On the Multivariate Circulant Rational Covariance Extension Problem

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Rational covariance extension

Ingredients

- Let $y = \{y(t) \in \mathbb{C}^m,\ t \in \mathbb{Z}\}$ be a zero-mean, multivariate, wide-sense stationary random process.
- We know its covariance lags

$$C_k := \mathbb{E}[y(t+k)y^*(t)] \in \mathbb{C}^{m\times m} \quad \text{for } k = 0, \dots, n$$

and the Toeplitz matrix

$$T_n = \begin{bmatrix} C_0 & C_1^* & C_2^* & \cdots & C_n^* \\ C_1 & C_0 & C_1^* & \cdots & C_{n-1}^* \\ C_2 & C_1 & C_0 & \cdots & C_{n-2}^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_n & C_{n-1} & C_{n-2} & \cdots & C_0 \end{bmatrix}$$

is positive definite

Rational covariance extension

Problem statement

Multivariate rational covariance extension

Given the sequence C_k, for $k = 0, \ldots, n$, find $C_{n+1}, C_{n+2}, \ldots$ up to infinity such that

$$\sum_{k=-\infty}^{+\infty} C_k e^{-jk\vartheta}, \quad C_{-k} = C_k^*$$

converges for all $\vartheta \in \mathbb{T}$ to a positive definite spectral density $\Phi(e^{j\vartheta})$ that has the rational form

$$\Phi(e^{j\vartheta}) = P(e^{j\vartheta})Q^{-1}(e^{j\vartheta}).$$

with $P(z), Q(z)$ are of the same kind as

$$M(z) = \sum_{k=-n}^{n} M_k z^{-k}, \quad M_{-k} = M_k^*$$

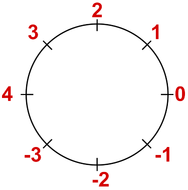
Circulant rational covariance extension

Covariance extension for periodic processes

Now assume y is a zero-mean, stationary m-dimensional process defined on $\mathbb{Z}_{2N}$, i.e. periodic of period $2N$.

Let

$\mathbf{y} := [y(-N+1)^\top, \dots, y(N)^\top]^\top$.



$\mathbb{Z}_{2N}$ for $N = 4$

Theorem

y is the restriction on $[-N+1, N]$ of a stationary, m-dimensional process $\tilde{y}$ periodic of period $2N$ if and only if its covariance matrix

$$\Sigma := \mathbb{E}[\mathbf{y}\mathbf{y}^*]$$

is Hermitian and block-circulant.

Preliminaries

Harmonic analysis in $\mathbb{Z}_{2N}$

- DFT : Let $\zeta_h := e^{jh\frac{\pi}{N}}$ and $\mathbf{g} := \{\mathbf{g}_k \in \mathbb{C}^m, k = -N+1, \dots, N\}$. Then DFT maps

$$\mathbf{g} \mapsto \mathbf{G}(\zeta_h) := \sum_{k=-N+1}^{N} \mathbf{g}_k {\zeta_h}^{-k}, \quad h = -N+1, \dots, N$$

- Inverse DFT:

$$\mathbf{g}_k = \frac{1}{2N} \sum_{h=-N+1}^{N} {\zeta_h}^{k} \mathbf{G}(\zeta_h), \quad k = -N+1, \dots, N$$

$$= \int_{-\pi}^{\pi} e^{jk\vartheta} \mathbf{G}(e^{j\vartheta}) d\nu(\vartheta), \quad d\nu(\vartheta) := \sum_{h=-N+1}^{N} \delta(e^{j\vartheta} - \zeta_h) \frac{d\vartheta}{2\pi}$$

Problem statement

Multivariate circulant rational covariance extension

Given the sequence C_k's with values in $\mathbb{C}^{m\times m}$, for $k = 0, \ldots, n$, for $n < N$, find a rational spectral density $\Phi = PQ^{-1}$ such that

$$\int_{\pi}^{\pi} e^{jk\vartheta}\Phi(e^{j\vartheta})d\nu(\vartheta) = \frac{1}{2N}\sum_{h=-N+1}^{N} {\zeta_h}^k \Phi(\zeta_h) = C_k, \quad k = 0, 1, \ldots, n.$$

- We require $P, Q \in \mathfrak{M}_+^{(m,n)}(N)$, i.e. the set of pseudo-polynomials

$$M(\zeta) = \sum_{k=-n}^{n} M_k \zeta^{-k}$$

such that $M_{-k} = M_k^*$, $M_k \in \mathbb{C}^{m\times m}$ and

$$M(\zeta_h) > 0, \text{ for } h = -N+1, \ldots, N$$

Circulant rational covariance extension

In terms of matrices...

- Covariance extension for periodic processes is equivalent to compute C_{n+1}, C_{n+2} up to C_N properly...
- It can be recast as a circulant matrix completion problem.
- Toy problem with $n = 2, N = 4$:

$$\underbrace{\begin{bmatrix} C_0 & C_1^* & C_2^* & 0 & 0 & 0 & C_2 & C_1 \\ C_1 & C_0 & C_1^* & C_2^* & 0 & 0 & 0 & C_2 \\ C_2 & C_1 & C_0 & C_1^* & C_2^* & 0 & 0 & 0 \\ 0 & C_2 & C_1 & C_0 & C_1^* & C_2^* & 0 & 0 \\ 0 & 0 & C_2 & C_1 & C_0 & C_1^* & C_2^* & 0 \\ 0 & 0 & 0 & C_2 & C_1 & C_0 & C_1^* & C_2^* \\ C_2^* & 0 & 0 & 0 & C_2 & C_1 & C_0 & C_1^* \\ C_1^* & C_2^* & 0 & 0 & 0 & C_2 & C_1 & C_0 \end{bmatrix}}_{=:\mathbf{C}} \mapsto \underbrace{\begin{bmatrix} C_0 & C_1^* & C_2^* & C_3^* & C_4 & C_3 & C_2 & C_1 \\ C_1 & C_0 & C_1^* & C_2^* & C_3^* & C_4 & C_3 & C_2 \\ C_2 & C_1 & C_0 & C_1^* & C_2^* & C_3^* & C_4 & C_3 \\ C_3 & C_2 & C_1 & C_0 & C_1^* & C_2^* & C_3^* & C_4 \\ C_4 & C_3 & C_2 & C_1 & C_0 & C_1^* & C_2^* & C_3^* \\ C_3^* & C_4 & C_3 & C_2 & C_1 & C_0 & C_1^* & C_2^* \\ C_2^* & C_3^* & C_4 & C_3 & C_2 & C_1 & C_0 & C_1^* \\ C_1^* & C_2^* & C_3^* & C_4 & C_3 & C_2 & C_1 & C_0 \end{bmatrix}}_{=:\Sigma}$$

- Recall that block-circulant matrices are block-diagonalized by DFT.
- $\Sigma > 0 \Leftrightarrow P, Q \in \mathfrak{M}_+^{(m,n)}(N)$

Circulant rational covariance extension

Our assumptions...

- First assume $P(\zeta)$ is fixed
- For technical reasons it has the form $P(\zeta) = p(\zeta)I$, with p scalar pseudo-polynomial in $\mathcal{P}_+^{(1,n)}(N)$
- The sequence $\{C_k\}_{k=0,\dots,n}$ is such that

$$C(\zeta) := \sum_{k=-n}^{n} C_k \zeta^{-k}, \quad C_{-k} = C_k^*$$

belongs to $\mathfrak{C}_+^{(m,n)}(N)$, defined as the *dual cone* of $\mathfrak{M}_+^{(m,n)}(N)$, i.e. the set of all $C(\zeta)$ such that

$$\langle C, M \rangle \geq 0, \quad \forall M(\zeta) \in \mathfrak{M}_+^{(m,n)}(N)$$

Circulant rational covariance extension

Main Result

Theorem

If the previous assumptions hold

- There exists a unique $\hat{Q}(\zeta) \in \mathfrak{M}_+^{(m,n)}(N)$ such that $\hat{\Phi}(\zeta) := P(\zeta)\hat{Q}(\zeta)^{-1}$ maximizes the generalized entropy

$$\mathbb{I}_P(\Phi) = \int_{-\pi}^{\pi} P(e^{j\vartheta}) \log\det \Phi(e^{j\vartheta}) d\nu(\vartheta)$$

  and solves the circulant covariance extension problem

$$\int_{\pi}^{\pi} e^{jk\vartheta}\Phi(e^{j\vartheta}) d\nu(\vartheta) = C_k, \quad \text{for } k = 0, \ldots, n$$

- $\hat{Q}(\zeta)$ is the unique minimizer of

$$\mathbb{J}_P(Q) := \langle C, Q \rangle - \int_{-\pi}^{\pi} P(e^{j\vartheta}) \log\det Q(e^{j\vartheta}) d\nu(\vartheta)$$

  over all $Q \in \mathfrak{M}_+^{(m,n)}(N)$

More on the computation of $\hat{Q}(\zeta)$ - 1

- DFT can be efficiently used in minimizing $\mathbb{J}_P(Q)$.
- Let

$$\mathbf{M} = \mathrm{Circ}(M_0, M_1, ..., M_N, M_{N-1}^*, \ldots, M_1^*)$$

We say that

$$M(\zeta) = \sum_{k=-N}^{N} M_k \zeta^{-k}$$

is the symbol of $\mathbf{M}$.

More on the computation of $\hat{Q}(\zeta)$ - 2

- Circulant covariance extension can be recast in terms of matrices
- Let $C(\zeta)$, $P(\zeta)$ be the symbols of the block-circulant matrices $\mathbf{C}$ and $\mathbf{P}$, respectively. Then, we can compute $\hat{Q}(\zeta)$ by finding $\hat{\mathbf{Q}}$ which minimizes

$$\mathbb{J}_{\mathbf{P}}(\mathbf{Q}) = \frac{1}{2N}\mathrm{tr}\left[\mathbf{C}\mathbf{Q}\right] - \frac{1}{2N}\mathrm{tr}\left[\mathbf{P}\log\mathbf{Q}\right]$$

over all

$$\mathbf{Q} = \mathrm{Circ}(Q_0, Q_1, \ldots, Q_n, 0, \ldots, 0, Q_n^*, \ldots, Q_1^*)$$

which are positive definite

Determining P from logarithmic moments - 1

- Aim: estimate P based on data only
- Idea: look for the spectral density Φ which maximizes the entropy gain

$$\int_{-\pi}^{\pi} \log\det\Phi(e^{j\vartheta})\ d\nu(\vartheta)$$

while satisfying the moment constraints which stem from the available covariance lags and the logarithmic moments

$$\gamma_k = \int_{-\pi}^{\pi} e^{jk\vartheta} \log\det\Phi(e^{j\vartheta}) d\nu(\vartheta), k = 1, 2, \dots, n$$

Determining P from logarithmic moments - 2

- Let $\Gamma(\zeta)$ be the pseudo-polynomial

$$\Gamma(\zeta) := \sum_{k=-n}^{n} \gamma_k \zeta^{-k}$$

- By duality theory this problem is problem can be solved by minimizing

$$\mathbb{J}(P,Q) := \langle C, Q\rangle - \int_{-\pi}^{\pi} P(e^{j\vartheta}) \log\det Q(e^{j\vartheta}) d\nu(\vartheta) \\ - \langle \Gamma, P\rangle + \int_{-\pi}^{\pi} P(e^{j\vartheta}) \log\det P(e^{j\vartheta}) d\nu(\vartheta)$$

over all the $(P,Q) \in \hat{\mathfrak{M}}_+^{(m,n)}(N) \times \mathfrak{M}_+^{(m,n)}(N)$, where

$$\hat{\mathfrak{M}}_+^{(m,n)}(N) := \left\{ M(\zeta) = m(\zeta) I \,|\, m(\zeta) \in \mathfrak{M}_+^{1,n} \, m_0 = 1 \right\}$$

Bilateral ARMA models

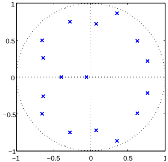
- After solving the rational circulant covariance extension problem we end up with a bilateral ARMA model:

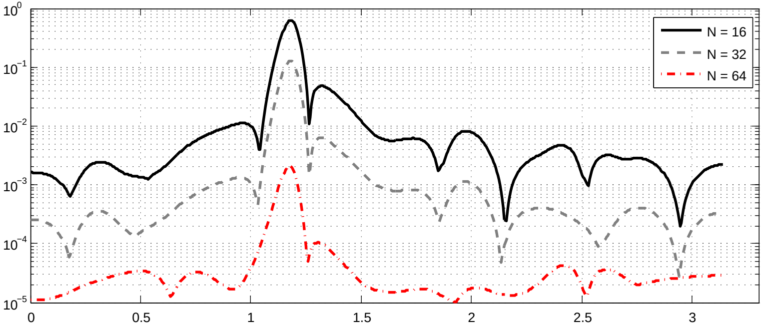
$$\sum_{k=-n}^{n} Q_k y(t-k) = \sum_{k=-n}^{n} P_k e(t-k), \quad t \in \mathbb{Z}_{2N}$$

- Open problem: do bilateral ARMA models generalize standard models for reciprocal processes?

Multivariate AR case

MVAR model of order 8

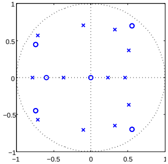


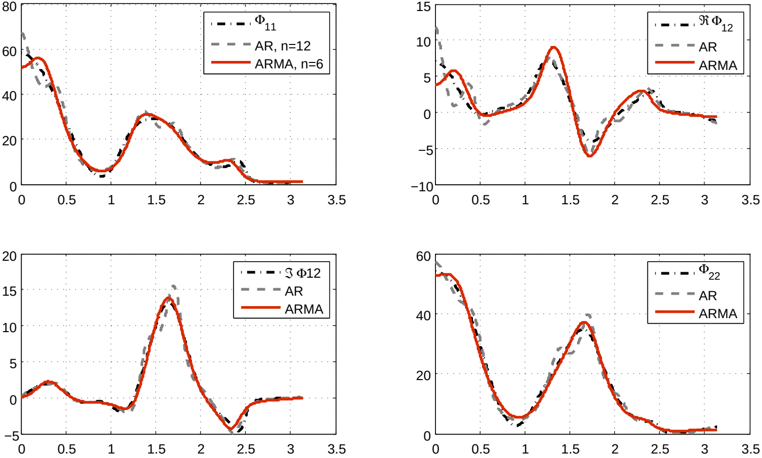


Estimation error

Multivariate ARMA case

Zero poles map





Comparison between AR (N=64, n=12) and ARMA (N=32, n=6)

Conclusions and Future Work

Conclusions

- A first step towards rational covariance extension for multivariate periodic processes
- Fast approximation of regular multivariate rational covariance extension

Future work

- Extension to rational models with general $P(\zeta)$
- Connection with reciprocal models
- Application to image processing (textures)

Thank you for your attention!

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Interpretation of bilateral ARMA models - 1

- After solving the rational circulant covariance extension problem we end up with a bilateral ARMA model:

$$\sum_{k=-n}^{n} Q_k y(t-k) = \sum_{k=-n}^{n} P_k e(t-k)$$

- Note that $e(t)$ is not white noise.
- Is there any connection with reciprocal processes?

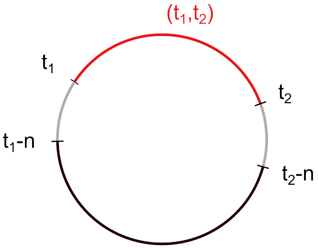
Interpretation of bilateral ARMA models - 2

Reciprocal processes

A reciprocal process y of order n defined on $[-N+1, N]$ is characterized by the following property:

$$\hat{\mathbb{E}}[y_{(t_1,t_2)} \mid y(s), s \in (t_1, t_2)^c] = \hat{\mathbb{E}}[y_{(t_1,t_2)} \mid y_{[t_1-n,t_1)} \vee y_{(t_2,t_2+n]}]$$

for $t_1, t_2 \in [-N+1, N]$.



Interpretation of bilateral ARMA models - 3

- Consider the case of bilateral AR models.
- Σ is the covariance matrix of a reciprocal process of order n the discrete group if and only if Σ^{-1} is a positive-definite, Hermitian, block-circulant matrix which is banded of bandwidth n.
  [Carli, Ferrante, Pavon and Picci, 2011]
- Idea: bilateral ARMA models somewhat generalize reciprocal processes. This point is the subject of current research.