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Since the beginnings of Calculus, differential equations have provided an effective mathematical model for a wide variety of physical phenomena. Consider a ‘state’ $x = (x_1, \ldots, x_n)$ and assume that the evolution $t \mapsto x(t)$ can be described by the ODE

$$\dot{x}(t) = g(t, x(t)).$$

(1)

Under suitable assumptions on $g$, if the state of the system is known at some initial time $t_0$:

$$x(t_0) = x_0,$$

(2)

then the future (and past) behavior of the state $x$ can be uniquely determined (Cauchy Problem). Celestial mechanics provides a typical example of this situation, where we can only predict but not modify the evolution of the system.
Control Theory provides a different paradigm. We now assume the presence of an external agent, i.e. a "controller", who can actively influence the evolution of the system. This new situation is modeled by a control system, namely

\[ \dot{x}(t) = f(t, x(t), u) \quad u(\cdot) \in \mathcal{U}, \]  

(3)

where \( \mathcal{U} \) is a family of admissible control functions. In this case, the rate of change \( \dot{x}(t) \) depends not only on the state \( x \) itself, but also on some external parameters, say \( u = (u_1, \ldots, u_m) \), which can also vary in time.

The control function \( u(\cdot) \), subject to some constraints, will be chosen by a controller in order to achieve certain preassigned goals:

- steer the system from one state to another (controllability),
- keep the state close to a given position in long time (stabilizability),
- minimize a certain cost functional (optimization), etc...
Open loop and closed loop controls

We may have

- **Open loop controls**: \( u = u(t) \),

  \[ u(\cdot) \in U := \{ u : [t_0, t_1] \to U = \overline{U} \subset \mathbb{R}^m, \text{measurable} \} . \]

  Given such \( u(\cdot) \), \( t \mapsto f(t, x(t), u(t)) \equiv g(t, x(t)) \) is a classical differential equation.

- **Closed loop or feedback controls**: \( u = u(x) \in U \).

  Usually, \( x \mapsto u(x) \) that performs a given task is *discontinuous*, hence \( t \mapsto f(t, x(t), u(x)) \equiv g(t, x(t)) \) is NOT a classical differential inclusion (discontinuous in \( x \)).

  **Finding open loops controls** is often easier, but feedback controls have the advantage that they do not need to be redetermined if the initial configuration is changed. Moreover, they are robust in the presence of random perturbations.

  **We will consider only open loop controls.**
Differential inclusions

The control system can then be written as a differential inclusion, namely

\[ \dot{x}(t) \in F(t, x(t)) \]  

where the set of possible velocities at any time \( t \) is given by

\[ F(t, x) := \{ f(t, x, u) : u \in U \} \]

Clearly, every admissible trajectory of the control system (3) is also a solution of (4). Under some regularity assumptions on \( f \), it turns out that the converse is also true.

Figure 1.1: A differential equation vs. a differential inclusion.
Examples

Born in the ’60s, Control Theory has a lot of applications in MECHANICS, ENGINEERING, ECONOMICS, BIOLOGY, . . .

Example 1 (Boat on a river).

Consider a river with straight course. Using a set of planar coordinates, assume that it occupies the horizontal strip

\[ S := \{(x_1, x_2) : \ x_1 \in \mathbb{R}, \ x_2 \in [-1, 1]\} \]

Assume that speed of the water is given by the velocity vector

\[ \mathbf{v}(x_1, x_2) = (1 - x_2^2, 0) \].

If the boat is powered by an engine, its motion can be modeled by the control system

\[
\begin{align*}
\dot{x}_1, \dot{x}_2 &= \mathbf{v} + \mathbf{u} = (1 - x_2^2 + u_1, u_2) \\
\end{align*}
\]

where the vector \( \mathbf{u} \) describes the velocity of the boat relative to the water.

Let the control set \( U \) be the closed ball in \( \mathbb{R}^2 \) of radius \( M > 0 \).
Given an initial condition \((x_1, x_2)(0) = (\bar{x}_1, \bar{x}_2)\), solving (5) one finds

\[
\begin{align*}
\begin{cases}
    x_1(t) &= \bar{x}_1 + t + \int_0^t u_1(s) \, ds - \int_0^t (\bar{x}_2 + \int_0^s u_2(s') \, ds')^2 \, ds, \\
    x_2(t) &= \bar{x}_2 + \int_0^t u_2(s) \, ds 
\end{cases}
\end{align*}
\]

In particular, \(u \equiv (-2/3, 1)\) takes the boat from a point \((\bar{x}_1, -1)\) on one side of the river to \((\bar{x}_1, -1)\) on the opposite side in \(t = 2\).

Actually, the boat can be steered from any point to any other point.

Equivalent differential inclusion:

\[
(\dot{x}_1, \dot{x}_2) \in F(x_1, x_2) := \left\{(y_1, y_2) : \frac{1}{2} \sqrt{(y_1 - 1 + x_2^2)^2 + y_2^2} \leq M \right\}
\]
Example 2 (Car parking).

Consider a car in a large parking lot. At a given time, its position is determined by the coordinates \((x, y)\) of its barycenter \(B\) and the angle \(\theta\) giving its orientation. The driver controls the car by acting on the gas pedal and on the steering wheel. The controls are thus the speed \(u(t)\) of the car and the turning angle \(\alpha(t)\). The motion is described by the nonlinear control system

\[
(\dot{x}, \dot{y}, \dot{\theta}) = (u \cos \theta, u \sin \theta, \alpha u).
\]

It is natural to assume \((u, \alpha) \in U := [-m, M] \times [-\bar{\alpha}, \bar{\alpha}]\). The typical maneuver needed for parallel parking is illustrated in the figure.
Call \( t \mapsto x(t, u) \) the solution of the Cauchy problem

\[
\dot{x}(t) = f(t, x(t), u(t)) \quad u(\cdot) \in \mathcal{U}, \quad x(0) = \bar{x}.
\] (7)

A wide range of mathematical questions can be formulated. We will address just some of them:

**Concerned with the dynamics of the system:**

- Fundamental properties of trajectories and of the set of solutions
- Reachable set at time \( t \): \( \mathcal{R}(t) := \{x(t, u) : u(\cdot) \in \mathcal{U}\} \)

**Concerned with optimal control:** among all strategies which accomplish a certain task, one seeks an optimal one, which minimizes/maximizes a given **performance criterion**, as, for instance, \( J = \phi(T, x(T)) \).

- Existence of optimal controls
- Necessary conditions for the optimality of a control
Some (incomplete!) bibliography


an the references therein....
Properties of the control system

Let us now begin a study of the control system:

\[ \dot{x}(t) = f(t, x(t), u(t)) \quad u(\cdot) \in \mathcal{U}, \]

where

\[ \mathcal{U} := \{ u(\cdot) \text{ measurable, } u(t) \in U \text{ for all } t \} \]

We shall assume:

(H) The control set \( U \subset \mathbb{R}^m \) is compact, \( \Omega \) is an open subset of \( \mathbb{R} \times \mathbb{R}^n \), \( f \) is continuous in all the variables and \( C^1 \) (continuously differentiable) in \( x \).

**Definition 3.**

An absolutely continuous function \( x(\cdot) : [a, b] \rightarrow \mathbb{R}^n \) \((x \in AC([a, b], \mathbb{R}^n))\) is a solution to (8) if

- \( \text{graph} \ x(\cdot) := \{(t, x(t)) : \ t \in [a, b]\} \subset \Omega; \)
- \( \dot{x}(t) = f(t, x(t), u(t)) \) for a.e. \( t \in [a, b]. \)
1. Equivalence between the controls system and a differential inclusion

Let us introduce the multi-function \( F : \Omega \to \mathcal{P}(\mathbb{R}^n) \) given by \( F(t, x) := \{ f(t, x, u) : u \in U \} \) and consider the differential inclusion

\[
\dot{x}(t) \in F(t, x(t)) \quad \text{a.e. } t
\]  

(9)

**Theorem 4 (Equivalence Filippov’s Thm.).**

Under assumption \((H)\), a function \( x \in AC([a, b], \mathbb{R}^n) \) is a solution to

\[
\dot{x}(t) = f(t, x(t), u(t))
\]  

(10)

if and only if it verifies (9) for a.e. \( t \in [a, b] \).

**Proof.** It is based on the use of a **Measurable Selection Thm.** for multifunctions.
Basic facts on differential inclusions

Let $X$, $Y$ be metric spaces. For any $x \in X$ and $A \subset X$, set

$$d(x, A) := \inf_{a \in A} d(x, a); \quad B(A, \varepsilon) := \{x \in X : d(x, A) < \varepsilon\}.$$ 

For any pair of nonempty, compact subsets $A, A' \subset X$, their **Hausdorff distance** is defined as

$$d_H(A, A') := \max\{d(x, A'), d(x', A) : x \in A, \ x' \in A'\}.$$ 

Equivalently,

$$d_H(A, A') := \inf\{\rho > 0 : A \subset B(A', \rho), \text{ and } A' \subset B(A, \rho)\}.$$ 

$$d_H(A, A') = \max\{\rho, \rho'\}$$
A multifunction $F : X \to \mathcal{P}(Y)$ has **compact values** if $F(x) \subset Y$ is compact. It is **Hausdorff-continuous** if for every $x \in X$,

$$\lim_{x' \to x} d_H(F(x'), F(x)) = 0$$

We say that the multifunction $F$ has **closed graph** if its graph $\text{graph}(F) := \{(x, y) : y \in F(x)\}$ is a closed subset of $X \times Y$.

This condition means that,

whenever $x_\nu \to x$, $y_\nu \to y$ and $y_\nu \in F(x_\nu)$ for every $\nu$, then we also have the inclusion $y \in F(x)$.
Given a multifunction \( t \mapsto F(t) \) with non-empty values, a natural problem is to construct a selection, i.e. a single-valued, function \( f \) such that \( f(t) \in F(t) \) for every \( t \) with some additional properties, such as continuity or at least measurability.

To this aim, we introduce the notion of lexicographical ordering on \( \mathbb{R}^n \).

Given \( x, y \in \mathbb{R}^n, x \neq y \), we write \( x \prec y \) if one of the following alternatives holds:

\[
\begin{align*}
&x_1 < y_1 \\
&x_1 = y_1, \quad x_2 < y_2 \\
&\ldots \\
&x_1 = y_1, \ x_2 = y_2, \ldots, x_{n-1} = y_{n-1}, \quad x_n < y_n.
\end{align*}
\]

Notice the analogy with the ordering of words in a dictionary.

Any compact set \( K \subset \mathbb{R}^n \) has one first point \( \xi \), w.r.t. the lexicographical order.
Examples of first points of a compact set $K$ w.r.t. the lexicographical order.

Proof.
If \( t \mapsto F(t) \subset \mathbb{R}^n \) is a multifunction with compact values, we can now define the lexicographic selection \( t \mapsto \xi(t) \in F(t) \), where \( \xi(t) \) is the first point of the compact set \( F(t) \) w.r.t. the lexicographical order.

**Theorem 5 (Measurable selection).**

Let \( t \mapsto F(t) \) be a bounded multifunction with closed graph, defined for \( t \in [a, b] \). For each \( t \), let \( \xi(t) \in F(t) \) be the lexicographic selection. Then the map \( t \mapsto \xi(t) \) is measurable.

**Proof.**

**Theorem 6 (Lusin’s Theorem).**

For any \( f : [a, b] \rightarrow \mathbb{R}^n \), \( f \) is measurable if and only if there exists a sequence of disjoint, compact subsets \( J_k \subset [a, b] \) with

\[
\text{meas} \left( [a, b] \setminus \bigcup_{k=1}^{+\infty} J_k \right) = 0
\]

and such that the restriction of \( f \) to each set \( J_k \) is continuous.
Global existence and continuous dependence

Under assumption (**H**), for any measurable control \( u(\cdot) \in \mathcal{U} \) and any \((t_0, \bar{x}) \in \Omega\), the Cauchy problem

\[
\dot{x}(t) = f(t, x(t), u(t)) =: g(t, x(t)), \quad x(t_0) = \bar{x}
\]  

admits a unique, local solution \( t \mapsto x(t, u) \) on \([t_0 - \delta, t_0 + \delta]\) for some \( \delta > 0 \), since \( g \) verifies the so-called Carathéodory conditions (i.e., it is measurable in \( t \) and \( C^1 \) in \( x \)... see e.g. [BP]).

To guarantee the existence of a global, bounded solution, let us assume

**(**H**)** The control set \( U \subset \mathbb{R}^m \) is compact, \( f \) is continuous on \( \mathbb{R} \times \mathbb{R}^n \times U, \ C^1 \) in \( x \), and such that, for some constants \( C \) and \( L \),

\[
|f(t, x, u)| \leq C, \quad \|D_x f(t, x, u)\| \leq L \quad \text{for all} \ (t, x, u).
\]
Theorem 7 (Global existence and continuous dependence).

Under assumption (H)*, for every $T > 0$,

i) for any $u(\cdot) \in \mathcal{U}$ the Cauchy problem

$$
\dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) = \bar{x}
$$

has a unique solution $x(\cdot, u)$ defined for all $t \in [0, T]$.

ii) the input-output map $u(\cdot) \mapsto x(\cdot, u(\cdot))$ is continuous from $L^1([0, T], \mathbb{R}^m)$ into $C^0([0, T], \mathbb{R}^n)$.

Proof. The main ingredients of the proof are:

- the Contraction Mapping Theorem and
- the Dominated Convergence Theorem

recalled below.
Theorem 8 (Contraction Mapping).

Let $X$ be a Banach space, $\Lambda$ a metric space, and let $\Phi : \Lambda \times X \rightarrow X$ be a continuous mapping such that, for some $\ell < 1$,

$$\|\Phi(\lambda, x) - \Phi(\lambda, y)\| \leq \ell \|x - y\|.$$ 

Then for each $\lambda \in \Lambda$ there exists a **fixed point** $x(\lambda) \in X$ such that

$$x(\lambda) = \Phi(\lambda, x(\lambda)).$$

The map $\lambda \mapsto x(\lambda)$ is continuous. Moreover, for any $\lambda \in \Lambda$, $y \in X$ one has

$$\|y - x(\lambda)\| \leq \frac{1}{1 - \ell} \|y - \Phi(\lambda, y)\|.$$ 

**Proof.**
Theorem 9 (Dominated Convergence).

Let $v, \gamma, v_1, \ldots, v_\nu, \ldots$ be functions in $L^1([0, T], \mathbb{R}^N)$ verifying

- $v_\nu(t) \to v(t)$ for a.e. $t \in [0, T]$;
- $|v_\nu(t)| \leq \gamma(t)$ for all $\nu \geq 1$ and a.e. $t \in [0, T]$.

Then

$$\lim_{\nu \to +\infty} \int_0^T v_\nu(t) \, dt = \int_0^T v(t) \, dt.$$ 

Moreover

If $v, v_1, \ldots, v_\nu, \ldots$ are functions in $L^1([0, T], \mathbb{R}^N)$ such that $v_\nu \to v$ in $L^1$, then one can extract a subsequence $(v_{\nu'})_{\nu'}$ such that

$$v_{\nu'}(t) \to v(t) \quad \text{for a.e. } t \in [0, T].$$
Proof of the existence of a first element w.r.t. the lexicographic order for any compact set $K \subseteq \mathbb{R}^n$.

- Set $g_1 := \min_{x \in K} \langle e_1, x \rangle$.
  - If it exists, since it is the minimum of the continuous map $x \mapsto \langle e_1, x \rangle$ on $K$ compact.
  - Set $K_1 = \{ x \in K : \langle e_1, x \rangle = g_1 \}$ and define $g_2 := \min_{x \in K_1} \langle e_2, x \rangle$.
  - Repeating the procedure, we arrive at $g_n = \min_{x \in K_n} \langle e_n, x \rangle$ and obtain $K_n = \{ x \in \mathbb{R}^n : \langle f_1, \ldots, f_n \rangle \leq K_{n-1} \subseteq \ldots \subseteq K \}$, with $g$ first element of $K$ by construction. □

Proof of the Measurable Selection Theorem.

**Lemma:** If $F : J \rightarrow P(\mathbb{R}^n)$ is a bounded multifunction with closed graph and $J$ is a closed subset of $\mathbb{R}$, then

- $\forall r \in \mathbb{R}^n$, the map $\varphi(r) := \min_{y \in F(t)} \langle r, y \rangle$ is b.s.c.

Recall: $\varphi$ is b.s.c. (lower semicontinuous) means:

- $\forall t \in J$, $\lim_{t \rightarrow t^+} \varphi(t) \geq \varphi(t^+)$ and $\varphi$ is measurable.

**Proof.** Let $r \in \mathbb{R}^n$. Fix $t \in J$ and consider a sequence $(t_j)_j \subseteq J$, $t_j \rightarrow t$ such that $\lim_{t \rightarrow t^+} \varphi(t) = \lim_{t \rightarrow t^+} \varphi(t_j)$.  

For any $j$ there exists $y_j \in F(t_j)$ such that

$$\varphi(t_j) = \langle r, y_j \rangle$$

Since $F(t_j)$ is compact, $(y_j)_j$ is bounded and admits a subsequence $(y_{j_k})_k$ such that $\lim_{k \rightarrow \infty} y_{j_k} = \bar{y}$. Moreover, $\bar{y} \in F(t)$, since $F$ has closed graph. Thus

$$\varphi(t) = \min_{y \in F(t)} \langle r, y \rangle \leq \langle r, \bar{y} \rangle = \langle r, \lim_{j \rightarrow \infty} y_{j_k} \rangle = \langle r, \bar{y} \rangle.$$
\[
\lim_{j \to \infty} \langle v_j, y_j \rangle = \lim_{j \to \infty} Y(t_j) = \lim_{n \to \infty} \min_{y \in F(t)} Y(n).
\]

This proves that \( Y \) is e.s.c. at any \( t \in I \). \( \blacksquare \)

**Step 2.** Going back to the proof of the selection Thm., for any \( t \in J \), \( \exists(t) \) is the first element of the compact set \( F(t) \). Our goal is now to show that each component \( t \mapsto \exists_j(t) \) is measurable on \( J \).

1. \( \exists_j(t) = \min_{y \in F(t)} \langle e_j, y \rangle \) is e.s.c. and thus measurable by Step 1 (Lemma).

Assume by induction that \( \exists_1, \ldots, \exists_j \) for \( 1 \leq j < m \) are measurable. By Luzin’s Thm., there exists a sequence of disjoint, closed subsets \( (J_k)_{k \in \mathbb{N}} \subseteq J \) such that \( \bigcup_{k=1}^{\infty} J_k = J \) and \( \exists_1, \ldots, \exists_j \) are continuous on each \( J_k \).

Set
\[
F_j(t) = \left\{ y \in F(t) : \langle e_j, y \rangle = \exists_j(t) \right\}
\]

By construction, \( F_j(t) \) is nonempty and compact. Moreover, it is closed (open) on each \( J_k \), by the continuity of the \( \exists_1, \ldots, \exists_j \).

By the lexicographic order, we have
\[
\exists_{j+1}(t) = \min_{y \in F_{j+1}(t)} \langle e_{j+1}, y \rangle
\]

which turns out to be measurable by Step 1.

At this point, \( t \mapsto (\exists_1(t), \ldots, \exists_m(t)) \) is measurable by induction. \( \blacksquare \)

**Proof of Filipponi’s Thm.**

\[
\Rightarrow \quad \text{If } x(t) \text{ is a solution to }
\]

1. \( x(t) = f(t, x(t)), u(t) \) for some \( u \in U \), \( t \mapsto (t, x(t)) \) \( \in \Omega \)
by definition it verifies

\[ x(t) \in F(t, x(t)) \quad \text{a.e.} 
\]

\( \Leftrightarrow \) Viceversa, let \( x(\cdot) \) verify (2). Fix an arbitrary \( \omega \in U \) and define the multifunction

\[ W(t) = \{ \omega \in U : f(t, x(t), \omega) = x(t) \} \quad \text{if} \ x(t) \in F(t, x(t)) \]

Clearly, \( W(t) = \{ \omega \} \) only if \( x \) is not differentiable at \( t \) or if \( x(t) \notin F(t, x(t)) \). Therefore the set of this points, \( \mathcal{N} \), has measure \( (\mathcal{N}) = 0 \).

The theorem is now proven if we show that there exists a measurable selection \( t \mapsto u(t) \in W(t) \).

**Step 1:** For any \( t \), \( W(t) \) is compact \( \Rightarrow \) we can choose

\[ u(t) \in W(t) \quad \text{as the first element of} \ W(t) \quad \text{in the lexicographic order.} \]

**Step 2:** In order to prove that such \( u(\cdot) \) is measurable, observe that by Lusin's theorem there exists a sequence \( (\mathcal{J}_k)_{k=1}^{\infty} \) of disjoint, compact subsets of \([a, b]\) such that

\[ \text{meas} \left( [a, b] \setminus \bigcup_{k=1}^{\infty} \mathcal{J}_k \right) = 0 \]

and \( x(\cdot) \) is well defined and continuous on each set \( \mathcal{J}_k \).

**Step 3:** On each \( \mathcal{J}_k \), the bounded multifunction

\[ t \mapsto W(t) \quad \text{has closed graph.} \]

Indeed, for any \( t \), \( t \in \mathcal{J}_k \), \( \omega \in W(t) \) \( \Leftrightarrow \)
\[ \dot{x}(t, \omega) = f(t, x(t), \omega, \omega) \]. Hence if \( \omega_n \to \omega \), by the continuity of \( \dot{x} \) (and also \( f \)), passing to the limit we get 
\[ \dot{x}(t) = f(t, x(t), \omega) \Rightarrow \omega \in W(t) \] 

**Step 4**: By the Mean Value Theorem, on each \( J_k \), the lexicographic selection \( J_k \to u(t) \in W(t) \) is measurable. Since the sets \( J_k \) cover \( \omega \in [a, b] \), this implies that \( u(\cdot) \in U \).

In conclusion, \( x(\cdot) \) is a solution to (1) corresponding to such \( u(\cdot) \). \( \square \)