Discontinuous time-dependent optimal control problems and Hamilton-Jacobi equations

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in honour of Giovanni Colombo and Franco Rampazzo

Outline of the talk

- Optimal control problems with discontinuous time dependence and Hamilton-Jacobi equation
- A characterization of the value function for extended valued terminal costs
- Adding an integral term
- Enter state constraints

- P.B.-Vinter, *The Hamilton Jacobi Equation For Optimal Control Problems with Discontinuous Time Dependence*, SiCON 2017

- Bernis-P.B., *Discontinuous time dependent oprimal control problems with an integral cost and Hamilton-Jacobi equations*, preprint

Optimal control problems - Value function

Consider the optimal control problem:

$$(P_{S,x_0}) \begin{cases} \text{Minimize } g(x(T)) \\ \text{over arcs } x(.) \in W^{1,1}([S,T];\mathbb{R}^n) \text{ satisfying} \\ \dot{x}(t) \in F(t,x(t)) \quad \text{a.e. } t \in [S,T] \\ x(S) = x_0, \end{cases}$$

Embed in a family of problems, parameterized by initial data

$$(P_{t,x}) \begin{cases} \text{Minimize } g(x(T)) \\ \text{over trajectories } x(.) \text{ s.t. } \dot{x}(s) \in F(s,x(s)) \;\; x(t) = x\,. \end{cases}$$

Define $\boxed{V(t,x) = \mathrm{Inf}(P_{t,x})}$ **Value Function**

Hamilton Jacobi equation

$$V(t,x) = \mathrm{Inf}(P_{t,x}) \begin{cases} \text{Minimize } g(x(T)) \\ \text{over trajectories } x(.) \text{ s.t. } \dot{x}(s) \in F(s,x(s)) \;\; x(t) = x \end{cases}$$

Principle of Optimality: it establishes some important monotonicity properties of the Value Function

PDE of Dynamic Programming: $V(.,.)$ is a solution to

$$(HJE) \begin{cases} V_t(t,x) + \min_{v \in F(t,x)} V_x(t,x) \cdot v \;=\; 0 \quad \forall (t,x) \in (S,T) \times \mathbb{R}^n \\ V(T,x) \;=\; g(x) \quad \forall x \in \mathbb{R}^n \,. \end{cases}$$

$\rightarrow$ Characterize the value function as solution to (HJE), in a generalized sense.
Mainly employed techniques: viscosity solutions theory, nonsmooth analysis, viability/invariance results,...

Value function $\leftrightarrow$ Hamilton-Jacobi equation

$\rightarrow$ **Viscosity solutions theory**:
- Crandall-Lions 1983 and 1989, Crandall-Evans-Lions 1984,...
- Ishii 1985, Barron-Jensen 1987, Lions-Perthame 1987: **dicontinuous/measurable time-dependent Hamiltonians**
- Soner 1986 **state constraints**
- ..... cf. the books Barles 1994, Bardi & Capuzzo-Dolcetta 1997

$\rightarrow$ **Nonsmooth theory, invariance/viability results**:
- Frankowska 1993, 1995
- Clarke-Ledyaev-Stern-Wolenski 1995,
- Frankowska-Plaskacz-Rzezuchowski 1995: **dicontinuous/measurable time-dependence problems**
- Frankowska-Vinter 2000, Frankowska-Mazzola 2013: **state constraints**
- ..... cf. books Clarke-Ledyaev-Stern-Wolenski 1998, Vinter 2000, Clarke 2013

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- Frankowska-Vinter 2000, Frankowska-Mazzola 2013: **state constraints**
- ..... cf. books Clarke-Ledyaev-Stern-Wolenski 1998, Vinter 2000, Clarke 2013

$\rightarrow$ Colombo-Palladino, *The minimum time function for the controlled Moreau's sweeping process*, SICON 2016

$\rightarrow$ Rampazzo, *Faithful representations for convex Hamilton-Jacobi equations*, SICON 2005

Our framework

$$V(t,x) = \text{Inf}(P_{t,x}) \begin{cases} \text{Minimize } g(x(T)) \\ \text{over trajectories } x(.) \text{ s.t. } \dot{x}(s) \in F(s,x(s)) \;\; x(t) = x \end{cases}$$

$\rightarrow g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is **extended valued**; incorporates an implicit **terminal constraint**

$$x(T) \in C\,,$$

where $C := \{x \in \mathbb{R}^n | g(x) < +\infty\}$ is a closed set.

$\Rightarrow$ It is necessary to consider lower semicontinuous solutions (lsc) to (HJE)

Our framework

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$\Rightarrow$ It is necessary to consider lower semicontinuous solutions (lsc) to (HJE)
$\rightarrow$ we impose the dicontinuous time-dependent hypothesis:

($*$) *the multifunction $t \rightarrow F(t,x)$ has everywhere left and right limits and is continuous on the complement of a set of measure zero.*

$\rightarrow$ We use analytical techniques based on the application of **invariance/viability results** to a differential inclusion in a higher dimensional space, solutions to which are required to evolve in the **epigraph** set of V.

The application of viability theory to characterize lsc value functions for optimal control problems with extended valued terminal costs was first achieved by Frankowska:

Theorem. [Frankowska, 1993 and 1995] F is required to be **continuous** w.r.t. time. Then, V is the unique lsc function satisfying the HJE, in the sense ($\rightarrow$ **Dini/contingent solution**):

(i): $\inf_{v \in F(t,x)} D_{\uparrow} V((t,x);(1,v)) \ \leq \ 0,$
for all $(t,x) \in ([S,T) \times \mathbb{R}^n) \cap \mathrm{dom}\, V$

(ii): $\sup_{v \in F(t,x)} D_{\uparrow} V((t,x);(-1,-v)) \ \leq \ 0,$
for all $(t,x) \in ((S,T] \times \mathbb{R}^n) \cap \mathrm{dom}\, V$

(iii): $V(T,x) = g(x)$ for all $x \in \mathbb{R}^n$.

$D_{\uparrow} V$ denotes the lower Dini directional derivative (also called contingent epi-derivative):

$$D_{\uparrow}\varphi(\bar{x};d) \ = \ \liminf_{h \downarrow 0,\ e \to d} h^{-1}\left[\varphi(\bar{x}+he) - \varphi(\bar{x})\right]$$

Rmk. Equivalent conditions involving generalized solutions to HJE in a Frêchet subgradient sense were also given in [Frankowska 1995]

Subsequently, in refined 'proximal subgradient' form,

Theorem. [Clarke-Ledyaev-Stern-Wolenski, 1995] F is required to be **continuous** w.r.t. time. Then, V is the unique lsc function satisfying the HJE, in the sense ($\rightarrow$ **proximal solution**):

(i) for all $(t, x) \in ((S, T) \times \mathbb{R}^n) \cap \mathrm{dom}\, V$, $(\xi^0, \xi^1) \in \partial_P V(t, x)$

$$\xi^0 + \inf_{v \in F(t,x)} \xi^1 \cdot v \; = \; 0,$$

(ii) for all $x \in \mathbb{R}^n$,

$$\liminf_{\{(t',x') \to (S,x) : t' > S\}} V(t', x') = V(S, x)$$

and

$$\liminf_{\{(t',x') \to (T,x) : t' < T\}} V(t', x') = V(T, x) = g(x).$$

$$\partial_P \varphi(\bar{x}) := \{\xi \,|\, (\xi, -1) \in N^P_{epi\ \varphi}(\bar{x}, \varphi(\bar{x}))\}.$$

Discontinuous time-dependent problems

Generalized solution to HJE in an 'almost everywhere w.r.t. time' sense?

Example. Consider

$$\begin{cases} \text{Minimize } g(x(1)) := x(1) \\ \text{over arcs } x(.) \in W^{1,1}([t_0, 1]; \mathbb{R}) \text{ s.t.} \\ \dot{x}(t) \; = \; 0 \quad \text{a.e. } t \in [0,1] \\ x(0) \; = \; x_0 \; , \end{cases}$$

The value function is $V(t,x) = x$ for all (t,x). However

$$W(t,x) := \begin{cases} x - 1 & \text{if } t \leq \frac{1}{2} \\ x & \text{if } t > \frac{1}{2} \end{cases}$$

is also an lsc function that satisfies the conditions (i) and (ii) above in the 'almost everywhere' sense: we exclude consideration of the troublesome point $\frac{1}{2}$ at which $W(t,x)$ fails to satisfy conditions (i) and (ii).

$\Rightarrow$ the value function is not the unique lsc function satisfying conditions (i), (ii) and (iii) in the almost everywhere sense.

The non-uniqueness issue

This issue can be circumvented by restricting candidate solutions $V(.,.)$ to (i), (ii) and (iii) to have the following regularity property Frankowska-Plaskacz-Rzezuchowski 1995:

(EPI) $t \to \text{epi}\, V(t,.)$ is absolutely continuous.

Here $\text{epi}\, V(t,.) := \{(\alpha, x) \,|\, \alpha \geq V(t,x)\}$ and ‘absolute continuity’ means that there exists an integrable function $\gamma(.) : [S,T] \to \mathbb{R}$ such that

$$d_H(\text{epi}\, V(s,.), \text{epi}\, V(t,.)) \leq \int_{[s,t]} \gamma(\sigma) d\sigma\,, \quad \text{for all } [s,t] \subset [S,T]\,.$$

($d_H(.,.)$ denotes the Hausdorff distance.)

A question was: necessary to impose (EPI)?

The ‘almost everywhere’ HJE theory of Frankowska et al. covers a broad class of optimal control problems for which $t \to F(t, x)$ is discontinuous. But it leaves open the following question:

> *For the special case, when* $t \to F(t, x)$ *has everywhere one-sided limits and is continuous on the complement of a zero-measure subset of* $[S, T]$*, can we provide a characterization of the value function as a unique lsc function* $V(.,.)$ *satisfying conditions similar to (i) and (ii), and also (iii), without imposing the a priori regularity condition (EPI) on* $V(.,.)$*?*

Hypotheses

The following hypotheses will be imposed:

(H1): $g(.) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is lsc, $F : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ takes closed, convex, non-empty values, $F(., x)$ is $\mathcal{L}(S, T)$-measurable for all $x \in \mathbb{R}^n$,

(H2): **(i)** there exists $c(.) \in L^1(S, T)$ such that

$$F(t,x) \subset c(t)(1+|x|)\,\mathbb{B} \quad \text{for all } x \in \mathbb{R}^n \quad \text{and for a.e. } t \in [S,T]\,,$$

and

(ii) for every $R_0 > 0$, there exists $c_0 > 0$ such that

$$F(t,x) \subset c_0\,\mathbb{B} \quad \text{for all } (t,x) \in [S,T] \times R_0\mathbb{B}\,,$$

Hypotheses...

(H3): **(i)** for every $R_0 > 0$, there exists a modulus of continuity $\omega(.) : \mathbb{R}^+ \to \mathbb{R}^+$ and $k_{F(.)} \in L^1(S,T)$ such that

$$d_H(F(t,x'),F(t,x)) \leq \omega(|x-x'|) \quad \text{for all } \; x,x' \in R_0\mathbb{B}\,,$$

and

(ii) $F(t,x') \subset F(t,x) + k_F(t)|x-x'|\,\mathbb{B}$ for all $x,x' \in R_0\mathbb{B}$ and a.e. $t \in [S,T]$,

(H4): **(i)** for each $s \in [S,T)$, $t \in (S,T]$ and $x \in \mathbb{R}^n$ the following one-sided set-valued limits exist and are non-empty:

$$F(s^+,x) := \lim_{s'\downarrow s} F(s',x) \quad \text{and} \quad F(t^-,x) := \lim_{t'\uparrow t} F(t',x)\,,$$

and

(ii) and for a.e. $s \in [S,T)$ and $t \in (S,T]$ we have

$$F(s^+,x) = F(s,x) \quad \text{and} \quad F(t^-,x) = F(t,x)\,, \qquad \text{for all } \; x \in \mathbb{R}^n\,.$$

Characterization of lsc Value Functions

Theorem 1. [P.B.-Vinter]
Take a function $V : [S, T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Then, assertions (a)–(c) below are equivalent:

(a) V is the value function for (P_S, x_0).

(b) V is lsc on $[S, T] \times \mathbb{R}^n$ and

(i) for all $(t, x) \in ([S, T) \times \mathbb{R}^n) \cap \mathrm{dom}\, V$

$$\inf_{v \in F(t^+, x)} D_\uparrow V((t, x); (1, v)) \leq 0,$$

(ii) for all $(t, x) \in ((S, T] \times \mathbb{R}^n) \cap \mathrm{dom}\, V$

$$\sup_{v \in F(t^-, x)} D_\uparrow V((t, x); (-1, -v)) \leq 0,$$

(iii) for all $x \in \mathbb{R}^n$

$$V(T, x) = g(x).$$

Characterization of lsc Value Functions...

(c) V is lsc on $[S,T]\times\mathbb{R}^n$ and

(i) for all $(t,x)\in((S,T)\times\mathbb{R}^n)\cap\mathrm{dom}\,V$, $(\xi^0,\xi^1)\in\partial_P V(t,x)\cup\partial_P^\infty V(t,x)$

$$\xi^0+\inf_{v\in F(t^+,x)}\xi^1\cdot v\ \le\ 0,$$

(ii) for all $(t,x)\in((S,T)\times\mathbb{R}^n)\cap\mathrm{dom}\,V$, $(\xi^0,\xi^1)\in\partial_P V(t,x)\cup\partial_P^\infty V(t,x)$

$$\xi^0+\inf_{v\in F(t^-,x)}\xi^1\cdot v\ge 0,$$

(iii) for all $x\in\mathbb{R}^n$,

$$\liminf_{\{(t',x')\to(S,x):t'>S\}}V(t',x')=V(S,x)$$

and

$$\liminf_{\{(t',x')\to(T,x):t'<T\}}V(t',x')=V(T,x)=g(x).$$

The *asymptotic proximal subdifferential* of φ at $\bar{x}\in\mathrm{dom}\,\varphi$:

$$\partial_P^\infty\varphi(\bar{x}):=\{\xi\,|\,(\xi,0)\in N^P_{epi\ \varphi}(\bar{x},\varphi(\bar{x}))\}.$$

Exchange the limits of F?

Example. Consider the optimal control problem

$$\begin{cases} \text{Minimize } g(x(1)) := x(1) \\ \text{over arcs } x(.) \in W^{1,1}([t_0, 1]; \mathbb{R}) \text{ s.t.} \\ \dot{x}(t) \in F(t) \quad \text{a.e. } t \in [t_0, 1] \\ x(t_0) = x_0 \,, \end{cases}$$

where $t_0 \in [0, 1]$, $x_0 \in \mathbb{R}$ and

$$F(t) := \begin{cases} [-\frac{1}{2}, \frac{1}{2}] & \text{if} \quad 0 \le t \le \frac{1}{2} \\ [-1, 1] & \text{if} \quad \frac{1}{2} < t \le 1 \,. \end{cases}$$

The value function $V : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is

$$V(t, x) := \begin{cases} x + \frac{t}{2} - \frac{3}{4} & \text{if} \quad 0 \le t \le \frac{1}{2} \\ x + t - 1 & \text{if} \quad \frac{1}{2} < t \le 1 \,. \end{cases}$$

We have, as the result of a routine calculation:

$$D_{\uparrow} V((1/2, 0); (1, v)) = 1 + v \quad \text{and} \quad D_{\uparrow} V((1/2, 0); (-1, -v)) = -\frac{1}{2} - v \,.$$

Exchange the limits of F?...

Consistent with conditions (b)(i) and (b)(ii) in Thm. above, V satisfies

$$\inf_{v \in F(\frac{1}{2}^+)} D_\uparrow V((1/2,0);(1,v)) \;=\; \inf_{v\in[-1,1]} (1+v) \;=\; 0 \;\; (\leq 0)\,,$$

$$\sup_{v \in F(\frac{1}{2}^-)} D_\uparrow V((1/2,0);(-1,-v)) \;=\; \sup_{v\in[-\frac{1}{2},\frac{1}{2}]} (-\frac{1}{2}-v) = \;0 \;\; (\leq 0)\,.$$

On the other hand, switching roles of $F(\frac{1}{2}^-)$ and $F(\frac{1}{2}^+)$ in these calculations would give:

$$\inf_{v \in F(\frac{1}{2}^-)} D_\uparrow V((1/2,0);(1,v)) \;=\; \inf_{v\in[-\frac{1}{2},\frac{1}{2}]} (1+v) \;=\; \frac{1}{2} \;\; (> 0)\,,$$

$$\sup_{v \in F(\frac{1}{2}^+)} D_\uparrow V((1/2,0);(-1,-v)) \;=\; \sup_{v\in[-1,1]} (-\frac{1}{2}-v) \;=\; \frac{1}{2} \;\; (> 0)\,.$$

This example shows that condition (b)(i) must involve the right limit $F(t^+,x)$ and (b)(ii) must involve the left limit $F(t^-,x)$ (similarly for condition (c)).

Proof structure

- (a) $\Rightarrow$ (b). Apply the Optimality Principle of the value function, and the definition of Dini/continget derivative
- (b) $\Rightarrow$ (c). Use standard properties of Dini/continget derivative and proximal normal cone
- (c) $\Rightarrow$ (a) (**the key step**) This involves showing that, for an arbitrary point (t, x) in the domain of a function V satisfying condition (c),

  (A): $V(t, x)$ is the cost of some state trajectory originating from (t, x) and

  (B): $V(t, x)$ is a lower bound on the cost of an arbitrary state trajectory.

  For both (A) and (B) we use the **weak invariance theorem**. The proof of (A) is standard. The proof of (B) employs techniques based on the **Steiner representation** of $\dot{x} \in F$ as a controlled differential equation, taking account of the possible discontinuities of $F(.,.)$ w.r.t. time.

Add an integral term in the cost

$$
(P_{S,x_0}) \begin{cases} \text{Minimize } g(x(T)) + \int_S^T L(t, x(t), \dot{x}(t))\, dt \\ \text{over arcs } x(.) \in W^{1,1}([S,T]; \mathbb{R}^n) \text{ satisfying} \\ \dot{x}(t) \in F(t, x(t)) \quad \text{a.e. } t \in [S,T] \\ x(S) = x_0, \end{cases}
$$

(H5): **(i)** $L : [S,T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is lower semicontinuous.

(ii) L is locally bounded

(iii) For every $t \in [S,T]$, $x \in \mathbb{R}^n$, $L(t, x, \cdot)$ is convex.

(iv) L is coercive: for all $(t, x, v) \in [S,T] \times \mathbb{R}^n \times \mathbb{R}^n$, $L(t, x, v) \geq \Theta(|v|) - \alpha|x|$, for some $\alpha \in \mathbb{R}_+$ and some convex function $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$
\lim_{r \to +\infty} \frac{\Theta(r)}{r} = +\infty.
$$

(H6): L is continuous w.r.t. x; $L(t^+, x, v)$ and $L(t^-, x, v)$ exist for every t , and $L(t^+, x, v) = L(t, x, v) = L(t^-, x, v)$ for a.e. t.

Characterization of lsc Value Functions - integral term

Theorem 2. [Bernis-P.B.]
Take a function $V : [S, T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Then, assertions (a)–(c) below are equivalent:

(a) V is the value function for (P_S, x_0).

(b) V is lsc on $[S, T] \times \mathbb{R}^n$ and

(i) for all $(t, x) \in ([S, T) \times \mathbb{R}^n) \cap \mathrm{dom}\, V$

$$\inf_{v \in F(t^+, x)} [D_\uparrow V((t, x); (1, v)) + L(t, x, v)] \leq 0,$$

(ii) for all $(t, x) \in ((S, T] \times \mathbb{R}^n) \cap \mathrm{dom}\, V$

$$\sup_{v \in F(t^-, x)} [D_\uparrow V((t, x); (-1, -v)) - L(t^-, x, v)] \leq 0,$$

(iii) for all $x \in \mathbb{R}^n$

$$V(T, x) = g(x).$$

Characterization of lsc Value Functions - integral term...

(c) V is lsc on $[S,T]\times\mathbb{R}^n$ and

(i) for all $(t,x)\in((S,T)\times\mathbb{R}^n)\cap\mathrm{dom}\,V$, $(\xi^0,\xi^1,-\lambda)\in N^P_{\mathrm{epi}\,V}((t,x),V(t,x))$

$$\xi^0+\inf_{v\in F(t^+,x)}[\xi^1\cdot v+\lambda L(t,x,v)]\ \le\ 0,$$

(ii) for all $(t,x)\in((S,T)\times\mathbb{R}^n)\cap\mathrm{dom}\,V$, $(\xi^0,\xi^1,-\lambda)\in N^P_{\mathrm{epi}\,V}((t,x),V(t,x))$

$$\xi^0+\inf_{v\in F(t^-,x)}[\xi^1\cdot v-\lambda L(t^-,x,v)]\ge 0,$$

(iii) for all $x\in\mathbb{R}^n$,

$$\liminf_{\{(t',x')\to(S,x):t'>S\}} V(t',x')=V(S,x)$$

and

$$\liminf_{\{(t',x')\to(T,x):t'<T\}} V(t',x')=V(T,x)=g(x).$$

Proof structure

- introduce an auxiliary Lagrangian
- (a) $\Rightarrow$ (b). Apply the Optimality Principle of the value function,
- (b) $\Rightarrow$ (c). Use standard properties of Dini/continget derivative and proximal normal cone
- (c) $\Rightarrow$ (a) (**the key step**) This involves showing that, for an arbitrary point (t, x) in the domain of a function V satisfying condition (c),

  (A): $V(t, x)$ is the cost of some state trajectory originating from (t, x) and

  (B): $V(t, x)$ is a lower bound on the cost of an arbitrary state trajectory.

  For both (A) and (B) we use a **NEW weak invariance theorem** (linear growth is violated for the differential inclusion). Invoke **the Steiner representation** argument.

Example. Consider the optimal control problem

$$(P_{t_0,x_0}) \begin{cases} \text{Minimize } g(x(1)) + \int_0^1 L(t, x(t), \dot{x}(t))t \\ \text{over arcs } x(.) \in W^{1,1}([t_0, 1]; \mathbb{R}) \text{ such that} \\ \dot{x}(t) \in F(t) \quad \text{for a.e. } t \in [t_0, 1], \\ x(t_0) = x_0 \ , \end{cases}$$

where $t_0 \in [0, 1]$, $x_0 \in \mathbb{R}$,

$$F(t) := \begin{cases} [-\frac{1}{2}, \frac{1}{2}] & \text{if} \quad 0 \leq t \leq \frac{1}{2} \\ [-1, 1] & \text{if} \quad \frac{1}{2} < t \leq 1 \ , \end{cases}$$

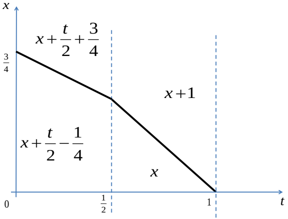
$$g(x) := \begin{cases} 1 + x & \text{if} \quad x > 0 \\ x & \text{if} \quad x \leq 0 \ , \end{cases}$$

and

$$L(t, x, v) := \begin{cases} 1 + (v + 1)^2 & \text{if} \quad \frac{1}{2} < t \leq 1 \\ (v + \frac{1}{2})^2 & \text{if} \quad 0 \leq t \leq \frac{1}{2} \ , \end{cases}$$

The value function $V : [0,1] \times \mathbb{R} \to \mathbb{R}$ is

$$V(t,x) := \begin{cases} x + \frac{t}{2} + \frac{3}{4} & \text{if} \quad 0 \le t \le \frac{1}{2} \quad \text{and} \quad x + t/2 - \frac{3}{4} > 0 \\ x + \frac{t}{2} - \frac{1}{4} & \text{if} \quad 0 \le t \le \frac{1}{2} \quad \text{and} \quad x + t/2 - \frac{3}{4} \le 0 \\ x + 1 & \text{if} \quad \frac{1}{2} < t \le 1 \quad \text{and} \quad x + t - 1 > 0 \\ x & \text{if} \quad \frac{1}{2} < t \le 1 \quad \text{and} \quad x + t - 1 \le 0\,. \end{cases}$$



Take the point $(t_0, x_0) = (\frac{1}{2}, \frac{1}{2})$. For every $(\xi^0, \xi^1, -\lambda) \in N^P_{\text{epi } V}((\frac{1}{2}, \frac{1}{2}), V(\frac{1}{2}, \frac{1}{2}))$:

$$\xi^0 + \inf_{v \in F(\frac{1}{2}^+)} \left[\xi^1 \cdot v + \lambda L\left(\frac{1}{2}, \frac{1}{2}, v\right)\right] \le 0,$$

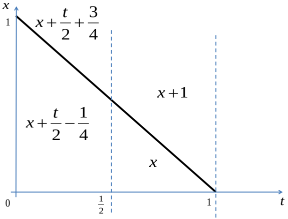
and

$$\xi^0 + \inf_{v \in F(\frac{1}{2}^-)} \left[\xi^1 \cdot v + \lambda L\left(\frac{1}{2}^-, \frac{1}{2}, v\right)\right] \ge 0.$$

The information provided by the 'asymptotic' vectors (proximal subdifferentials) $(\xi^0, \xi^1, 0) \in N^P_{\text{epi } V}((\frac{1}{2}, \frac{1}{2}), V(\frac{1}{2}, \frac{1}{2}))$ says how the epigraph of the value function bends at the point $(t_0, x_0) = (\frac{1}{2}, \frac{1}{2})$.

Consider now

$$W(t,x) := \begin{cases} x + \frac{t}{2} + \frac{3}{4} & \text{if} \quad 0 \le t \le \frac{1}{2} \quad \text{and} \quad x + t - 1 > 0 \\ x + \frac{t}{2} - \frac{1}{4} & \text{if} \quad 0 \le t \le \frac{1}{2} \quad \text{and} \quad x + t - 1 \le 0 \\ x + 1 & \text{if} \quad \frac{1}{2} < t \le 1 \quad \text{and} \quad x + t - 1 > 0 \\ x & \text{if} \quad \frac{1}{2} < t \le 1 \quad \text{and} \quad x + t - 1 \le 0 \,. \end{cases}$$



- Taking just vectors $(\xi^0, \xi^1, -1) \in N^P_{\text{epi}\, W}((t,x), W(t,x))$ all the 'restricted conditions' (c)(i)-(ii)-(iii) would be satisfied.
- (c)(i) is clearly violated when we check the inequality for the asymptotic vector $(1,1,0) \in N^P_{\text{epi}\, W}((t,x), W(t,x))$, for $t \in (0, 1/2)$ and $x + t = 1$:

$$\xi^0 + \inf_{v \in F(\frac{1}{2}^+)} \left[\xi^1 \cdot v + 0\right] = 1 - 1/2 > 0 \,.$$

$\to$ asymptotic vectors cannot be neglected...

Enter state constraints

$$(SC_{S,x_0}) \begin{cases} \text{Minimize } g(x(T)) \\ \text{over arcs } x \in W^{1,1}([S,T];\mathbb{R}^n) \text{ satisfying} \\ \dot{x}(t) \in F(t,x(t)) \quad \text{a.e. } t \in [S,T] \\ x(t) \in A \quad \text{for all } \ t \in [S,T] \qquad \leftarrow \quad \textbf{state constraint} \\ x(S) = x_0 \,. \end{cases}$$

Impose the additional 'bounded variation w.r.t time' condition:

(BV): For each $R_0 > 0$, $F(.,x)$ has bounded variation uniformly over $x \in R_0\mathbb{B}$, in the following sense: there exists a bounded variation function $\eta(.) : [S,T] \to \mathbb{R}$ such that, for every $[s,t] \subset [S,T]$ and $x \in R_0\mathbb{B}$,

$$d_H(F(s,x),F(t,x)) \leq \eta(t) - \eta(s) \,.$$

Characterization of Value Functions for State Constrained Problems (I): Outward-Pointing Condition

Theorem 3. [P.B.-Vinter]
Assume (H1), (H2), (H3) and (BV). Suppose in addition that

(CQ)$_{\text{outward}}$: for each $s \in [S,T)$, $t \in (S,T]$ and $x \in \partial A$,

$$F(t^-,x) \cap \big(-\operatorname{int} T_A(x)\big) \neq \emptyset \quad \text{and} \quad F(s^+,x) \cap \big(-\operatorname{int} T_A(x)\big) \neq \emptyset\,.$$

Take a $V : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$; (a)–(c) are equivalent:

(a) V is the value function for (SC_{S,x_0}).

(b) V is lsc on $[S,T] \times \mathbb{R}^n$, $V(t,x) = +\infty$ if $x \notin A$, and

(i) for all $(t,x) \in ([S,T) \times A) \cap \operatorname{dom} V$

$$\inf_{v \in F(t^+,x)} D_\uparrow V((t,x);(1,v)) \leq 0,$$

(ii) for all $(t,x) \in ((S,T] \times \operatorname{int} A) \cap \operatorname{dom} V$

$$\sup_{v \in F(t^-,x)} D_\uparrow V((t,x);(-1,-v)) \leq 0,$$

(iii) for all $x \in A$

$$\liminf_{\{(t',x') \to (T,x) : t' < T, x' \in \operatorname{int} A\}} V(t',x') = V(T,x) = g(x).$$

(c) V is lsc on $[S,T]\times\mathbb{R}^n$, $V(t,x)=+\infty$ if $x\notin A$, and

(i) for all $(t,x)\in((S,T)\times A)\cap\operatorname{dom}V$, $(\xi^0,\xi^1)\in\partial_P V(t,x)\cup\partial_P^\infty V(t,x)$

$$\xi^0+\inf_{v\in F(t^+,x)}\xi^1\cdot v\ \le\ 0,$$

(ii) $(t,x)\in((S,T)\times\operatorname{int}A)\cap\operatorname{dom}V$, $(\xi^0,\xi^1)\in\partial_P V(t,x)\cup\partial_P^\infty V(t,x)$

$$\xi^0+\inf_{v\in F(t^-,x)}\xi^1\cdot v\ge 0,$$

(iii) for all $x\in A$,

$$\liminf_{\{(t',x')\to(S,x):t'>S\}}V(t',x')=V(S,x)$$

and

$$\liminf_{\{(t',x')\to(T,x):t'<T,\ x'\in\operatorname{int}A\}}V(t',x')=V(T,x)=g(x).$$

Characterization of Value Functions for State Constrained Problems (II): Inward-Pointing Condition

Theorem 4. [P.B.-Vinter]
Assume (H1), (H2), (H3) and (BV). Suppose in addition that $g(.)$ is continuous on A and

(CQ)$_{\text{inward}}$: for each $s \in [S, T)$, $t \in (S, T]$ and $x \in \partial A$,

$$F(t^-, x) \cap \mathrm{int}\, T_A(x) \neq \emptyset \quad \text{and} \quad F(s^+, x) \cap \mathrm{int}\, T_A(x) \neq \emptyset .$$

Take a function $V : [S, T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Then assertions (a)–(c) below are equivalent:

(a) V is the value function for (SC_{S,x_0}).

(b) V is lsc on $[S, T] \times \mathbb{R}^n$, $V(t, x) = +\infty$ if $x \notin A$, and

(i) for all $(t, x) \in ([S, T) \times A) \cap \mathrm{dom}\, V$

$$\inf_{v \in F(t^+, x)} D_\uparrow V((t, x); (1, v)) \leq 0,$$

(ii) for all $(t, x) \in ((S, T] \times \mathrm{int}\, A) \cap \mathrm{dom}\, V$

$$\sup_{v \in F(t^-, x)} D_\uparrow V((t, x); (-1, -v)) \leq 0,$$

(iii) for all $x \in A$, $\quad V(T, x) = g(x)$.

(c) V is lsc on $[S,T]\times\mathbb{R}^n$, $V(t,x)=+\infty$ if $x\notin A$, and

(i) for all $(t,x)\in((S,T)\times A)\cap\mathrm{dom}\,V$, $(\xi^0,\xi^1)\in\partial_P V(t,x)\cup\partial_P^\infty V(t,x)$

$$\xi^0+\inf_{v\in F(t^+,x)}\xi^1\cdot v\ \le\ 0,$$

(ii) $(t,x)\in((S,T)\times\mathrm{int}\,A)\cap\mathrm{dom}\,V$, $(\xi^0,\xi^1)\in\partial_P V(t,x)\cup\partial_P^\infty V(t,x)$

$$\xi^0+\inf_{v\in F(t^-,x)}\xi^1\cdot v\ge 0,$$

(iii) for all $x\in A$,

$$\liminf_{\{(t',x')\to(S,x):t'>S\}} V(t',x')=V(S,x)$$

and $$V(T,x)=g(x).$$

Enter state constraints and integral cost

$$(SC_{S,x_0})\begin{cases} \text{Minimize } g(x(T)) + \int_S^T L(t,x(t),\dot{x}(t))\,dt \\ \text{over arcs } x \in W^{1,1}([S,T];\mathbb{R}^n) \text{ satisfying} \\ \dot{x}(t) \in F(t,x(t)) \quad \text{a.e. } t \in [S,T] \\ x(t) \in A \quad \text{for all } \ t \in [S,T] \qquad \leftarrow \quad \textbf{state constraint} \\ x(S) = x_0\,. \end{cases}$$

Theorem 5 [Bernis-P.B.]
Impose additional assumptions guaranteeing neighbouring feasible trajectories results with $W1,1$-estimates: Then assertions (a)-(c) are equivalent.

Rmk. Neighbouring feasible trajectories theorems are useful/important analytical tools to obtain results for state constrained problems:

- L^∞ estimates used for problem: **Minimize** $g(x(T))$
- $W^{1,1}$ estimates used for problem:
  Minimize $g(x(T)) + \int_S^T L(t, x(t), \dot{x}(t))\, dt$

$\rightarrow$ Rampazzo-Vinter IMA 1999, Rampazzo-Vinter SICON 2000, Frankowska-Rampazzo JDE 2000:
L^∞ **and** $W^{1,1}$ **estimates**, assuming 'standard' inward pointing condition

$\rightarrow$ Colombo-Khalil-Rampazzo, in preparation,
estimates assuming 'second order' inward pointing condition

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