 3.3.1.

2) Assume that for $\mathcal{D}_{M-1}(\mathbb{R}^{M+K})$, the Compactness Theorem is valid.

3) For $T_j \in \mathcal{I}_M(\mathbb{R}^{M+K})$, then by Weak Polyhedral Approximation Theorem 3.2.4, there exists a sequence $\{P_j^l\}$ of currents of the form

$$P_j^l = \sum_{F \in \mathcal{L}_M} p_F^{(l)} \eta_{\rho_l \#}[|F|], \quad p_F^{(l)} \in \mathbb{Z}, \tag{3.52}$$

such that P_j^l and ∂P_j^l converge weakly to T_j and ∂T_j, respectively, in U as $\rho_l \downarrow 0$. Since $\partial P_j^l \in \mathcal{I}_{M-1}(\mathbb{R}^{M+K})$, by assumption in 2) we have that $\partial T_j \in \mathcal{I}_{M-1}(\mathbb{R}^{M+K})$, then $\partial T \in \mathcal{I}_{M-1}(\mathbb{R}^{M+K})$. This result is called the *boundary rectifiability theorem.* By Proposition 3.1.1, $\delta_0 ⋇ \partial T_j$ and $\delta_0 ⋇ \partial T$ are also Integer-Multiplicity currents.

Next we show that without loss of generality, we can assume that $\partial T_j = 0$. If $\partial T_j \neq 0$, letting $\widetilde{T}_j = T_j - \delta_0 ⋇ \partial T_j$, we have

$$\begin{aligned}\partial \widetilde{T} &= \partial T_j - \partial(\delta_0 ⋇ \partial T_j) \\ &= \partial T_j - \partial T_j - \delta_0 ⋇ \partial^2 T_j \\ &= 0.\end{aligned}$$

So, if $\widetilde{T}_j$ is an Integer-Multiplicity current, T_j is also an Integer-Multiplicity current. Then it is enough to consider the case that $\partial T_j = 0$ and obviously $\partial T = 0$.

We also observe that $\partial[T_j \lfloor \mathbb{B}(x,r)]$ is an Integer-Multiplicity current, by assumption in 2), $\partial[T \lfloor \mathbb{B}(x,r)]$ is also an Integer-Multiplicity current. This allows us to use the Density Lemma 3.3.7: there exists $\delta > 0$ such that

$$\Theta_*^M(\|T\|, x) = \liminf_{r \downarrow 0} \frac{||T||[\mathbb{B}(x,r)]}{\omega^M r^M} > \delta$$

holds for $\|T\|$-almost every $x \in \mathbb{R}^{M+K}$. By Lemma 2.3.1, we see that $||T||$ is absolutely continuous with respect to $\mathcal{H}^M$ on $\mathbb{R}^{M+K}$. By Radon-Nikodym Theorem, we conclude that there exists a real-valued function $\theta > \delta$ such that $||T|| = \mathcal{H}^M \lfloor \theta$. (One can find more details in Remark 2.37, Theorem 3.24 and Theorem 8.1 in [3].)

Next let $A = \{x \in \mathbb{R}^{M+K} : \theta(x) > 0\}$. Since$||T||(A) < \infty$, we have $\mathcal{H}^M(A) < \infty$. Consider α a multi-index with

$$1 \leq \alpha_1 < \cdots < \alpha_M \leq M + K. \tag{3.53}$$

Let

$$\begin{aligned}\mathbf{p}_\alpha : \mathbb{R}^{M+K} &\longrightarrow \mathbb{R}^M \\ (x_1, ..., x_{M+K}) = x &\longmapsto (x_{\alpha_1}, ..., x_{\alpha_M}).\end{aligned}$$

be the orthogonal projection. By Theorem 3.3.5, we see that $u(y) = \langle T, \mathbf{p}_\alpha, y \rangle$ is an MBV function. By Theorem 3.3.8, we see that there is a set $E_\alpha \subseteq \mathbb{R}^M$ with $\mathcal{L}^M(E_\alpha) = 0$ such that

$$S_\alpha = \bigcup_{y \in \mathbb{R}^M \setminus E_\alpha} \operatorname{spt}[u(y)]$$

is a countably M-rectifiable subset of $\mathbb{R}^{M+K}$. Also set

$$B_\alpha = A \cap \mathbf{p}_\alpha^{-1}(E_\alpha).$$

We have $A \subseteq S_\alpha \cup B_\alpha$.

Letting I denote the set of all the multi-indices as in (3.53), we see that

$$A \subseteq \bigcap_{\alpha \in I} [S_\alpha \cup B_\alpha] \subseteq S \cup B,$$

where

$$S = \bigcup_{\alpha \in I} S_\alpha, \quad B = \bigcap_{\alpha \in I} B_\alpha.$$

By Lemma 2.3.1, $T \lfloor B = 0$, so $T = T \lfloor S$.

We may suppose that $A \subseteq S$. By Theorem 3.3.5 we know that, for each $\alpha \in I$ and for $\mathcal{L}^M$-almost every $x \in \mathbb{R}^M$, $\langle T, \mathbf{p}_\alpha, x \rangle$ is an Integer-Multiplicity 0-current. So we conclude that θ is integer-valued.

Finally, Theorem 3.3.4 tells us that T is an Integer-Multiplicity current. $\square$

3.4 Minimizing Mass and Plateau's Problem

Thanks to the Compactness Theorem, we can now easily reach our final goal: prove the existence of solutions to the Plateau's problem for Integral currents. Using the argument in Section 2.1.2, by compactness, any minimizing sequence of Integral currents with a fixed boundary admits a weakly convergent subsequence. Combined with the weak lower semicontinuity of mass and the continuity of the boundary operator, the limit current inherits both the prescribed boundary and minimality. This framework bridges geometric intuition with the abstract measure-theoretic tools.

The next definition of *mass-minimizing* formalizes the goal in Plateau's problem, where solutions represent surfaces of "least area" constrained by fixed boundaries.

Definition 3.4.1. Suppose that $U \subseteq \mathbb{R}^N$ is open and $T \in \mathcal{I}_M(\mathbb{R}^N)$ is an Integer-Multiplicity current. For a subset $B \subseteq U$, we say that T is *mass-minimizing* in B if

$$M_W[T] \leq M_W[S] \tag{3.54}$$

holds whenever $S \in \mathcal{I}_M(\mathbb{R}^N)$ and

$$\begin{aligned} &W \subset\subset U, \quad \partial S = \partial T, \\ &\operatorname{spt}[S - T] \text{ is a compact subset of } B \cap W. \end{aligned}$$

Remark 3.4.1. In case $B = \mathbb{R}^N$, we say simply that T is *mass-minimizing*. If, additionally, T has compact support, then Definition 3.4.1 reduces to the requirement that

$$M[T] \leq M[S]$$

hold whenever $\partial S = \partial T$.

If R is a nontrivial $(M-1)$-dimensional current that is the boundary of some Integer-Multiplicity current, then it makes sense to ask whether there exists a mass-minimizing Integer-Multiplicity current with R as its boundary. The next theorem tells us that indeed, such a mass-minimizing current does exist.

Theorem 3.4.1 (Plateau's Problem)**.** *Suppose that $1 \leq M \leq N$. If $R \in \mathcal{D}_{M-1}(\mathbb{R}^N)$ has compact support and if there exists an Integral current $Q \in I_M(\mathbb{R}^N)$ with $R = \partial Q$, then there exists a mass-minimizing Integral current $T \in I_M(\mathbb{R}^N)$ such that $\partial T = R$.*

Proof. Let $\{T_i\}_{i=1}^{\infty} \in I_M(\mathbb{R}^N)$ be a sequence of Integral currents with $\partial T_i = R$, for $i = 1, 2, ...$, and with

$$\lim_{i\to\infty} M[T_i] = \inf\{M[S] : \partial S = R,\ S \in \mathcal{I}_M(\mathbb{R}^N)\}.$$

Set $m = \operatorname{dist}(\operatorname{spt} R, 0)$ and let $f : \mathbb{R}^N \to \overline{\mathbb{B}(0,m)}$ be the nearest-point retraction:

$$f(x) = \begin{cases} x & \text{if } x \in \overline{\mathbb{B}(0,m)} \\ y & \text{if } x \notin \overline{\mathbb{B}(0,m)} \end{cases}$$

where $y \in \overline{\mathbb{B}(0,m)}$ is the unique point such that $\operatorname{dist}(x,y) = \operatorname{dist}(x, \mathbb{B}(0,m))$. Because the boundary operator and the pushforward operator commute, we have

$$\partial(f_{\#}T_i) = f_{\#}(\partial T_i) = f_{\#}R = R$$

for $i = 1, 2, \ldots$. Noting that $\operatorname{Lip}(f) = 1$, we conclude that

$$M[f_{\#}T_i] \leq M[T_i] < \infty$$

holds, for $i = 1, 2, \ldots$. Thus, by replacing T_i with $f_{\#}T_i$ if need be, we may suppose that $\operatorname{spt} T_i \subseteq \overline{\mathbb{B}}(0,m)$ holds for $i = 1, 2, \ldots$.

Now we consider the sequence of Integral currents $\{S_i\}_{i=1}^{\infty}$ defined by setting $S_i = T_i - Q$, for each $i = 1, 2, \ldots$. Noting that $\partial S_i = 0$ for each i, we see that the sequence $\{S_i\}_{i=1}^{\infty}$ satisfies the conditions of the Compactness Theorem 3.3.1. We conclude that there exists a subsequence $\{S_{i_k}\}_{k=1}^{\infty}$ of $\{S_i\}_{i=1}^{\infty}$ and an Integral current S^* such that $S_{i_k} \to S^*$ as $k \to \infty$. We conclude also that $\partial S^* = 0$.

Setting $T = S^* + Q$, we see that $T_{i_k} = S_{i_k} + Q \to S^* + Q = T$ as $k \to \infty$ and that $\partial T = \partial(S^* + Q) = \partial S^* + \partial Q = \partial Q = R$. By the lower semicontinuity of the mass, we have

$$M[T] = \inf\{M[S] : \partial S = R,\ S \in \mathcal{I}_M(\mathbb{R}^N)\}.$$

Then, $T \in I_M(\mathbb{R}^N)$ is the desired mass-minimizing. $\square$

Bibliography

[1] Steven Krantz, Harold Parks, *Geometric Integration Theory*, Birkhäuser, Boston, MA, 2008.

[2] Ilkka Holopainen, *Geometric Measure Theory*, University of Helsinki, 2021. `https://www.mv.helsinki.fi/home/iholopai/GMT.pdf`

[3] Leon Simon, *Introduction to Geometric Measure Theory*, Stanford University , 2014. `https://web.stanford.edu/class/math285/ts-gmt.pdf`

[4] Lin, Fang-Hua; Yang, Xiaoping, *Geometric Measure Theory: An Introduction*, Science Press, Beijing, 2002.

[5] H. Federer, *Geometric Measure Theory*, Springer Verlag, 1969.

[6] Stefan Wenger, *Introduction to Geometric Measure Theory*, University of Fribourg, 2019. `https://homeweb.unifr.ch/wengerst/pub/GMT2019/IntroGMT.pdf`

[7] E. M. Stein, G. L. Weiss, *Introduction to Fourier Analysis on Euclidean Space*, Princeton University Press, Princeton, NJ, 1971.

[8] Evans, L.C, *Measure Theory and Fine Properties of Functions, Revised Edition*, CRC Press, 2015.

[9] Walter Rudin, *Functional Analysis, Second Edition*, McGraw-Hill, 1991.

[10] Steven Krantz, Harold Parks, *The Geometry of Domains in Space*, Birkhäuser, Boston, Cambridge, MA, 1999.

[11] L. Ambrosio, B. Kirchheim, Currents in metric spaces, Acta Math. 185 2000, 1-80.

[12] Urs Lang. Local currents in metric spaces. J. Geom. Anal., 21(3):683–742, 2011.

[13] E. De Giorgi, Problema di Plateau generale e funzionali geodetici. Atti Sem. Mat. Fis. Univ. Modena, 43, 1995, 285-292.

[14] J. Douglas, Solutions of the Plateau problem. Trans. Amer. Math. Soc. 33 (1931), no. 1, 263–321.

[15] Garnier, René. Le problème de Plateau. Annales scientifiques de l'École Normale Supérieure 45 (1928): 53-144.

[16] T. Radó, The problem of least area and the problem of Plateau. Math. Z. 32(1930), 763–796.

[17] E. R. Reifenberg, Solution of the Plateau Problem for m-dimensional surfaces of varying topological type. Acta Math. 104 (1960), 1–92.

[18] H. Federer, The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension. Bull. Amer. Math. Soc. 76 (1970), 767–771.

[19] F. J. Almgren, Q-valued functions minimizing Dirichlet's integral and the regularity of area minimizing rectifiable currents up to codimension two. Bull. Amer. Math. Soc. (N.S.) 8 (1983), no. 2, 327–328.

[20] Wendell H. Fleming, On the oriented Plateau problem. Rend. Circ. Mat. Palermo (2) 11 (1962), 69–90.

[21] F. J. Almgren, Some interior regularity theorems for minimal surfaces and an extension of Bernstein's Theorem. Ann. of Math. (2), Vol. 84 (1966), 277–292.

[22] F. Morgan, *Geometric Measure Theory. A Beginner's Guide*. 4th edition, Academic Press, Inc., San Diego, CA, 2009. 264pp.

[23] Fefferman, C., Ionescu, A. D., Phong, D. H., and Wainger, S. *Advances in Analysis: The Legacy of Elias M. Stein*. Princeton University Press, 2014. Chapter 8.

[24] De Lellis, C., Spadaro, E. (2016). Regularity of area minimizing currents II: center manifold. Annals of Mathematics, 183(2), 499–575.

[25] De Lellis, C., Spadaro, E. (2016). Regularity of area minimizing currents III: blow-up. Annals of Mathematics, 183(2), 577–617.

[26] Lytchak, Alexander, Wenger, Stefan. (2017). Area Minimizing Discs in Metric Spaces. Archive for Rational Mechanics and Analysis. Vol. 223, 1123–1182.

[27] Almgren, F., Browder, W., Caldini, G., De Lellis, C. (2024). Optimal smooth approximation of integral cycles. arXiv:2411.17678v1 [math.DG].