A nonsmooth Chow-Rashevski's theorem

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Optimization, State Constraints and Geometric Control
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References

1. F. Rampazzo & H. Sussmann, *Set-valued differentials and a nonsmooth version of Chow-Rashevski's theorem*, Proceedings of the 40th IEEE Conference on Decision and Control, Orlando, FL, December 2001.
2. F. Rampazzo and H. Sussmann, *Commutators of flow maps of nonsmooth vector fields*, J. Differential Equations 2007

Sketch of results in

1. E. Feleqi & F. Rampazzo, *Integral representations for bracket-generating multi-flows*, Discrete Contin. Dyn. Syst. Ser. A., 2015.
2. E. Feleqi & F. Rampazzo, *Iterated Lie brackets for nonsmooth vector fields*, NoDEA - Nonlinear Differential Equations Appl., 2017.
3. E. Feleqi & F. Rampazzo, *An L^∞-Chow-Rashevski's Theorem*, work in progress.

Controllability

Given

$$\mathcal{X} = (X_1, \ldots, X_p) \quad \text{vector fields}\,.$$

on some open set $\Omega \subset \mathbb{R}^n$.

$\mathcal{X}$ – **trajectory** := concatenation of a finite no. of integral curves of $X_1, \ldots, X_p,\ -X_1, \ldots, -X_p$.

Definition

$\mathcal{X}$ **controllable** in Ω

if $\forall x, y \in \Omega\ \exists \mathcal{X}$ – trajectory $\xi \colon [t_1, t_2] \to \Omega$

s.t. $\xi(t_1) = x,\ \xi(t_2) = y.$

Lie bracket

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Given two vector fields X, Y

$$[X, Y] = XY - YX \equiv DY \cdot X - DX \cdot Y\,.$$

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Main fact needed here

$$e^{-tY} \circ e^{-tX} \circ e^{tY} \circ e^{tX}(x) = x + t^2[X, Y](x_*) + o(t^2)$$

as $(t, x) \to (0, x_*)$.

Iterated Lie brackets

Iterated brackets of a family $X_1, \ldots, X_p$ of vector fields:

- degree 1

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- degree 2 (**Lie Bracket or Commutator**)

$$[\mathbf{X_i}, \mathbf{X_j}] := \mathbf{X_i X_j} - \mathbf{X_j X_i} \equiv \nabla \mathbf{X_j X_i} - \nabla \mathbf{X_i X_j}$$

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$$\Big[[X_i, X_j] \ , \ X_k \Big]$$

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$$\Big[[X_i, X_j]\ ,\ X_k\Big]$$

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$$[[[X_i, X_j], X_k], X_\ell] \quad \ldots \quad [[X_i, X_j], [X_k, X_\ell]]$$

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$$[[[X_i, X_j], X_k], X_\ell] \quad \dots \quad [[X_i, X_j], [X_k, X_\ell]]$$

- et cetera....

LARC, Chow-Rashevski's theorem and other deep results

Theorem

Assume $X_1, \ldots, X_p$ *satisfy* **L***ie* **A***lgebra* **R***ank* **C***ondition or* **H***örmander's* **C***ondition or, that is,*

$$span\Big\{iterated\ Lie\ brackets\ at\ x\Big\} = \mathbb{R}^n\,. \qquad \text{(LARC)}$$

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2. *(Hörmander)*
$$\mathcal{L} = \sum_{j=1}^{p} X_j^2 \quad \text{is hypoelliptic}$$
3. *(Bony)* $\mathcal{L}$ *satisfies the strong maximum principle.*

A set-valued-bracket (Franco and Hector)

If X_1, X_2 are $C^{0,1}$,we set

$$[\mathbf{X_1}, \mathbf{X_2}]_{\mathbf{set}}(\mathbf{x}) := \overline{\mathbf{co}} \left\{ \mathbf{v} = \lim_{\mathbf{j} \to \infty} [\mathbf{X_1}, \mathbf{X_2}](\mathbf{x_j}), \right\}$$

where

1. $x_j \in \mathcal{D}iff(X_1) \cap \mathcal{D}iff(X_2)$ for all j,
2. $\lim_{j \to \infty} x_j = x$

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Asymptotic formula: As $|t| + |x - x_*| \to 0$,

$$e^{-tX_2} \circ e^{-tX_1} \circ e^{tX_2} \circ e^{tX_1}(x) - x \in t^2 [X_1, X_2](x_*) + t^2 o(1)$$

Higher-order set-valued brackets

If X_1, X_2 are $C^{1,1}$ and X_3 is $C^{0,1}$, we set

$$[[\mathbf{X_1}, \mathbf{X_2}], \mathbf{X_3}]_{\mathbf{set}}(\mathbf{x})$$

$$:= \overline{\mathbf{co}} \left\{ \mathbf{v} = \lim_{\mathbf{j}\to\infty} \mathbf{D}\mathbf{X_3}(\mathbf{y_j}) \cdot [\mathbf{X_1}, \mathbf{X_2}](\mathbf{x_j}) - \mathbf{D}[\mathbf{X_1}, \mathbf{X_2}](\mathbf{x_j}) \cdot \mathbf{X_3}(\mathbf{y_j}), \right\}$$

where

1. $x_j \in \mathcal{D}iff(DX_1) \cap \mathcal{D}iff(DX_2)\ \forall j$, $y_j \in \mathcal{D}iff(X_3)\ \forall j$,
2. $\lim_{j\to\infty}(x_j, y_j) = (x, x)$.

Properties: Chart-invariant, robust, u.s.c. with comp, conv values

Asymptotic formula: As $|t| + |x - x_*| \to 0$

$$e^{-tX_3} \circ \underbrace{e^{-tX_1} \circ e^{-tX_2} \circ e^{tX_1} \circ e^{tX_2}}_{\Psi^{-1}} \circ e^{tX_3} \circ \underbrace{e^{-tX_2} \circ e^{-tX_1} \circ e^{tX_2} \circ e^{tX_1}}_{\Psi}(x) - x$$

$$\in t^3[X_1, [X_2, X_3]](x_*) + t^3 o(1)\,.$$

Theorem (A generalization of Chow-Rashevski's theorem)

Assume $\exists$ *iterated brackets* $B_1, \ldots, B_r$*, possibly set-valued, of the vector fields* $X_1, \ldots, X_p$ *s.t. at* x_*

$$\operatorname{span}\left\{v_1, \ldots, v_r\right\} = \mathbb{R}^n \quad \forall v_1 \in B_1, \ldots v_r \in B_r \,. \qquad \text{(GHC)}$$

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Then *every point* x *in a neighborhood of* x_* *is reached by a* $\mathcal{X}$*-trajectory in minimum time*

$$T(x, x_*) \leq C|x - x_*|^{1/k} \,,$$

where $k = \max\left\{\deg B_j \ : \ j = 1, \dots, r\right\}$, *C ind. of* x.

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If (GHC) *holds at every* $x_* \in \Omega$, *and* Ω *is connected,* **then** *every two points of* Ω *can be connected by a* $\mathcal{X}$*-trajectory and*

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GHC is acronym for **G**eneralized **H**örmander's **C**ondition.
k is step of HGC at x_*.

EXAMPLES

Heisenberg group Lie algebra generators
Nonholonomic integrator (Brocket)

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ -y \end{pmatrix} \equiv \partial_x - y\partial_z, \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix} \equiv \partial_y + x\partial_z\,,$$

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We see that

$$[X_1, X_2] = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} .$$

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We see that

$$[X_1, X_2] = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} .$$

Thus LARC holds at every point of $\mathbb{R}^3$.
The system is controllable in $\mathbb{R}^3$ and has locally (1/2)-Hölder continuous minimum time function.

A modified nonholonomic integrator in dim $= 4$

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ -y \\ 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ x \\ 0 \end{pmatrix} \quad X_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \alpha \end{pmatrix}$$

with α a nonvanishing **continuous** function.

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(LARC) is verified:

$$span\left\{X_1, X_2, [X_1, X_2], X_3\right\} = \mathbb{R}^4$$

$\Longrightarrow$ 1/2-Hölder minimum time.

Another modification of the nonholonomic integrator

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ -2y + |y| \end{pmatrix} \equiv \partial_x + (|y| - 2y)\partial_z, \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ 2x + |x| \end{pmatrix} \equiv \partial_y + (|x| + 2x)\partial_z\,,$$

Simple calculations yield

$$[X_1, X_2] = \left\{ \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} \;:\; h \in [2, 6] \right\} \quad \text{for } x = y = 0.$$

In any case **LARC** of step 2 at every point of $\mathbb{R}^3$.
System $\dot{x} = u_1 X_1 + u_2 X_2 + u_3 X_3$, $|u_i| \leq 1$, **controllable**
Minimum time 1/2-Hölder continuous.

Higher order brackets

Let

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 \\ 2x^k - x|x|^{k-1} \end{pmatrix}$$

One checks (for k even)

$$\underbrace{[X_i, [X_i, [\cdots [X_i, X_{n+i}]]]]_{set}}_{k \text{ bracketings}} = \left\{ \begin{pmatrix} 0 \\ h \end{pmatrix} \; : \; h \in [k!, 3k!] \right\} \quad \text{at } x = 0\,.$$

Hörmander of step $k+1$.

Hence $1/(k+1)$ minimum time.

Proof: integral formulas

If $X_1, X_2 \in C^1$

$$xe^{t_1X_1}e^{t_2X_2}e^{-t_1X_1}e^{-t_2X_2} = x + \int_0^{t_1}\int_0^{t_2} xe^{t_1X_1}e^{s_2X_2}e^{(s_1-t_1)X_1}[X_1, X_2]e^{-s_1X_1}e^{-s_2X_2}ds_1$$

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If $X_1, X_2 \in C^2$, $X_3 \in C^1$, then

$$xe^{t_1X_1}e^{t_2X_2}e^{-t_1X_1}e^{-t_2X_2}e^{t_3X_3}e^{t_2X_2}e^{t_1X_1}e^{-t_2X_2}e^{-t_1X_1}e^{-t_3X_3} - x =$$
$$\int_0^{t_1}\int_0^{t_2}\int_0^{t_3} xe^{t_1X_1}e^{t_2X_2}e^{-t_1X_1}e^{-t_2X_2}e^{s_3X_3}e^{t_2X_2}e^{t_1X_1}e^{(s_2-t_2)X_2}e^{-t_1X_1}e^{-s_2X_2}$$
$$[e^{s_2X_2}e^{s_1X_1}[X_1, X_2]e^{-s_1X_1}e^{-s_2X_2}, X_3]e^{s_2X_2}e^{t_1X_1}e^{-s_2X_2}e^{-t_1X_1}e^{-s_3X_3}ds_1\, ds_2\, ds_3\,.$$

E. Feleqi & F. Rampazzo, *Integral representations for bracket-generating multi-flows*, Discrete Contin. Dyn. Syst. Ser. A., 2015.

Proof: asymptotic formulas

(i) If $f_1, f_2 \in C^1$, $x_* \in M$,

$$xe^{t_1 f_1} e^{t_2 f_2} e^{-t_1 f_1} e^{-t_2 f_2} = x + t_1 t_2 [f_1, f_2](x_*) + t_1 t_2 o(1)$$

as $|x - x_*| + |(t_1, t_2)| \to 0$.

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as $|x - x_*| + |(t_1, t_2)| \to 0$.
(ii) If $f_1, f_2, \in C^2$, $f_3 \in C^1$, $x_* \in M$,

$$xe^{t_1 f_1} e^{t_2 f_2} e^{-t_1 f_1} e^{-t_2 f_2} e^{t_3 f_3} e^{t_2 f_2} e^{t_1 f_1} e^{-t_2 f_2} e^{-t_1 f_1} e^{-t_3 f_3}$$
$$= x + t_1 t_2 t_3 [[f_1, f_2], f_3](x_*) + (t_1 t_2 t_3) o(1)$$

as $|x - x_*| + |(t_1, t_2, t_3)| \to 0$.

E. Feleqi & F. Rampazzo, *Iterated Lie brackets for nonsmooth vector fields*, NoDEA - Nonlinear Differential Equations Appl., 2017.

Proof: Generalized Differential Quotients

If $X \in C^{-1,1}$, $(I_n \; X(x_*)) \stackrel{\text{def}}{=} \{(I_n \; v) \;:\; v \in X(x_*)\}$,
is a GDQ $(t,x) \mapsto xe^{tX}$,

If $X_1, X_2 \in C^{0,1}$, $(I_n \; [X_1, X_2]_{set}(x_*)) \stackrel{\text{def}}{=} \{(I_n \; v) \;:\; v \in [X_1, X_2]_{set}(x_*)\}$

and if $X_1, X_2 \in C^{1,1}$, $X_3 \in C^{0,1}$
$(I_n \; [[X_1, X_2], X_3]_{set}(x_*)) \stackrel{\text{def}}{=} \{(I_n \; v) \;:\; v \in [[X_1, X_2], X_3]_{set}(x_*)\}$ are GDQs of, respectively, $\Sigma^{(X_1,X_2)}_{[\cdot,\cdot]}$, $\Sigma^{(X_1,X_2,X_3)}_{[[\cdot,\cdot],\cdot]}$ at $(x_*, 0)$ in the direction of $\Omega \times \mathbb{R}$,
where

$$\Sigma^{(X_1,X_2)}_{[\cdot,\cdot]}(x,t) := \begin{cases} x\Psi^{(X_1,X_2)}_{[\cdot,\cdot]}(\sqrt{t},\ \sqrt{t})\Psi^{(X_1,X_2)}_{[\cdot,\cdot]}(-\sqrt{t}, -\sqrt{t}) & \text{if } t \geq 0 \\ x\Psi^{(X_1,-X_2)}_{[\cdot,\cdot]}(\sqrt{-t},\ \sqrt{-t})\Psi^{(X_1,-X_2)}_{[\cdot,\cdot]}(-\sqrt{-t}, -\sqrt{-t}) & \text{if } t < 0, \end{cases}$$

$$\Sigma^{(X_1,X_2,X_3)}_{[[\cdot,\cdot],\cdot]}(x,t) := x\Psi^{(X_1,X_2,X_3)}_{[[\cdot,\cdot],\cdot]}(\sqrt[3]{t},\ \sqrt[3]{t},\ \sqrt[3]{t}) \qquad \forall t \in \mathbb{R}\,.$$

Proof: conclusion

We assume generalized LARC at x_* for $X_1, X_2 \in C^{1,1}$, $X_3 \in C^{0,1}$, $X_4 \in C^{-1,1}$, that is,

$$\text{span}\Big\{X_1(x_*), X_2(x_*), X_3(x_*), [X_1, X_2], [X_1, X_3]_{set}(x_*), [X_2, X_3]_{set}(x_*), [[X_1, X_2], X_3]_{set}(x_*), X_4(x_*)\Big\} = T_{x_*}\Omega \equiv \mathbb{R}^n .$$

Consider $\mathbb{R}^8 \ni (t_1, \ldots, t_8) \mapsto$
$x_* e^{t_1 X_1}\, e^{t_2 X_2}\, e^{t_3 X_3}\, \Sigma^{(X_1,X_2)}_{[\cdot,\cdot]}(t_4)\, \Sigma^{(X_1,X_3)}_{[\cdot,\cdot]}(t_5) \Sigma^{(X_2,X_3)}_{[\cdot,\cdot]}(t_6)\, \Sigma^{(X_1,X_2,X_3)}_{[[\cdot,\cdot],\cdot]}(t_7)\, e^{t_8 X_4} \in \Omega\,;$

By the **chain rule**, its GDQ at $0 \in \mathbb{R}^8$ is

$$\Big(X_1(x_*)\; X_2(x_*)\; X_3(x_*)\; [X_1, X_2](x_*)\; [X_1, X_3]_{set}(x_*)\; [X_2, X_3]_{set}(x_*)\; [[X_1, X_2], X_3]_{set}(x_*)\; X_4(x_*)\Big) .$$

The LARC implies that the open mapping for GDQs applies to this map and hence the conclusion.

E. Feleqi & F. Rampazzo, *An L^∞-Chow-Rashevski's Theorem*, work in

Best wishes

Franco and Giovanni!!

Thank you!