Optimization, State Constraints and Geometric Control

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# On the regularity of solutions to a class of variational problems, including the p-Laplace equation for $2 \leq p < 4$.

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## 1 An introduction

Consider the classical problem of the calculus of variations, i.e., in the problem of minimizing a functional of the kind

$$I(u) = \int_{\Omega} [L(|\nabla u(x)|) + f(x)u(x)]dx$$

with prescribed boundary conditions. A problem that has been of interest to me is to investigate the properties of its solution, in particular the (possible) higher differentiability of a solution. We minimize an integral functional over a subset of the space of functions having (weak) first order derivatives. The solution $\tilde{u}$ simply gives a value to the integral that is the smallest among the values computed along the competing functions.

A strange phenomenon might appear: under certain conditions, the solution $\tilde{u}$, instead of having only first order derivatives, has, in addition, second order derivatives.

## 2 The case of one dimensional integration set.

Assume we are minimizing the integral

$$\int_{\alpha}^{\beta}[\frac{1}{2}|y'(t)|^2 + f(t)y(t)]dt, y(\alpha) = a, y(\beta) = b,$$

where $f \in L^2((\alpha, \beta))$, among the functions that are in $W^{1,2}((\alpha, \beta))$ with the given boundary conditions.

The reasoning that leads to the proof of the additional regularity of the solution is basically very simple. Begin by assuming that we already knew that a second derivative exists, and try to learn something about its properties.

The Euler-Lagrange equation gives

$$\int_{\alpha}^{\beta}[x'(t)\eta'(t) + f(t)\eta(t)]dt = 0$$

for every variation $\eta$ that is zero at the boundary and that it is sufficiently regular.

Assume that $\eta$ is itself a derivative, $\phi'$, where both $\phi$ and $\phi'$ are zero at the boundary; then, we obtain

$$\int_{\alpha}^{\beta} [x'(t)\phi''(t) + f(t)\phi'(t)]dt = 0;$$

an integration by parts then gives

$$\int_{\alpha}^{\beta} x''(t)\phi'(t)dt = \int_{\alpha}^{\beta} f(t)\phi'(t)dt$$

Fix a point $t^0 \in (\alpha, \beta)$ and let $\delta > 0$ be such that $[t^0 - 2\delta, t^0 + 2\delta] \subset (\alpha, \beta)$ and let a variation $\eta$ be twice differentiable, such that: $\eta(t) \equiv 1$ on $(t^0 - \delta, t^0 + \delta)$, $\eta(t) \equiv 0$ on $(\alpha, \beta) \setminus [t^0 - 2\delta, t^0 + 2\delta]$ and $0 \leq \eta(t) \leq 1$ everywhere. For $\phi$ take $\phi(t) = \eta^2(t)x'(t)$; we have $\phi'(t) = 2\eta(t)\eta'(t)x'(t) + \eta^2(t)x''(t)$ and the assumptions on $x$ and $\eta$ imply that $\phi \in C^2$ so that Euler-Lagrange gives

$$\int_{\alpha}^{\beta} x''(t)[2\eta(t)\eta'(t)x'(t) + \eta^2(t)x''(t)]dt = \int_{\alpha}^{\beta} f(t)\phi'(t)dt,$$

i.e.,

$$\int_{\alpha}^{\beta} \eta^2(t)(x''(t))^2 dt = -\int_{\alpha}^{\beta} x''(t)2\eta(t)\eta'(t)x'(t)dt + \int_{\alpha}^{\beta} f(t)\phi'(t)dt.$$

This equality is the basis of the proof of regularity. At the left we have a positive integrand; hence, we have the natural inequality

$$\int_\alpha^\beta \eta^2(t)(x''(t))^2 dt \leq \int_\alpha^\beta |x''(t)2\eta(t)\eta'(t)x'(t)|dt + \int_\alpha^\beta |f(t)\phi'(t)|dt.$$

For any pair $y, z$ we have that $2zy = 2z\lambda\frac{1}{\lambda}y \leq (z\lambda)^2 + (\frac{1}{\lambda}y)^2$; let $z = |x''(t)\eta(t)|$, $y = |\eta'(t)x'(t)|$ and $\lambda = \frac{1}{2}$, to obtain

$$\int_\alpha^\beta |x''(t)2\eta(t)\eta'(t)x'(t)|dt \leq \frac{1}{4}\int_\alpha^\beta |x''(t)\eta(t)|^2 dt + 4\int_\alpha^\beta |\eta'(t)x'(t)|^2 dt,$$

and, with a few computations, we have

$$\int_\alpha^\beta |x''(t)\eta(t)|^2 dt \leq \int_\alpha^\beta [(8|\eta'(t)|+2\eta(t)^2)|x'(t)|^2+(8\eta(t)^2+2\eta'(t)^2)|f(t)|^2]dt.$$

In the meantime, you have somewhere proved that the r.h.s. is finite. Hence, on the interval where $\eta \equiv 1$, you have obtained an $L^2$ estimate on $x''$ depending only on the data of the problem.

For the true problem, the existence of $x''$ is what we want to prove. The idea is to obtain bounds as before not on $x''$ but on the difference quotient of $x'$ and use this information to prove the existence of $x''$.

In the multi-dimensional case (that is much more involved), this method of proving regularity goes back to Louis Nirenberg.

It also depends on the remarkable property of Sobolev spaces that implies that integral bounds on the difference quotients and integral bounds on the derivatives are essentially the same thing.

## 3 Going non-linear.

Assume that, under the integral sign, instead of $\frac{1}{2}(x')^2$, we have, more generally, $l(x')$. The same reasoning as before, keeping the same variation, will lead to an inequality of the kind

$$\int_\alpha^\beta l''(x'(t))|x''(t)\eta(t)|^2 dt \leq K$$

and, to obtain an inequality on $x''$ we must ‘’divide” by $l''(x'(t))$, hence we must require that $l''$ be bounded below by a positive constant.

In the multi-dimensional case, this require the condition that the quadratic form of the matrix of the second derivatives of $l$ be (uniformly) positive definite, i.e., the condition of ellipticity.

Consider the multi-dimensional problem of minimising

$$\int_\Omega [\frac{1}{2}|\nabla u(x)|^2 + f(x)u(x)]\,dx$$

The solution satisfies the Euler-Lagrange equation

$$\int_\Omega \langle \nabla \tilde{u}(x), \nabla \eta(x) \rangle + f(x)\eta(x)]\,dx = 0$$

for every $\eta \in W_0^{1,2}(\Omega)$

In this case, by the reasoning we have presented, we obtain that $f \in L^2$ implies that $\tilde{u} \in W_{loc}^{2,2}(\Omega)$ or, in the language of PDE's, that any $W^{1,2}(\Omega)$ weak solution to the equation

$$\mathrm{div} \nabla u = f$$

is actually in $W_{loc}^{2,2}(\Omega)$

Could this result possibly be true for the problem of minimizing

$$\int_\Omega [\frac{1}{p}|\nabla u(x)|^p + f(x)u(x)]\, dx$$

for values of $p$ other than $p = 2$?

For instance, what happens if $p = 2.01$?

The second derivative of $\frac{1}{p}|t|^p$ is $(p-1)|t|^{p-2}$, that equals 0 at $t = 0$.

The problem is not elliptic and the previous reasoning breaks down.

## 4 What one finds in books.

In books the problem of the regularity of solutions for variationaol problems of $p$-growth is presented as follows.

Introduce the function

$$V(z) = \sqrt{1 + |z|^2}$$

since both the conditions and the result will be phrased in terms of $V$. The conditions on $L$ are:

a growth condition ($L$ grows like $|z|^p$)

and a "strict convexity" condition:

$$\xi^T H_L \xi \geq \nu V(z)^{p-2} |\xi|^2$$

with $\nu > 0$,

and the result is

$$\int_\omega V^{p-2}|D^2u|^2 \leq K$$

However, in the case

$$L = \frac{1}{p}|z|^p$$

with $p > 2$, the condition

$$\xi^T H_L \xi \geq \nu V(z)^{p-2}|\xi|^2 \text{ with } \nu > 0$$

is NOT verified.

Hence, the results that are presented for $p-growth$ do not apply to $L(t) = \frac{1}{p}t^p$

## 5 An idea.

Let us go back to the one-dimensional case. How about using, as a variation, instead of $\eta^2(t)x'(t)$, the variation

$$\eta^2(t)\gamma(x'(t))?$$

In this case, instead of the inequality

$$\int_\alpha^\beta l''(x'(t))|x''(t)\eta(t)|^2 dt \le K$$

we would obtain the inequality

$$\int_\alpha^\beta l''(x'(t))\gamma'(x'(t))|x''(t)\eta(t)|^2 dt \le K$$

and, inside the integral, we could fight the ''zero'' of $l''$ with an ''infinity'' coming from $\gamma'$.

In other words: it not the ellipticity that counts, but the positivity of $l''\gamma'$.

## 6 Putting the idea to work.

In A.C., The regularity of solutions to some variational problems, including the p-Laplace equation for $2 \leq p < 3$. ESAIM Control Optim. Calc. Var. 23 (2017)

it is proved that, for $2 \leq p < 3$, any $W^{1,p}(\Omega)$ weak solution to the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f \tag{(6.1)}$$

is in $W^{2,2}_{loc}(\Omega)$. .

## 7 Why $p < 3$?

We have $l''(t) = (p-1)t^{p-2}$; hence, it is natural to require $\gamma'(t) = \frac{1}{t^{p-2}}$ to obtain

$$l''(t)\gamma'(t) = c > 0.$$

This leads to the choice $\gamma(t) = \int^t \frac{1}{s^{p-2}} ds$: this equality defines a continuous $g$ defined on $\mathbb{R}$ only if $p < 3$: at $p = 3$, $\gamma$ would be $\ln(t)$, unbounded at $t = 0$.

The actual proof is unfortunatly much longer, but the idea remains!

## 8 What happens at $p = 3$

The function $x(t) = t^{\frac{3}{2}}$ does not belong to $W^{2,2}((-1,1))$ but it is a solution to the problem of minimizing

$$\int_{-1}^{1} (\frac{1}{3}(x'(t))^3 + \frac{9}{4}x(t))dt.$$

Hence, the regularity result holds for $p < 3$ but not for $p = 3$!

## 9 An opinion

‘’ The new regularity questions must be addressed in connections with the mathematical problems rather than by inconsequential extension of the existing results.”

## So, what is it that holds true for $p = 3$?

In A.C., The regularity of solutions to some variational problems, including the $p$-Laplace equation for $3 \leq p < 4$., Cont. Dis. Dyn. Systems, to appear

it is proved that: for $p = 3$, the solution belongs to the fractional Sobolev space

$$W^{s,2}_{loc}(\Omega)$$

for every $s < 2$; more generally

$$u \in W^{1+s,2}_{loc}(\Omega)$$

for $0 < s < 4 - p$

## 10 An idea of the proof

We cannot really use $\ln(|\nabla u|)$ in the variation. We consider a family of variations, depending on the parameter $\varepsilon$; in the one dimensional case, this family would be $\eta^2(t)\phi(t,\varepsilon)$ where

$$\phi(t,\varepsilon) = \begin{cases} \frac{t}{\varepsilon} \text{ for } t \leq \varepsilon \\ \ln(\frac{et}{\varepsilon} \text{ for } \varepsilon < t \leq 1 \\ t + \ln\frac{e}{\varepsilon} - 1 \text{ for } t \geq 1 \end{cases}$$

(In the true $N$ dimensional case, it is somewhat more complex: for $j = 1, ..., N$, we have $\phi_j^{\tau,T}(x,\varepsilon)$, given by

$$\phi_j^{\tau,T}(x,\varepsilon) = \begin{cases} \dfrac{u_{x_j}^{\tau,T}(x)}{\varepsilon} & \text{for } x \in A \\ \ln\dfrac{e|\nabla u^{\tau,T}(x)|}{\varepsilon}\dfrac{u_{x_j}^{\tau,T}(x)}{|\nabla u^{\tau,T}(x)|} & \text{for } x \in B \\ u_{x_j}^{\tau,T}(x) + (\ln\frac{e}{\varepsilon} - 1)\dfrac{u_{x_j}^{\tau,T}(x)}{|\nabla u^{\tau,T}(x)|} & \text{for } x \in C \end{cases}$$

Each of these variations, for $\varepsilon > 0$, gives a finite bound for the integral that estimates the derivatives of the gradient of the solution. However, this finite bound depends on $\varepsilon$ and goes to $+\infty$ as $\varepsilon \to 0$.

Still, this "going to infinity" is slow (it goes like the logarithm); this gives the idea that one can lift the gradient of the solution to a function that is $L^2$ in the product variable $(x, \varepsilon)$ and such that the gradient of the solution is the trace on $\Omega$ of the lifted function. This would give that the gradient of the solution is in $W^{\frac{1}{2},2}$; a more precise proof gives the full result.

## 11 However, as such it cannot be done

The family of variations introduced before contains the gradient of the solution. Since in the Euler-Lagrange equation enters the gradient of the variation, to use this variation we must assume that the solution has second order derivatives, something we do not know.

A way out would be to use difference quotients; however, the computations become very difficult.

What we do is to modify the original Lagrangian by a family of Lagrangians "uniformly elliptic"; the solutions to these problems are in $W^{2,2}_{loc}$, hence we are free to use their gradients in the variations; then, we obtain the estimates for this family and eventually we show that we can pass the estimates to the true solution.

Then

what happens at $p = 4$?

## 12 and what happens when, instead, $l''(t) \to 0$ as $t \to \infty$?

As it is well known, the only growth requirement in order to apply the Direct Method of the Calculus of Variations to prove existence of solutions is that of *superlinear growth: there must exist a function* $\psi(\xi)$ *satisfying*

$$\lim_{|\xi|\to\infty} \frac{\psi(\xi)}{|\xi|} = +\infty$$

*that bounds the Lagrangean* $L(x,u,\xi)$ *from below, i.e., such that* $L(x,u,\xi) \geq \psi(\xi)$.

Hence a growth like

$$l(|t|) \sim |t| \ln(\ln(\dots \ln(|t|)))$$

(together with convexity) is enough to provide existence of solutions.

For a function $l$ growing superlinearly but less than $\frac{1}{2}t^2$, it is natural to expect (assuming some additional regularity) that

$$\lim_{t\to\infty} l''(t) = 0.$$

For instance, when, for $t$ large,

$$l(t) = t\ln(\ln(t)))$$

we have

$$l''(t) = \frac{1}{t\ln(t)}(1 - \frac{1}{\ln(t)})$$

that goes to zero faster than $\frac{1}{t}$.

I have been interested (work in progress) in showing that superlinearity plus regularity of $l$ is enough to show the regularity of the solution, without " quantitative" bounds on ''how far from linearity" is the superlinear growth.

The difficulty consists in separating the cases of linear and superlinear growth without introducing quantitative conditions.

In other words: it should work for

$$l(|t|) \sim |t| \ln(\ln(\dots \ln(|t|)))$$

without counting the number of ln.

A key assumption (besides regularity, $l''(t) > 0$) is

$$\lambda = \lim_{t \to +\infty} \frac{t l'''(t)}{l''(t)}$$

exists and

$$-1 \leq \lambda < 0.$$

The assumption

$$\lim_{t \to +\infty} \frac{t l'''(t)}{l''(t)} < 0$$

depends on the technique used in the proof and expresses the fact that we deal with SLOW superlinear growth: it is satisfied by $l(|t|) = \frac{1}{p}|t|^p$ only for $p < 2$.

The assumption

$$\lim_{t \to +\infty} \frac{t l'''(t)}{l''(t)} \geq -1$$

separates the cases of superlinear from those of linear growth, and IT IS SHARP: For every $\varepsilon$ there are maps $l$ of linear growth such that

$$\lim_{t \to +\infty} \frac{t l'''(t)}{l''(t)} \geq -(1 + \varepsilon)$$

In fact, it is enough to consider the family of maps of linear growth (introduced by A.C., Giulia Treu and Sandro Zagatti)

$$l(|t|) \sim |t| - |t|^{1-\varepsilon}$$

to have the result.

There is an additional assumption, to separate the case of linear and superlinear growth, i.e. that for every $\alpha > 1$ we have

$$\lim_{t\to\infty} \frac{t^{\alpha} l''(t)}{l'(t)} = +\infty \qquad ((12.2))$$

The condition

$$\lim_{t\to\infty} \frac{t l''(t)}{l'(t)} = 0$$

is fulfilled both by maps of linear and superlinear growth. The assumption ((12.2)) we use separates the two cases.

Under these conditions (work in progress!) the higher local differentiability of a solution holds true.

Un abbraccio, Giovanni e Franco!

ma come avete fatto a diventare così vecchi?