Controllability of large scale networks

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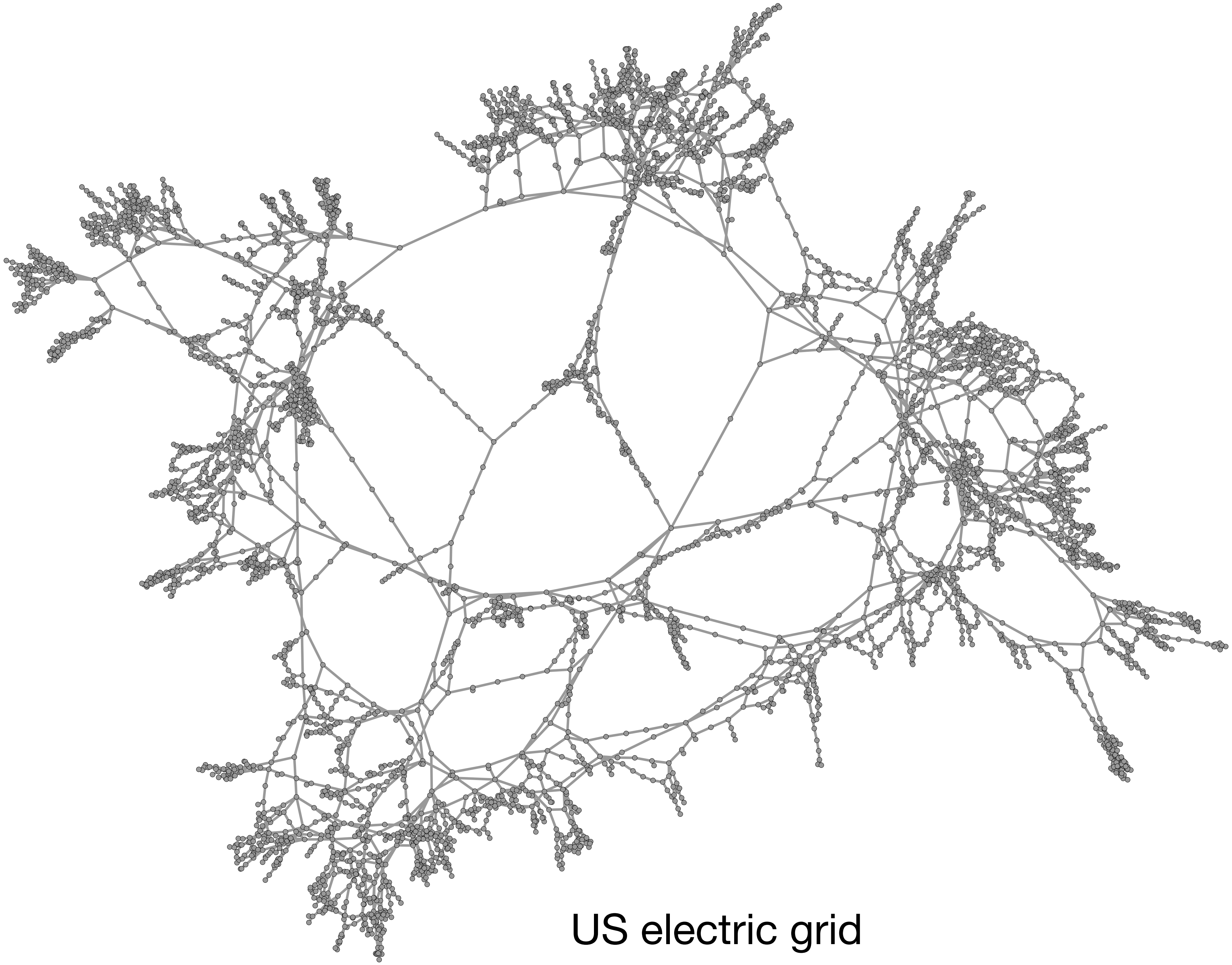
Large scale networks



US electric grid

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Problem formulation

Consider a linear system

$$x(t+1) = Ax(t) + Bu(t)$$

where A is sparse $n \times n$ matrix (the interactions between the states is described by a graph) and

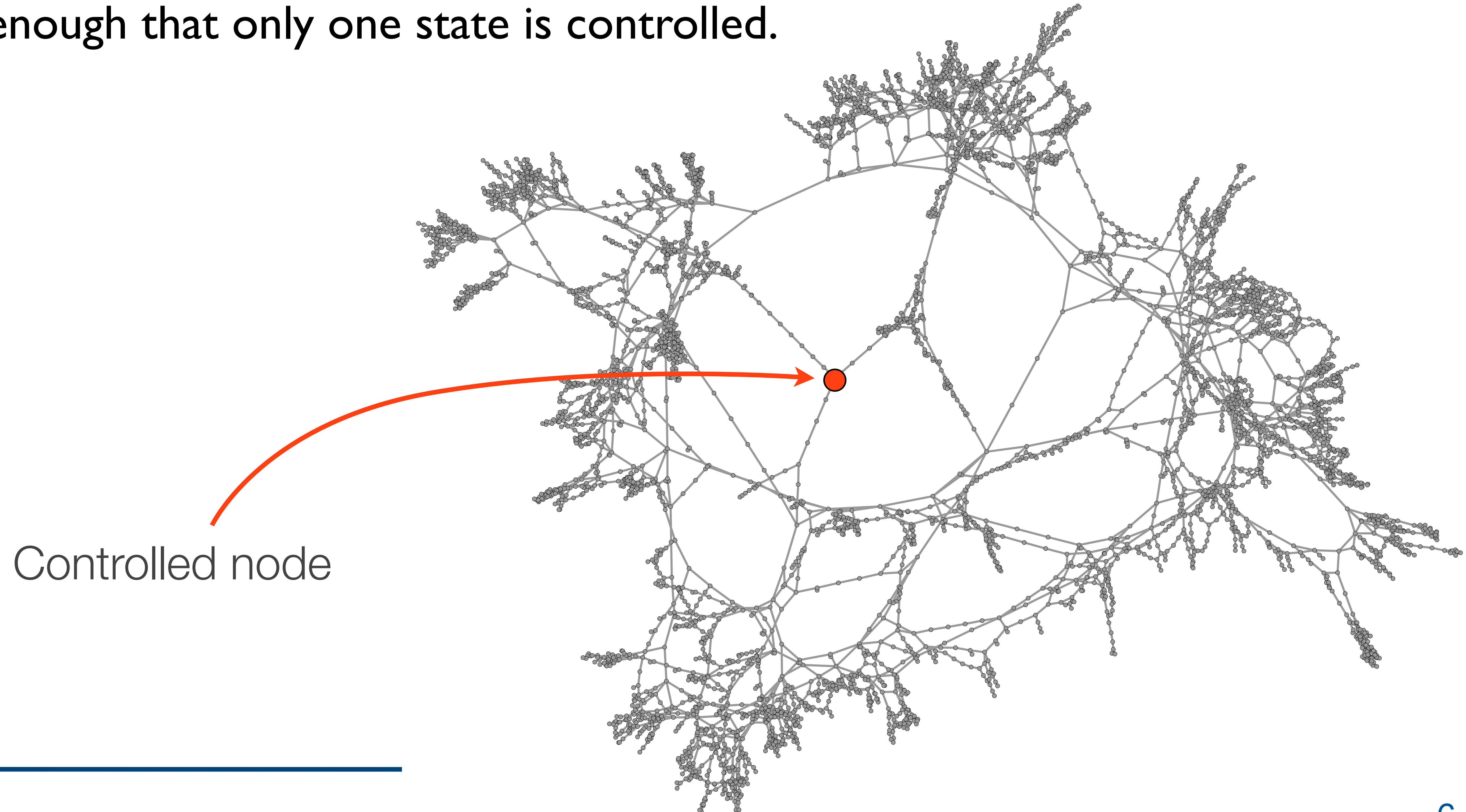
$$B = \begin{bmatrix} e_{i_1} & \cdots & e_{i_m} \end{bmatrix}$$

where

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$$

A controllability metric

If the graph is strongly connected and there the self-loops then to have that the resulting system is controllable generically (in the non-zero entries of A) it is enough that only one state is controlled.



A controllability metric

QUESTION: How controllable is the resulting system?

A controllability metric

There are various choices of metrics describing how controllable a system is. We choose the minimum eigenvalue of the controllability Gramian $\lambda_{min}(W_T)$ where

$$W_T := \sum_{t=0}^{T-1} A^t B B^T A^t$$

we are assuming that A is symmetric.

The energy to drive the state from zero to a norm one state (in the worst case) is given by

$$E = \frac{1}{\lambda_{min}(W_T)}$$

low $\lambda_{min}(W_T)$ ⟷ low controllability

high $\lambda_{min}(W_T)$ ⟷ high controllability

Conditions ensuring low controllability

Fix any constant $0 < C < 1$ and let

$$n(C) := |\{\lambda \in \lambda(A) \ : \ |\lambda| \leq C\}|$$

Then

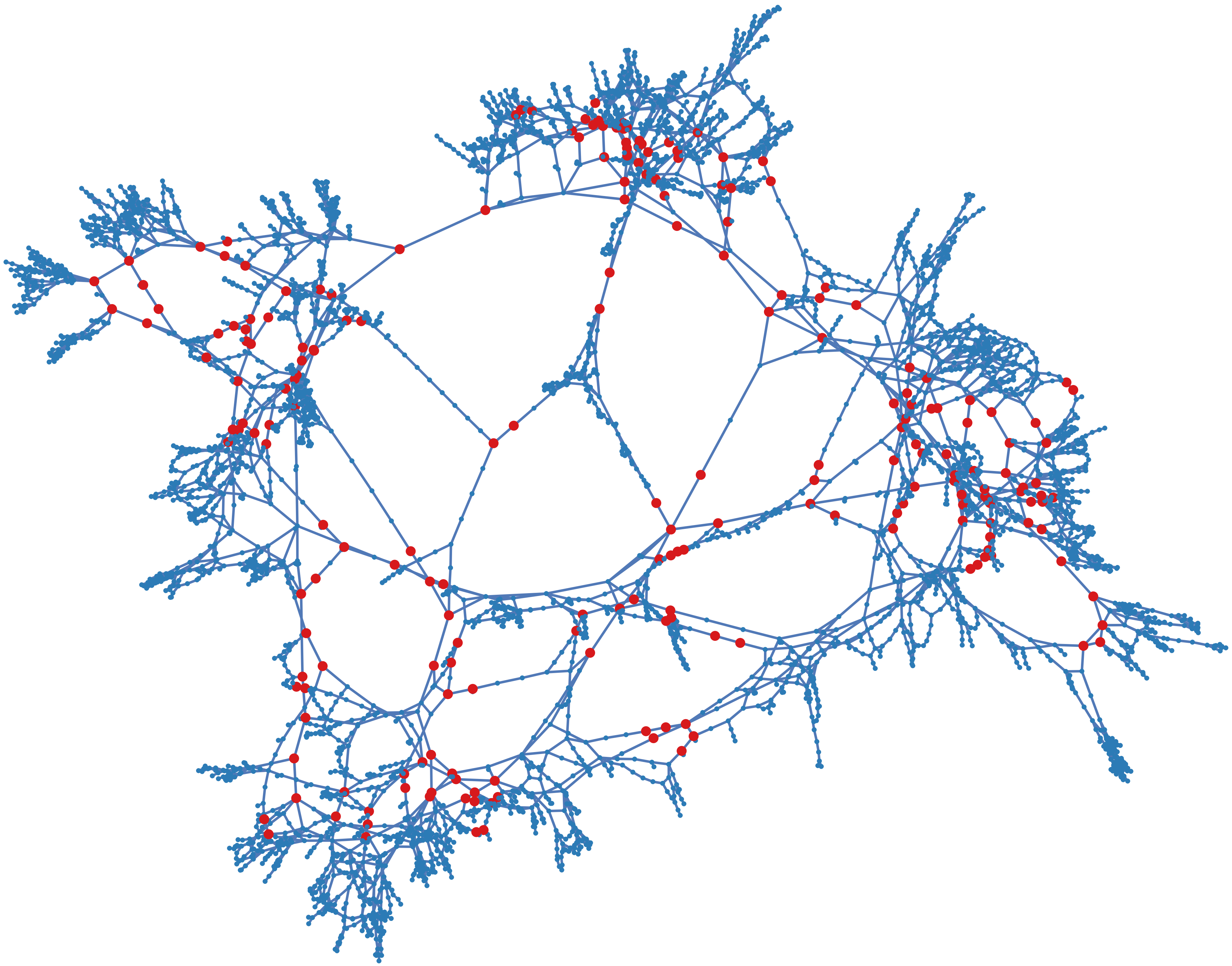
$$\lambda_{min}(W_T) \leq \frac{1}{C^2(1 - C^2)} C^{2\frac{n(C)}{m}}$$

Consequences Since $n(C)$ which typically grow linearly in n, then

1. for fixed m the degree of controllability decreases at least exponentially in $n(C)$. Therefore typically for fixed m the degree of controllability decreases at least exponentially in n.

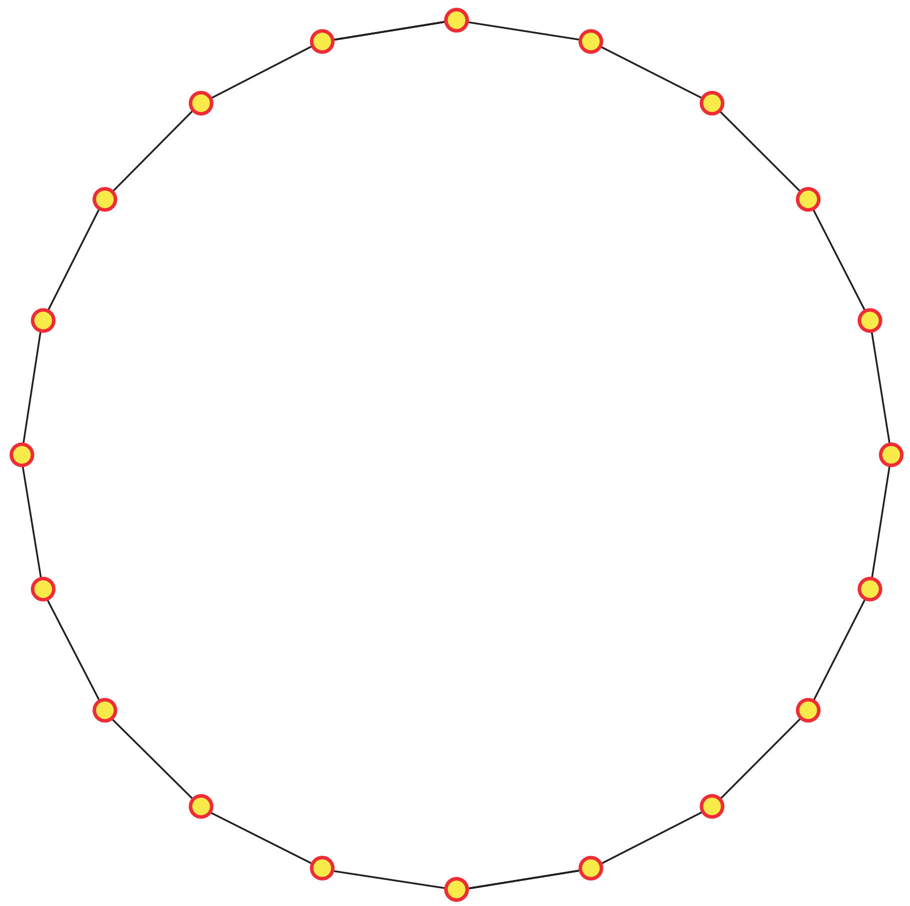
2. In order to have a fixed degree of controllability we need to control a fixed fraction of states.

Conditions ensuring low controllability

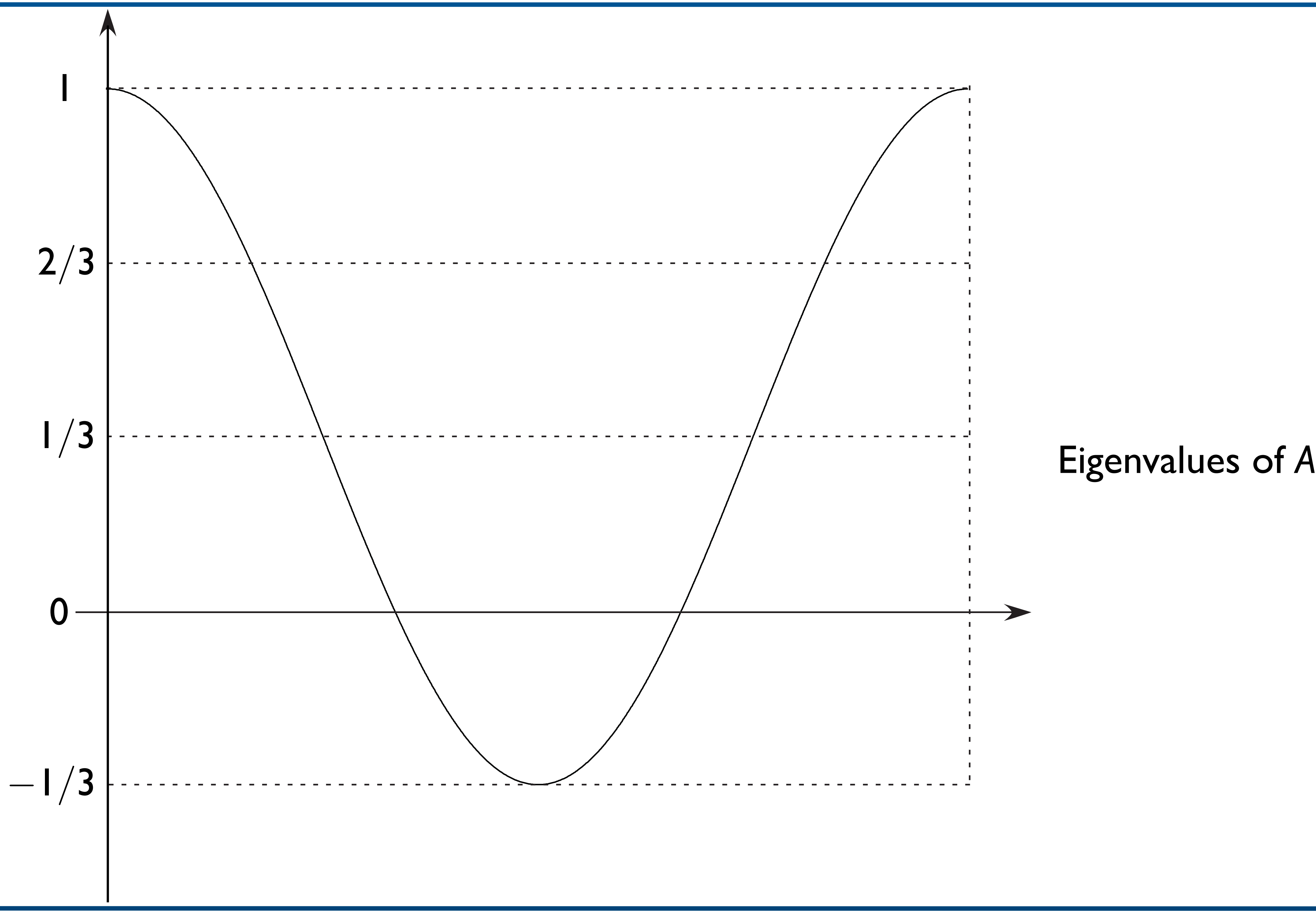


Example: consensus with circle graph

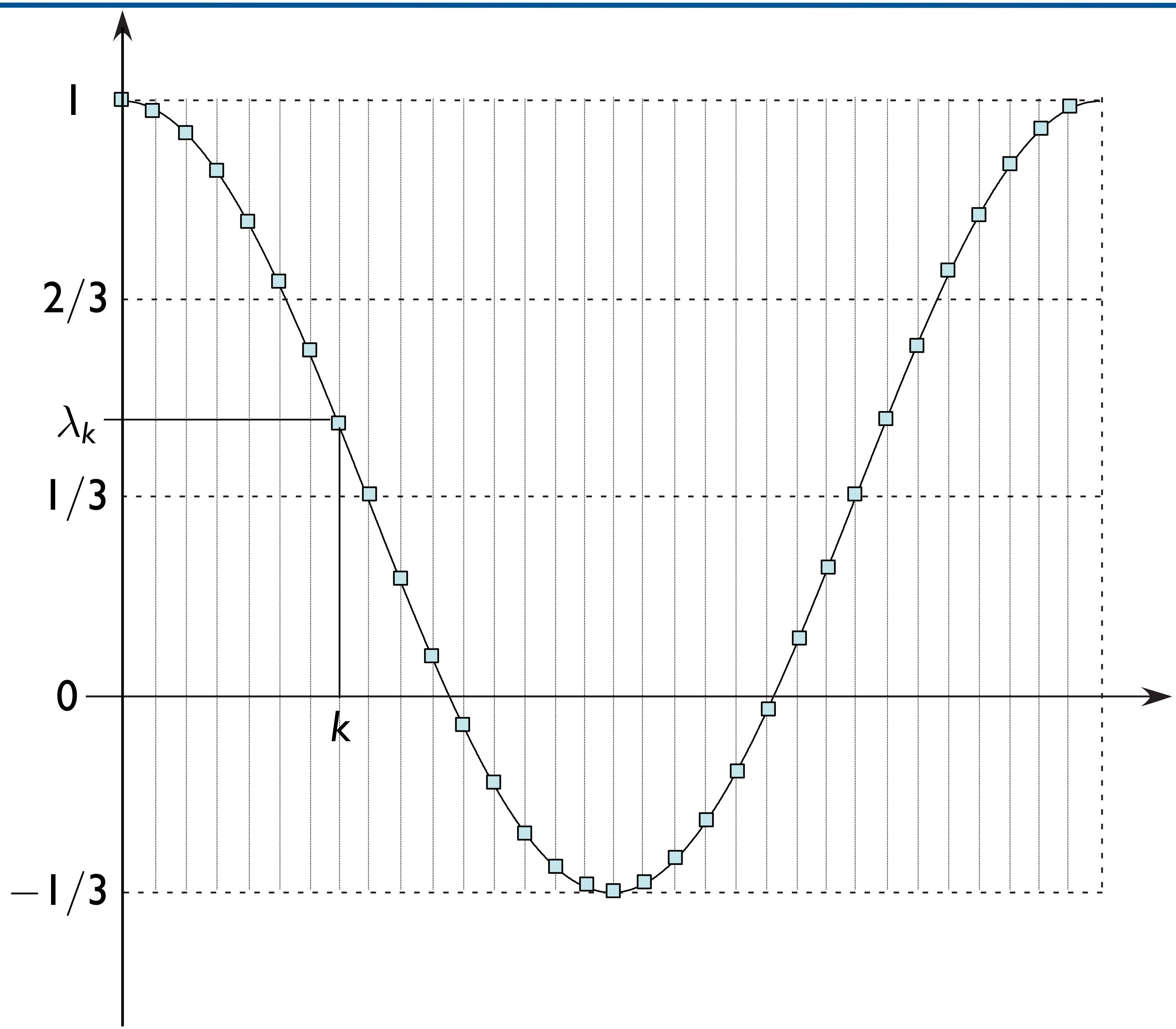
$$A = \begin{bmatrix} 1/3 & 1/3 & 0 & \cdots & \cdots & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & \cdots & \cdots & 0 & 0 \\ 0 & 1/3 & 1/3 & \ddots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \ddots & 1/3 & 1/3 \\ 1/3 & 0 & 0 & \cdots & \cdots & 1/3 & 1/3 \end{bmatrix}$$



Example: consensus with circle graph

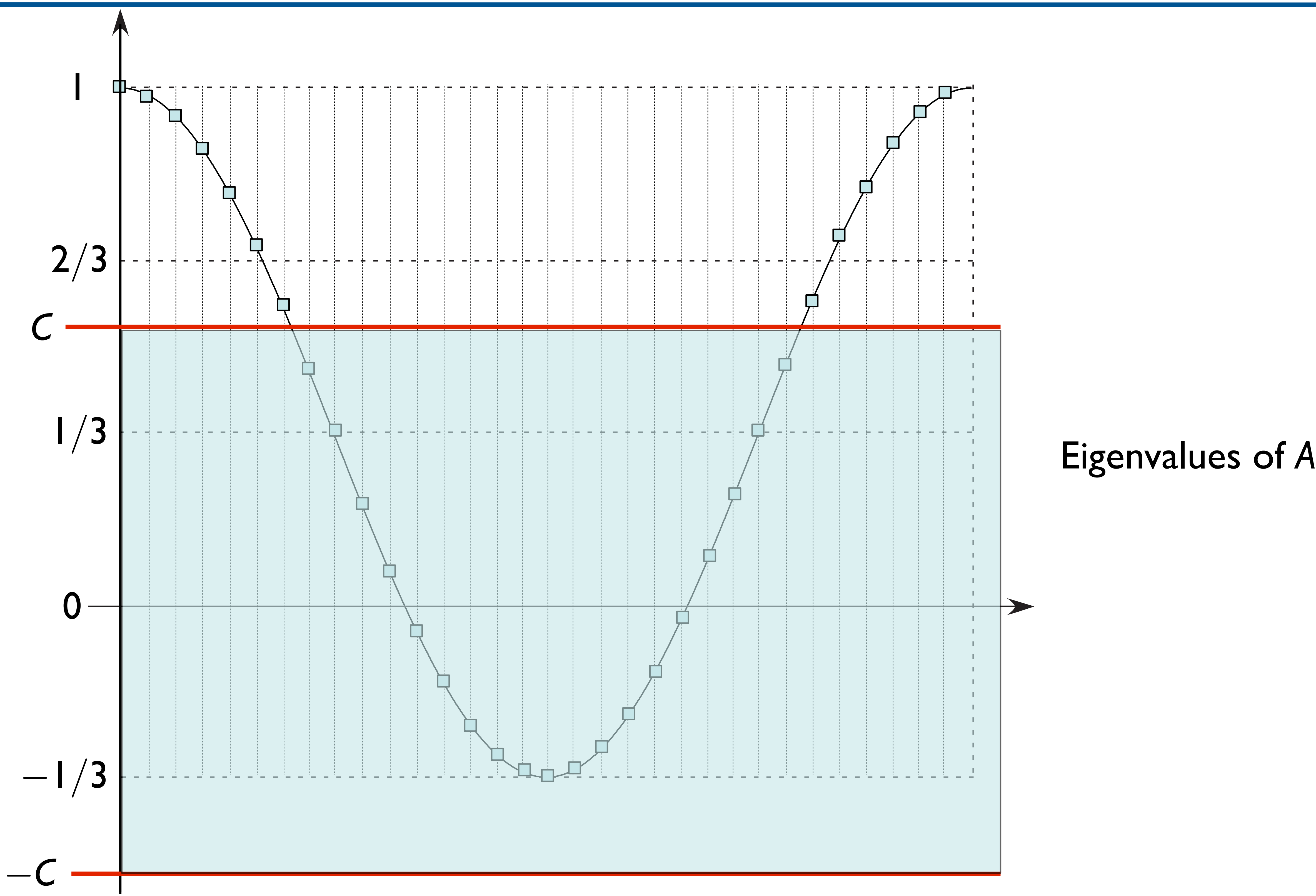


Example: consensus with circle graph



Eigenvalues of A

Example: consensus with circle graph



Decoupled control strategy

Network partitioning Partition $\mathcal{V} = \{1, \ldots, n\}$ into N disjoint sets $\mathcal{V}_1, \ldots, \mathcal{V}_N$. After relabeling of states and inputs, the matrices read as

$$A = \begin{bmatrix} A_1 & \cdots & A_{1N} \\ \vdots & \vdots & \vdots \\ A_{N1} & \cdots & A_N \end{bmatrix}, \; B = \begin{bmatrix} B_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_N \end{bmatrix},$$

The networks dynamics can be written as the interconnection of N subsystems of the form

$$x_i(t+1) = A_i x_i(t) + \sum_{j \in \mathcal{N}_i} A_{ij} x_j(t) + B_i u_i(t),$$

where $i \in \{1, \ldots, N\}$ and $\mathcal{N}_i := \{j \; : \; A_{ij} \neq 0\}$.

Decoupled control strategy

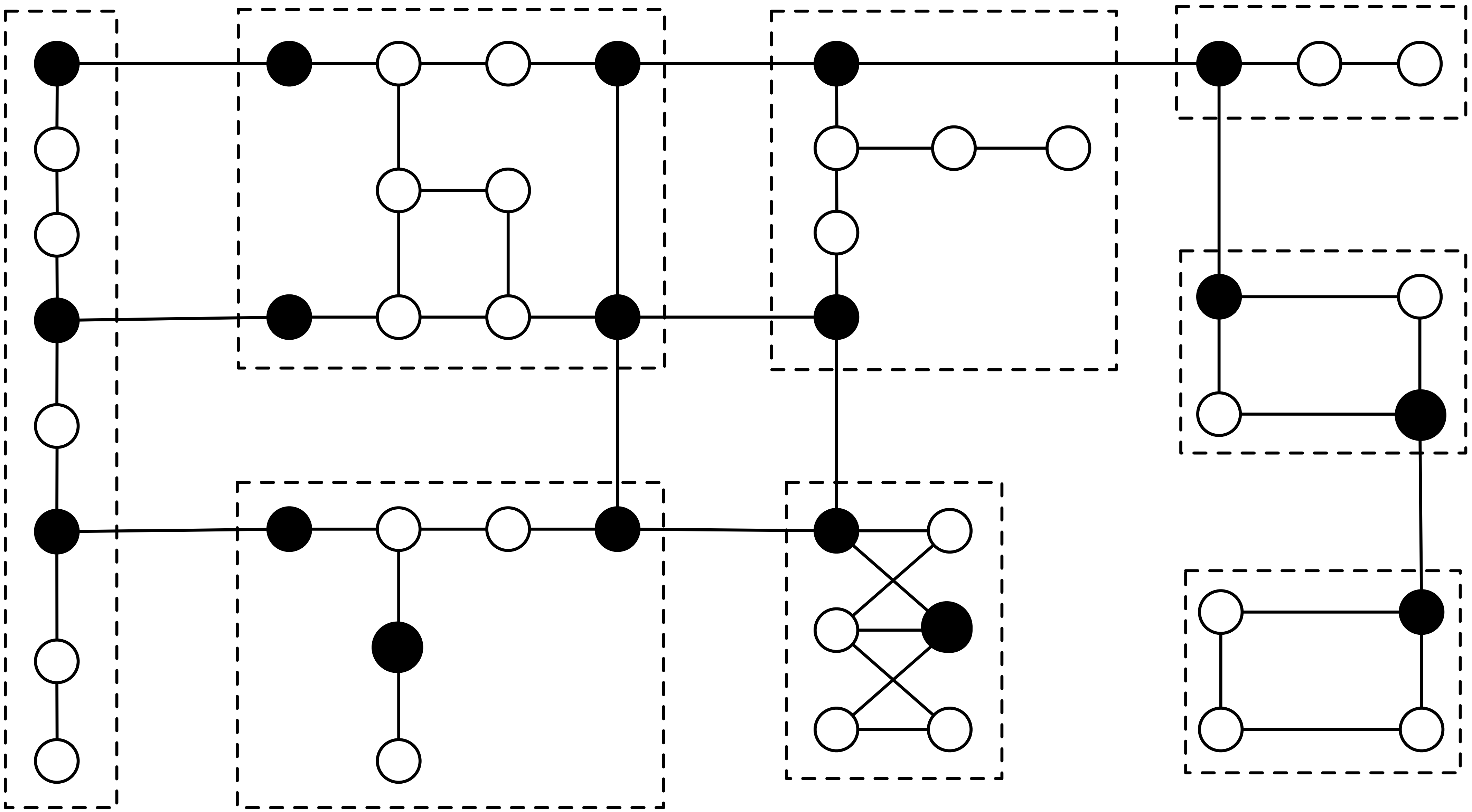
Selection of the control nodes We say that a node $i \in \mathcal{V}_k$ is a *boundary* node if $a_{ij} \neq 0$ for some node $j \in \mathcal{V}_\ell$, with $k, \ell \in \{1, \ldots, N\}$ and $k \neq \ell$.

Let $\mathcal{B}_i \subseteq \mathcal{V}_i$ be the set of boundary nodes in the *i*-th cluster, and let

$$\mathcal{B} = \bigcup_{i=1}^{N} \mathcal{B}_i$$

be the set of all the boundary nodes of the partition. We select the set of control nodes to be a set containing the boundary nodes.

High controllability and controllers positioning



Decoupled control strategy

The decoupled control law For the previous partitioned system consider the inputs

$$u_i(t) := v_i(t) - \sum_{j \in \mathcal{N}_i} B_i^T A_{ij} x_j(t)$$

Notice that this control law yields a new system composed by N decoupled subsystems

$$x_i(t+1) = A_i x_i(t) + B_i v_i(t)$$

The final step is to choose v_i which minimizes the energy to steer the subsystem to the desired substate. This will depend on the controllability Gramian $W_{i,T}$ the the i-th subsystem.

High controllability and controllers positioning

Define

$$\Lambda := \text{diag}(\lambda_{\min}^{-1}(W_{1,T}), \ldots, \lambda_{\min}^{-1}(W_{N,T})),$$

$$\Gamma := \begin{bmatrix} 1 & \gamma_{12} & \cdots & \gamma_{1N} \\ \gamma_{21} & 1 & \cdots & \gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{N1} & \gamma_{N2} & \ddots & 1 \end{bmatrix},$$

where

$$\gamma_{ij} = \|B_i^T A_{ij}(zI - A_j)^{-1} B_j\|_{H\infty}$$

Theorem If we choose a decoupled control law then we obtain

$$\lambda_{\min}(W_T) \geq \frac{1}{\|\Gamma\Lambda^{1/2}\|_2^2},$$

High controllability and controllers positioning

Define

$$\Delta := \begin{bmatrix} 1 & \|A_{12}\|_2 & \cdots & \|A_{1N}\|_2 \\ \|A_{21}\|_2 & 1 & \cdots & \|A_{2N}\|_2 \\ \vdots & \vdots & \ddots & \vdots \\ \|A_{N1}\|_2 & \|A_{N2}\|_2 & \cdots & 1 \end{bmatrix}.$$

and assume that

$$\bar{\lambda}_{\max} = \max\{\lambda_{\max}(A_i) : i \in \{1, \ldots, N\}\} < 1$$

Theorem If we choose a decoupled control law then we obtain

$$\lambda_{\min}(W_T) \geq \frac{(1 - \bar{\lambda}_{\max})^2}{\|\Lambda\|_\infty \|\Delta\|_\infty^2},$$

High controllability and controllers positioning

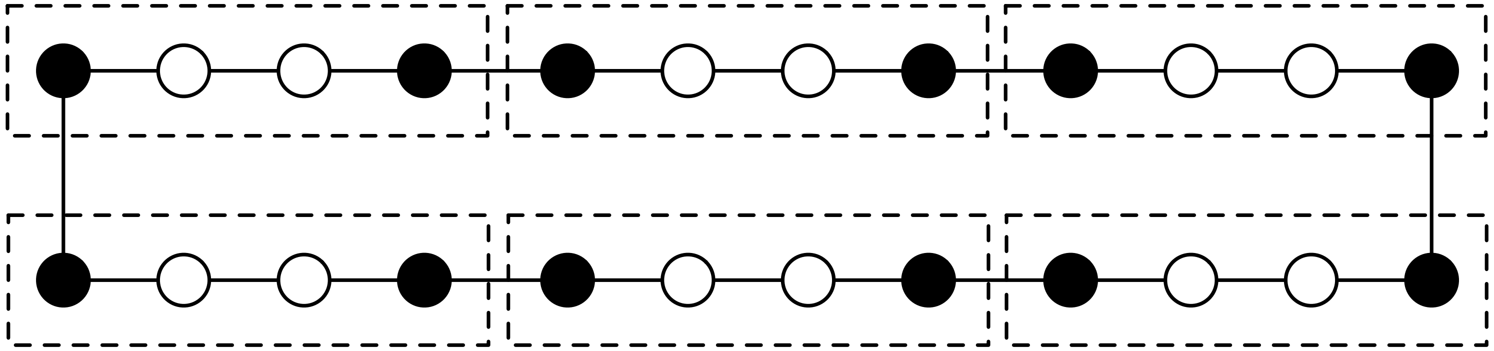
$$\lambda_{\min}(W_T) \geq \frac{1}{\|\Gamma\Lambda^{1/2}\|_2^2}, \qquad \lambda_{\min}(W_T) \geq \frac{(1-\bar{\lambda}_{\max})^2}{\|\Lambda\|_\infty \|\Delta\|_\infty^2},$$

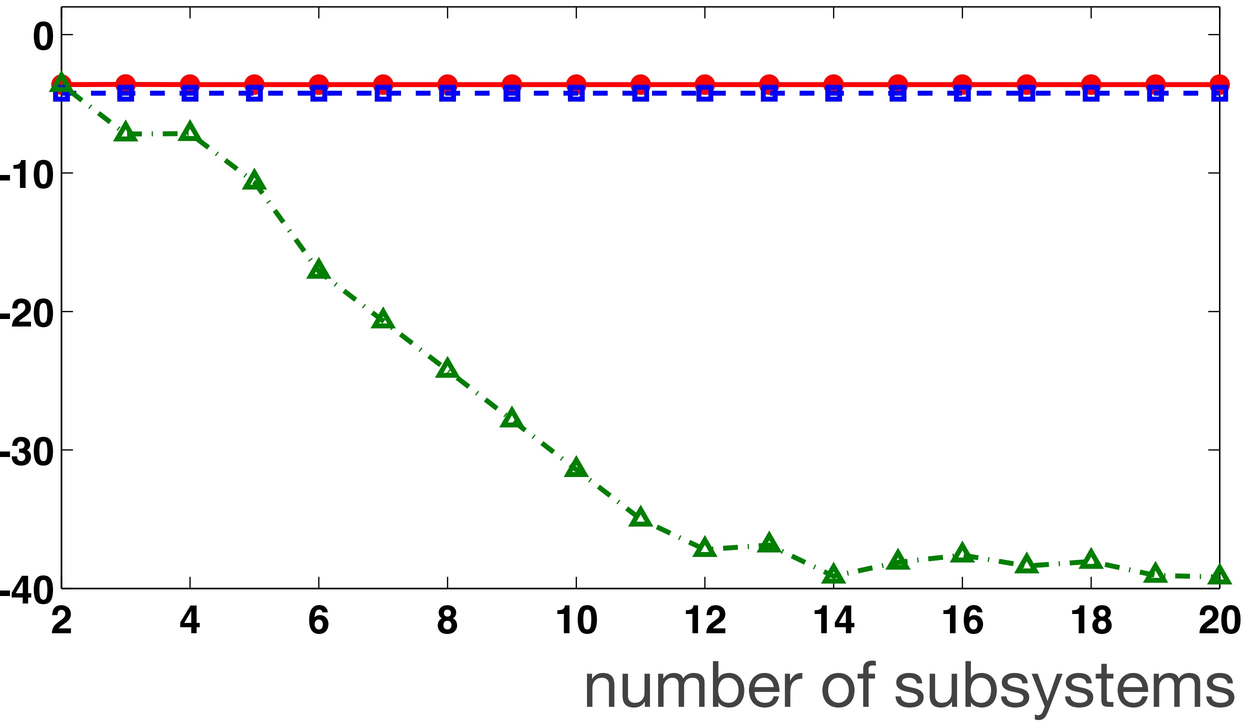
Consequences In order to have high controllability it is convenient to position the controllers in such a way that

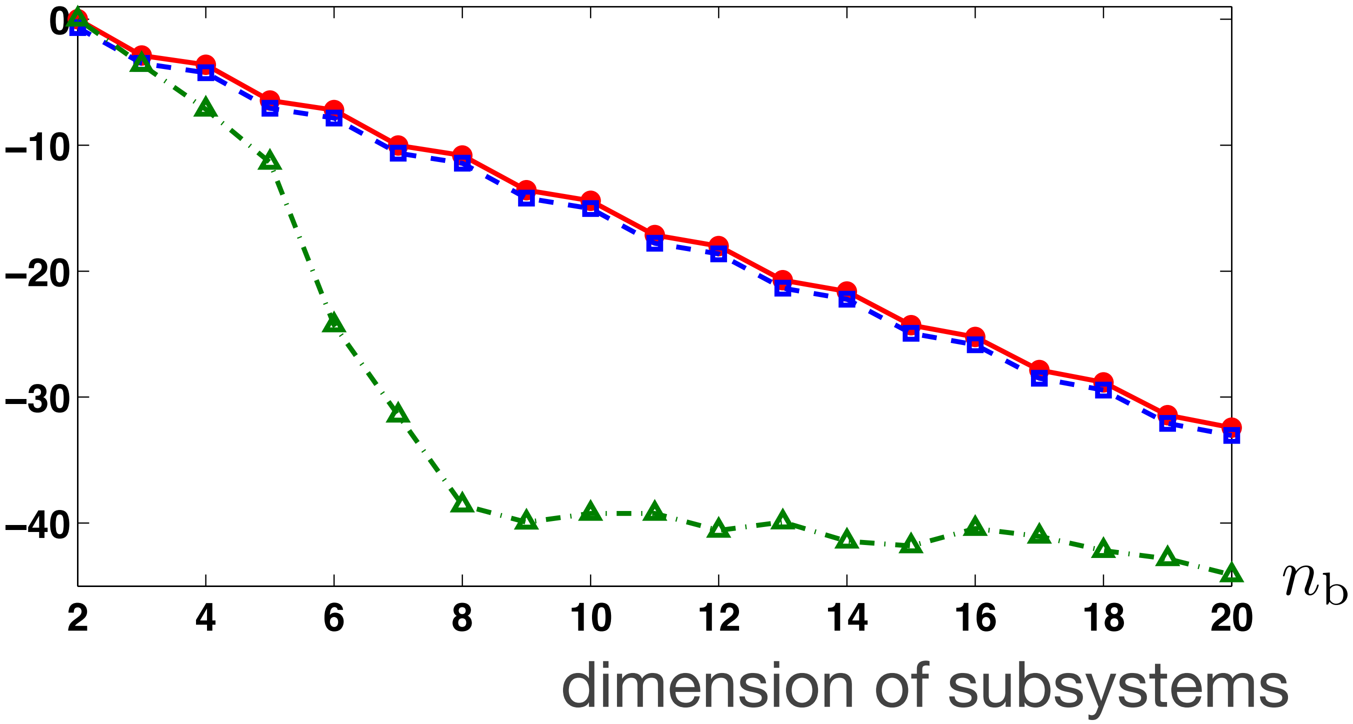
- The matrices Λ or Δ are small, which means choosing subsystems which are weakly interconnected.

- The matrix Δ is small, which means choosing controllers in the subsystems making them highly controllable.

Circulant graph





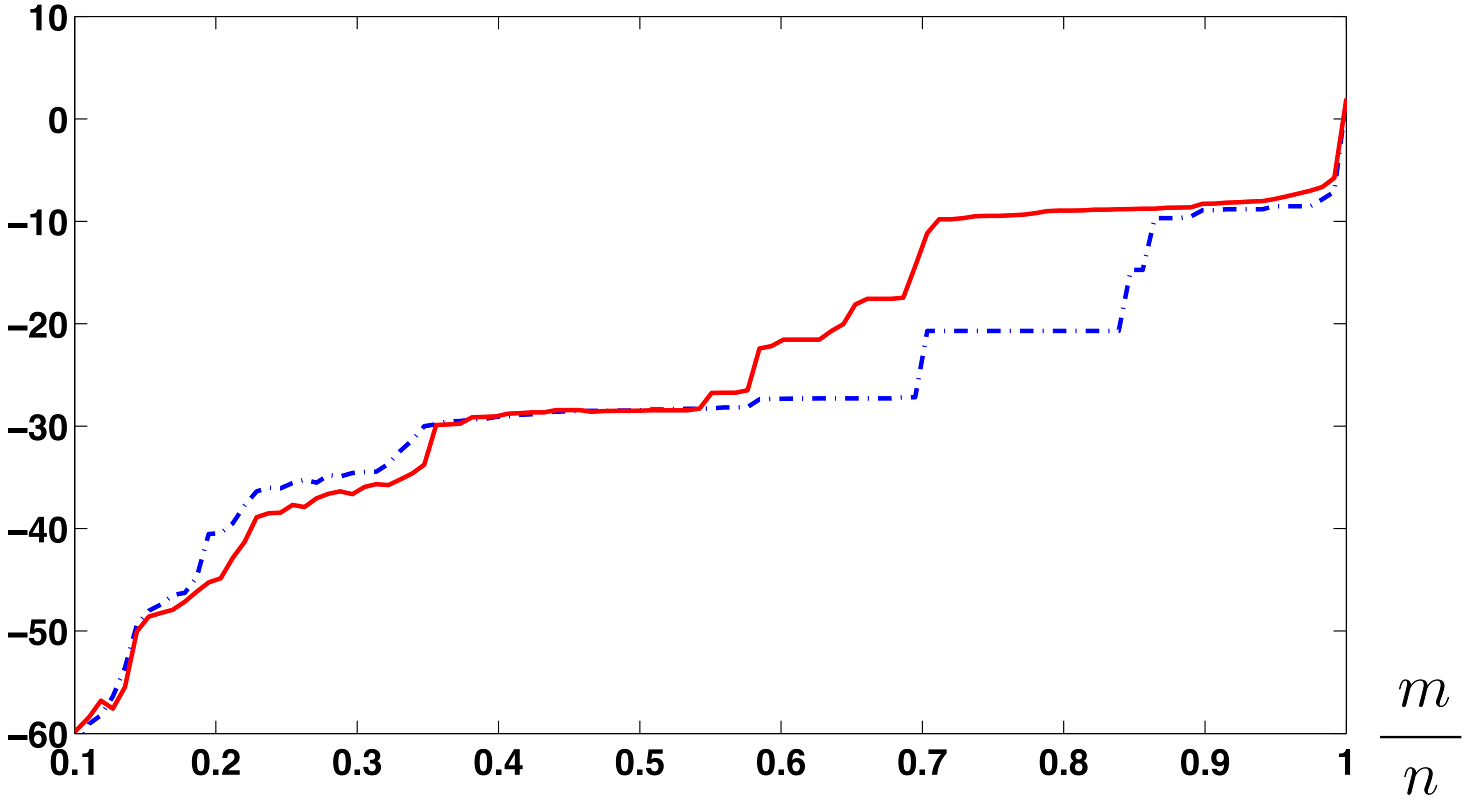


$\lambda_{\min}(W_T)$ with the decoupled control strategy

lower bound of $\lambda_{\min}(W_T)$ with the decoupled control strategy

$\lambda_{\min}(W_T)$ with random positioning

Power grid with 118 nodes

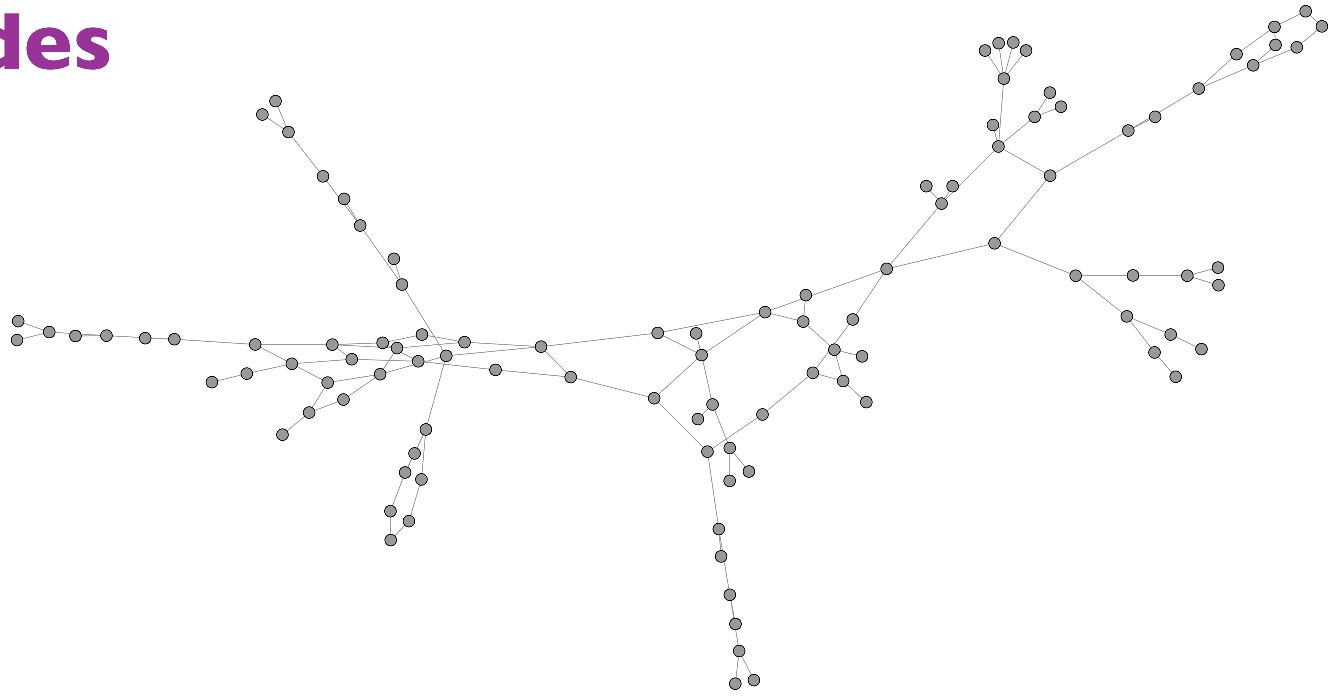


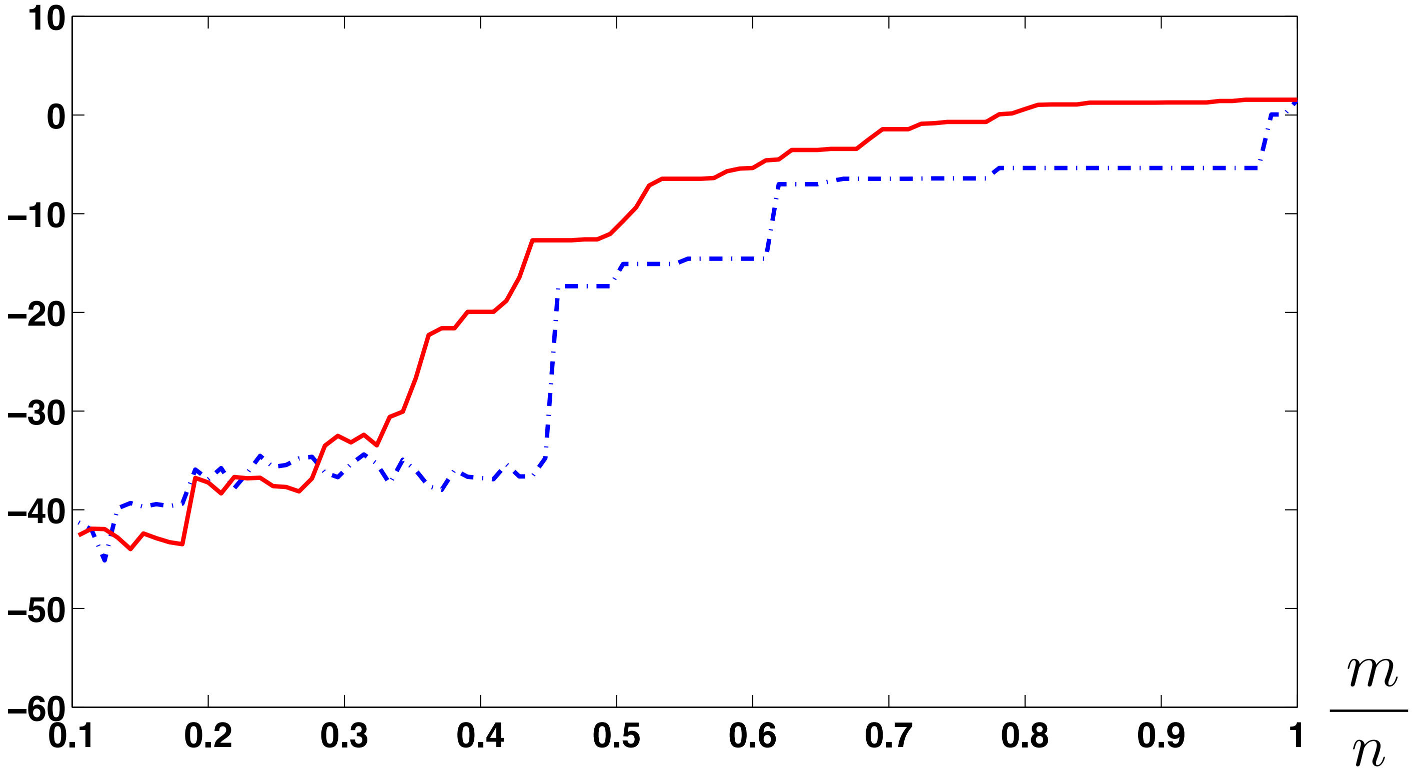
$\lambda_{\min}(W_T)$ with the decoupled control strategy

$\lambda_{\min}(W_T)$ with random positioning

Examples

Epidemics network with 86 nodes



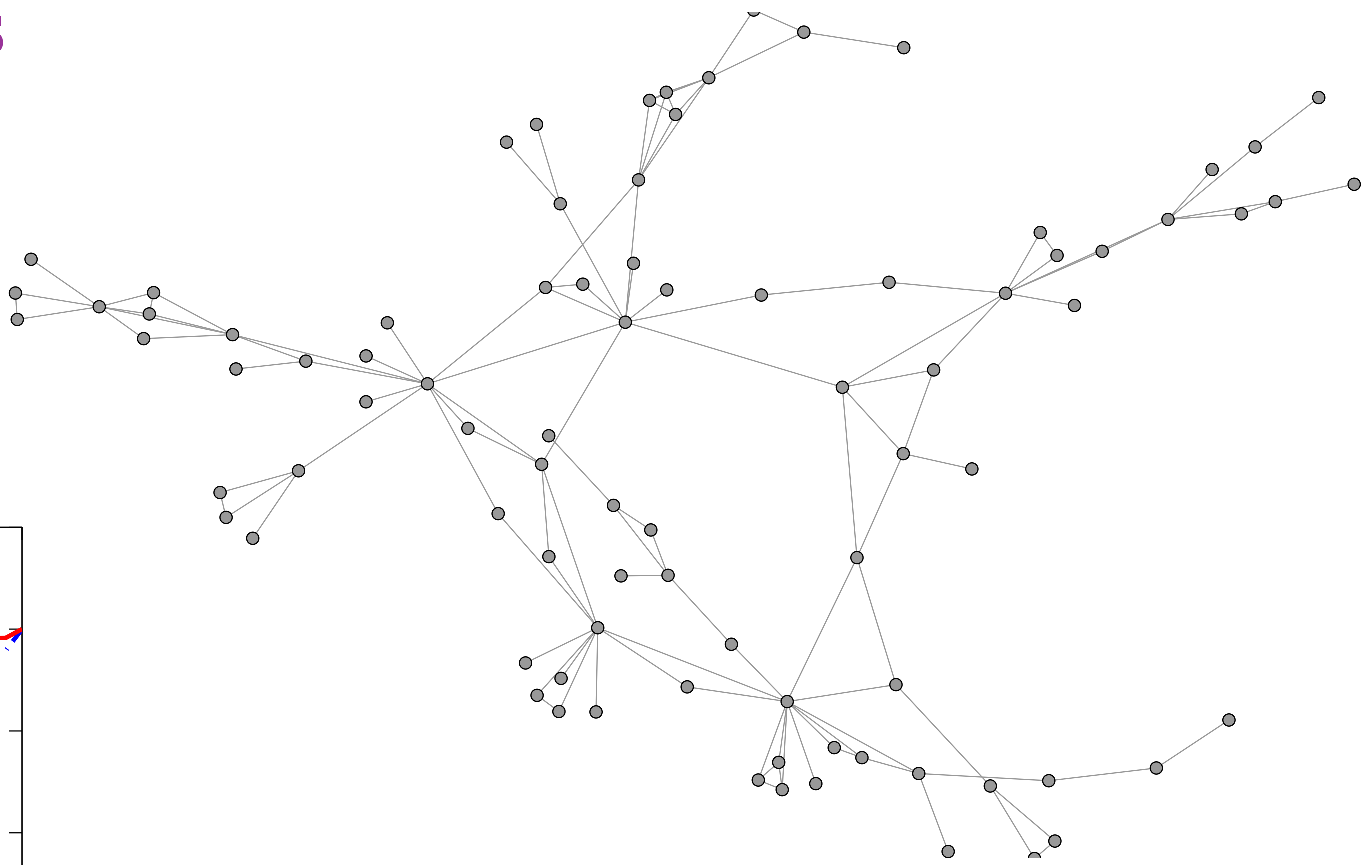


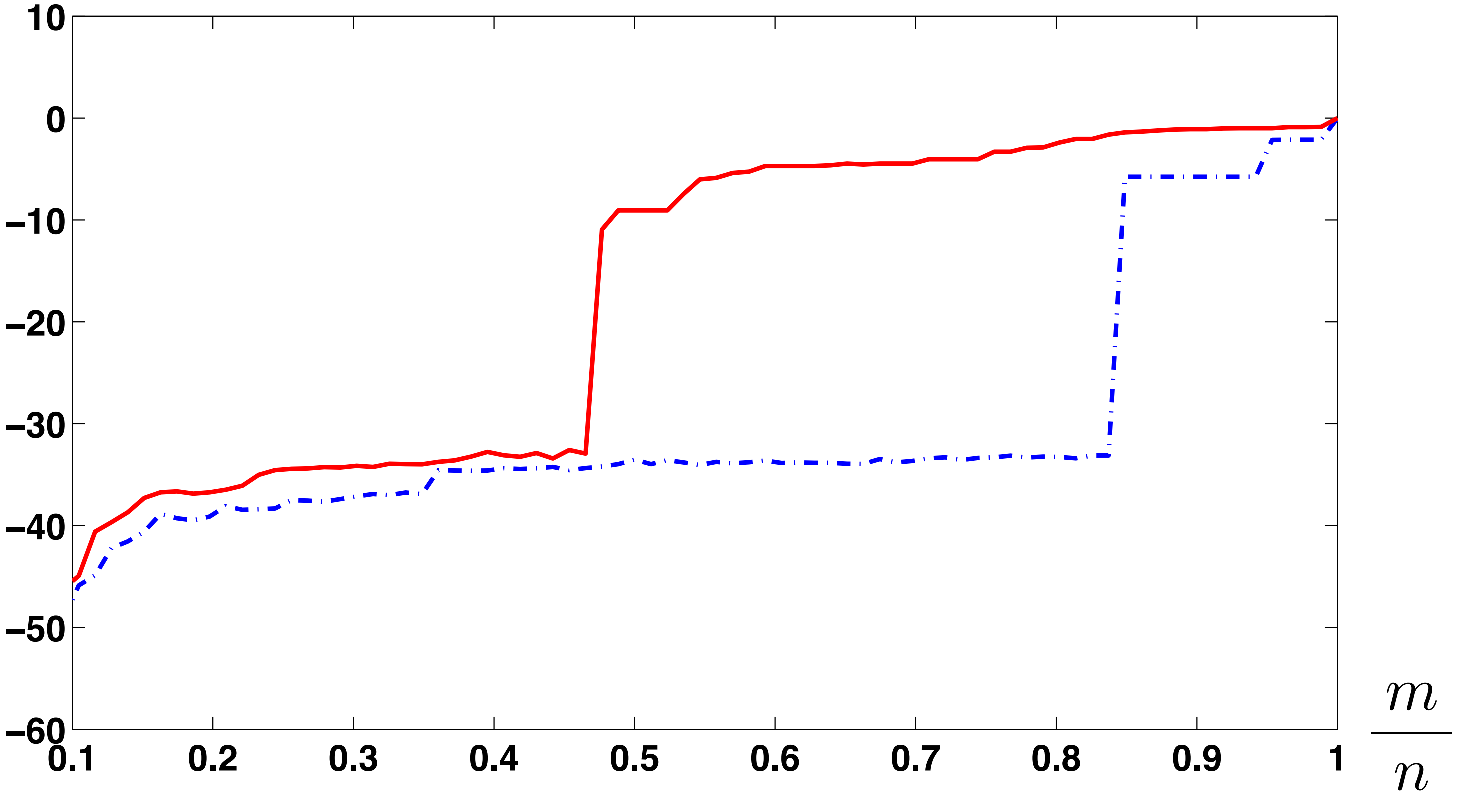
$\lambda_{\min}(W_T)$ with the decoupled control strategy

$\lambda_{\min}(W_T)$ with random positioning

Examples

Social network with 105 nodes





$\lambda_{\min}(W_T)$ with the decoupled control strategy

$\lambda_{\min}(W_T)$ with random positioning

Conclusions

- Similar results for **observability**
- For controllability we need to **control a fixed fraction** of nodes
- The decoupled control strategy works well for graph that are **partitionable**
- The decoupled control strategy admits a **decoupled control synthesis**
- **Random** positioning works pretty well
- **Phase transition** can be noticed (critical fraction of controlled nodes)
- There are a lot of open problems:
  - Controllability of random and of structured graphs
  - Performance of random positioning
  - Phase transition
  - Different metrics

Thank you