EXTENDED EULER-LAGRANGE AND HAMILTONIAN CONDITIONS IN OPTIMAL CONTROL OF SWEEPING PROCESSES WITH CONTROLLED MOVING SETS

BORIS MORDUKHOVICH

Wayne State University

Talk given at the conference

Optimization, State Constraints and Geometric Control

Tribute to Giovanni Colombo and Franco Rampazzo

Joint work with Nguyen Hoang (Univ. Concepción, Chile)

Padova, Italy, May 2018

Supported by NSF grants DMS-1512846 and DMS-1808978
and by Air Force grant 15RT0462

CONTROLLED SWEEPING PROCESS

This talk addresses the following sweeping process

$$\dot{x}(t) \in f\Big(t, x(t)\Big) - N\Big(g(x(t)); C(t, u(t))\Big) \text{ a.e. } t \in [0, T]$$

with $x(0) = x_0 \in C(0, u(0))$, where

$$C(t, u) := \Big\{x \in I\!R^n \Big| \psi(t, x, u) \in \Theta\Big\}, \quad (t, u) \in [0, T] \times I\!R^m$$

with $f\colon [0, T] \times I\!R^n \to I\!R^n$, $g\colon I\!R^n \to I\!R^n$, $\psi\colon [0, T] \times I\!R^n \times I\!R^m \to I\!R^s$, and $\Theta \subset I\!R^s$. The feasible pairs $(u(\cdot), x(\cdot))$ are absolutely continuous. The normal cone is defined via the projector by

$$N(\bar{x}; \Omega) := \Big\{v \in I\!R^n \Big| \exists x_k \to \bar{x},\ \alpha_k \geq 0,\ w_k \in \Pi(x_k; \Omega),\ \alpha_k(x_k - w_k) \to v\Big\}$$

if $\bar{x} \in \Omega$ and $N(\bar{x}; \Omega) = \emptyset$ otherwise

The major assumption is that $\nabla_x \psi$ is surjective

OPTIMAL CONTROL

Problem $(\mathbf{P})$

minimize $J[x,u] := \varphi\Big(x(T)\Big) + \int_0^T \ell\Big(t, x(t), u(t), \dot{x}(t), \dot{u}(t)\Big)dt$

over the sweeping control dynamics subject to the intrinsic pointwise state-control constraints

$$\psi\Big(t, g(x(t)), u(t)\Big) \in \Theta \quad \text{for all} \quad t \in [0,T]$$

From now on

$$F = F(t,x,u) := f(t,x) - N\Big(g(x); C(t,u)\Big)$$

LOCAL MINIMIZERS

DEFINITION Let the pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ be feasible to (P)

(i) We say that $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a local $W^{1,2} \times W^{1,2}$-minimizer if $\bar{x}(\cdot) \in W^{1,2}([0,T]; I\!R^n)$, $\bar{u}(\cdot) \in W^{1,2}([0,T]; I\!R^m)$, and

$J[\bar{x}, \bar{u}] \leq J[x, u]$ for all $x(\cdot) \in W^{1,2}([0,T]; I\!R^n)$, $u(\cdot) \in W^{1,2}([0,T]; I\!R^m)$ sufficiently close to $(\bar{x}(\cdot), \bar{u}(\cdot))$ in the norm topology of the corresponding spaces

(ii) Let the running cost $\ell(\cdot)$ in do not depend on $\dot{u}$. We say that the pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a local $W^{1,2} \times \mathcal{C}$-minimizer if $\bar{x}(\cdot) \in W^{1,2}([0,T]; I\!R^n)$, $\bar{u}(\cdot) \in \mathcal{C}([0,T]; I\!R^m)$, and

$J[\bar{x}, \bar{u}] \leq J[x, u]$ for all $x(\cdot) \in W^{1,2}([0,T]; I\!R^n)$, $u(\cdot) \in \mathcal{C}([0,T]; I\!R^m)$ sufficiently close to $(\bar{x}(\cdot), \bar{u}(\cdot))$ in the norm topology of the corresponding spaces

DISCRETE APPROXIMATIONS

For local $W^{1,2} \times W^{1,2}$-minimizers $(\bar{x}, \bar{u})$. Problem $(\mathbf{P_k^1})$

$$\text{minimize} \;\; J_k[z^k] := \varphi(x_k^k) + h_k \sum_{j=0}^{k-1} \ell\Big(x_j^k, u_j^k, \frac{x_{j+1}^k - x_j^k}{h_k}, \frac{u_{j+1}^k - u_j^k}{h_k}\Big)$$

$$+h_k \sum_{j=0}^{k-1} \int_{t_j^k}^{t_{j+1}^k} \Big(\Big\|\frac{x_{j+1}^k - x_j^k}{h_k} - \dot{\bar{x}}(t)\Big\|^2 + \Big\|\frac{u_{j+1}^k - u_j^k}{h_k} - \dot{\bar{u}}(t)\Big\|^2\Big)\, dt$$

over $z^k := (x_0^k, \ldots, x_k^k, u_0^k, \ldots, u_k^k)$ s.t. $(x_k^k, u_k^k) \in \psi^{-1}(\Theta)$ and $x_{j+1}^k \in x_j^k + h_k F(x_j^k, u_j^k),\; j = 0, \ldots, k-1,\; \big(x_0^k, u_0^k\big) = \big(x_0, \bar{u}(0)\big)$

$$\sum_{j=0}^{k-1} \int_{t_j^k}^{t_{j+1}^k} \Big(\Big\|\frac{x_{j+1}^k - x_j^k}{h_k} - \dot{\bar{x}}(t)\Big\|^2 + \Big\|\frac{u_{j+1}^k - u_j^k}{h_k} - \dot{\bar{u}}(t)\Big\|^2\Big)\, dt \le \frac{\varepsilon}{2}$$

DISCRETE APPROXIMATIONS (cont.)

For local $W^{1,2} \times \mathcal{C}$-minimizer-minimizers $(\bar{x}, \bar{u})$. Problem $(\mathbf{P_k^2})$

$$\text{minimize } \; J_k[z^k] := \varphi(x_k^k) + h_k \sum_{j=0}^{k-1} \ell\Big(x_j^k, u_j^k, \frac{x_{j+1}^k - x_j^k}{h_k}\Big)$$

$$+ \sum_{j=0}^{k} \Big\| u_j^k - \bar{u}(t_j^k) \Big\|^2 + \sum_{j=0}^{k-1} \int_{t_j^k}^{t_{j+1}^k} \Big\| \frac{x_{j+1}^k - x_j^k}{h_k} - \dot{\bar{x}}(t) \Big\|^2 dt$$

over $z^k = (x_0^k, \ldots, x_k^k, u_0^k, \ldots, u_k^k)$ s.t. $(x_k^k, u_k^k) \in \psi^{-1}(\Theta)$ and $x_{j+1}^k \in x_j^k + h_k F(x_j^k, u_j^k),\; j = 0, \ldots, k-1,\; \big(x_0^k, u_0^k\big) = \big(x_0, \bar{u}(0)\big)$

$$\sum_{j=0}^{k} \Big\| u_j^k - \bar{u}(t_j^k) \Big\|^2 + \sum_{j=0}^{k-1} \int_{t_j^k}^{t_{j+1}^k} \Big\| \frac{x_{j+1}^k - x_j^k}{h_k} - \dot{\bar{x}}(t) \Big\|^2 dt \le \frac{\varepsilon}{2}$$

STRONG CONVERGENCE OF DISCRETE APPROXIMATIONS

THEOREM (i) If $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a local $W^{1,2} \times W^{1,2}$-minimizer for (P), then any sequence of piecewise linear extensions on $[0, T]$ of the optimal solutions $(\bar{x}^k(\cdot), \bar{u}^k(\cdot))$ to (P_k^1) converges to $(\bar{x}(\cdot), \bar{u}(\cdot))$ in the norm topology of $W^{1,2}([0, T]; I\!R^n) \times W^{1,2}([0, T]; I\!R^m)$

(ii) If $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a local $W^{1,2} \times \mathcal{C}$-minimizer for (P), then any sequence of piecewise linear extensions on $[0, T]$ of the optimal solutions $(\bar{x}^k(\cdot), \bar{u}^k(\cdot))$ to (P_k^2) converges to $(\bar{x}(\cdot), \bar{u}(\cdot))$ in the norm topology of $W^{1,2}([0, T]; I\!R^n) \times \mathcal{C}([0, T]; I\!R^m)$

GENERALIZED DIFFERENTIATION

Subdifferential of an l.s.c. function $\varphi\colon I\!R^n \to (-\infty, \infty]$ at $\bar{x}$

$$\partial\varphi(\bar{x}) := \Big\{ v \Big|\, (v, -1) \in N((\bar{x}, \varphi(\bar{x})); \text{epi}\,\varphi) \Big\}, \quad \bar{x} \in \text{dom}\,\varphi$$

Coderivative of a set-valued mapping F

$$D^*F(\bar{x}, \bar{y})(u) := \Big\{ v \Big|\, (v, -u) \in N((\bar{x}, \bar{y}); \text{gph}\,F) \Big\}, \quad \bar{y} \in F(\bar{x})$$

Generalized Hessian of φ at $\bar{x}$

$$\partial^2\varphi(\bar{x}) := D^*(\partial\varphi)(\bar{x}, \bar{v}), \quad \bar{v} \in \partial\varphi(\bar{x})$$

Enjoy **FULL CALCULUS** and **PRECISELY COMPUTED** in terms of the given data of (P)

FURTHER STRATEGY

• For each k reduce problems (P_k^1) and to (P_k^2) a problems of mathematical programming (MP) with functional and increasingly many geometric constraints. The latter are generated by the graph of the mapping $F(z) := f(x) - N(x; C(u))$, and so (MP) is intrinsically nonsmooth and nonconvex even for smooth initial data

• Use variational analysis and generalized differentiation (first- and second-order) to derive necessary optimality conditions for (MP) and then discrete control problems (P_k^1) and (P_k^2)

• Explicitly compute the coderivative of $F(z)$ entirely in terms of the given data of (P)

• By passing to the limit as $k \to \infty$, to derive necessary optimality conditions for the sweeping control problem (P)

EXTENDED EULER-LAGRANGE CONDITIONS

THEOREM If $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a local $W^{1,2} \times W^{1,2}$-minimizer, then there exist a multiplier $\lambda \geq 0$, an adjoint arc $p(\cdot) = (p^x, p^u) \in W^{1,2}([0,T]; I\!R^n \times I\!R^m)$, a signed vector measure $\gamma \in C^*([0,T]; I\!R^s)$, as well as pairs $(w^x(\cdot), w^u(\cdot)) \in L^2([0,T]; I\!R^n \times I\!R^m)$ and $(v^x(\cdot), v^u(\cdot)) \in L^\infty([0,T]; I\!R^n \times I\!R^m)$ with

$$\Big(w^x(t), w^u(t), v^x(t), v^u(t)\Big) \in \text{co}\,\partial\ell\Big(\bar{x}(t), \bar{u}(t), \dot{\bar{x}}(t), \dot{\bar{u}}(t)\Big)$$

satisfying the collection of necessary optimality conditions

• **Primal-dual dynamic relationships**

$$\dot{p}(t) = \lambda w(t) + \begin{bmatrix} \nabla^2_{xx}\Big\langle \eta(t), \psi\Big\rangle\Big(\bar{x}(t), \bar{u}(t)\Big) \\ \nabla^2_{xw}\Big\langle \eta(t), \psi\Big\rangle\Big(\bar{x}(t), \bar{u}(t)\Big) \end{bmatrix} \Big(-\lambda v^x(t) + q^x(t)\Big)$$

$$q^u(t) = \lambda v^u(t) \;\; \text{a.e.} \;\; t \in [0,T]$$

where $\eta(\cdot) \in L^2([0,T]; I\!R^s)$ is uniquely defined by

$$\dot{\bar{x}}(t) = -\nabla_x \psi\Big(\bar{x}(t), \bar{u}(t)\Big)^* \eta(t), \;\; \eta(t) \in N(\psi(\bar{x}(t), \bar{u}(t)); \Theta)$$

where $q\colon [0,T] \to I\!R^n \times I\!R^m$ is of bounded variation with

$$q(t) = p(t) - \int_{[t,T]} \nabla\psi\Big(\bar{x}(\tau), \bar{u}(\tau)\Big)^* d\gamma(\tau)$$

• **Measured coderivative condition**: Considering the t-dependent outer limit

$$\operatorname*{Lim\,sup}_{|B|\to 0} \frac{\gamma(B)}{|B|}(t) := \Big\{ y \in I\!R^s \Big| \;\; \exists \text{ seq. } \;\; B_k \subset [0,1], \; t \in I\!B_k, \; |B_k| \to 0, \; \frac{\gamma(B_k)}{|B_k|} \to y \Big\}$$

over Borel subsets $B \subset [0,1]$, for a.e. $t \in [0,T]$ we have

$$D^*N_\Theta\Big(\psi(\bar{x}(t), \bar{u}(t)), \eta(t)\Big)\Big(\nabla_x\psi(\bar{x}(t), \bar{u}(t))(q^x(t) - \lambda v^x(t))\Big) \bigcap \operatorname*{Lim\,sup}_{|B|\to 0} \frac{\gamma(B)}{|B|}(t) \neq \emptyset$$

- **Transversality condition**

$$-\Big(p^x(T), p^u(T)\Big) \in \lambda\Big(\partial\varphi(\bar{x}(T)), 0\Big) + \nabla\psi\Big(\bar{x}(T), \bar{u}(T)\Big)N_{\Theta}\Big((\bar{x}(T), \bar{u}(T)\Big)$$

- **Measure nonatomicity condition**: Whenever $t \in [0, T)$ with $\psi(\bar{x}(t), \bar{u}(t)) \in \text{int}\,\Theta$ there is a neighborhood V_t of t in $[0, T]$ such that $\gamma(V) = 0$ for any Borel subset V of V_t

- **Nontriviality condition**

$$\lambda + \sup_{t\in[0,T]} \|p(t)\| + \|\gamma\| \neq 0 \quad \text{with} \quad \|\gamma\| := \sup_{\|x\|_{C([0,T])}=1} \int_{[0,T]} x(s)d\gamma$$

- **Enhanced nontriviality**: If $\theta = 0$ is the only vector satisfying

$$\theta \in D^*N_{\Theta}\Big(\psi(\bar{x}(T), \bar{u}(T)), \eta(T)\Big)(0), \; \nabla\psi\Big(\bar{x}(T), \bar{u}(T)\Big)^*\theta \in \nabla\psi\Big(\bar{x}(T), \bar{u}(T)\Big)$$

then we have

$$\lambda + \text{mes}\Big\{t \in [0,T] \Big|\ q(t) \neq 0\Big\} + \|q(0)\| + \|q(T)\| > 0$$

(ii) If $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a local $W^{1,2} \times \mathcal{C}$-minimizer, then all the above conditions hold with

$$\Big(w^x(t), w^u(t), v^x(t)\Big) \in \text{co}\, \partial \ell\Big(\bar{x}(t), \bar{u}(t), \dot{\bar{x}}(t)\Big)$$

HAMILTONIAN FORMALISM

Consider here for simplicity a particular case of the orthant $\Theta = I\!R^s_-$ and put

$$I(x,u) := \Big\{ i \in \{1,\ldots,s\} \Big| \, \psi_i(x,u) = 0 \Big\}$$

For each $v \in -N(x;C(u))$ there is a unique $\{\alpha_i\}_{i \in I(x,u)}$ with $\alpha_i \le 0$ and $v = \sum_{i \in I(x,u)} \alpha_i [\nabla_x \psi(x,u)]_i$. Define $[\nu, v] \in I\!R^n$ by

$$[\nu, v] := \sum_{i \in I(x,u)} \nu_i \alpha_i \Big[\nabla_x \psi(x,u) \Big]_i, \; \nu \in I\!R^s$$

and introduce the **modified Hamiltonian**

$$H_\nu(x,u,p) := \sup \Big\{ \Big\langle [\nu, v], p \Big\rangle \Big| \, v \in -N\Big(x; C(u)\Big) \Big\}$$

MAXIMUM PRINCIPLE

THEOREM In addition to the extended Euler-Lagrange conditions there is a measurable function $\nu\colon [0,T] \to \mathbb{R}^s$ such that

$$\nu(t) \in \operatorname*{Lim\,sup}_{|B|\to 0} \frac{\gamma(B)}{|B|}(t)$$

and the maximum condition holds

$$\Big\langle \Big[\nu(t), \dot{\bar{x}}(t)\Big], q^x(t) - \lambda v^x(t) \Big\rangle = H_{\nu(t)}\Big(\bar{x}(t), \bar{u}(t), q^x(t) - \lambda v^x(t)\Big) = 0$$

The conventional maximum principle with

$$H(x,p) := \sup \Big\{ \langle p, v \rangle \Big|\ v \in F(x) \Big\}$$

fails!

REFERENCES

N. D. Hoang and B. S. Mordukhovich, Extended Euler-Lagrange and Hamiltonian formalisms in optimal control of sweeping processes with controlled sweeping sets, preprint (2018); arXiv:1804.10635