A higher dimensional Poincaré - Birkhoff theorem for Hamiltonian flows

Alessandro Fonda

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Annales de l'Institut Henri Poincaré (2017)

But before starting...

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let me show you two recent photos...

Oberwolfach, 1985

Along the Adriatic, 1987

Ok, let’s start now

Jules Henri Poincaré (1854 – 1912)

SUR UN THÉORÈME DE GÉOMÉTRIE.

Par M. **H. Poincaré** (Paris).

Adunanza del 10 marzo 1912.

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Note: Poincaré died on July 17th, 1912

RENDICONTI

DEL

CIRCOLO MATEMATICO

DI PALERMO

DIRETTORE: G. B. GUCCIA.

TOMO XXXIII

(1° SEMESTRE 1912).

COMITATO DI REDAZIONE

PEL TRIENNIO 1912-1913-1914.

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Introduction.

Je n'ai jamais présenté au public un travail aussi inachevé; je crois donc nécessaire d'expliquer en quelques mots les raisons qui m'ont déterminé à le publier, et d'abord celles qui m'avaient engagé à l'entreprendre. J'ai démontré, il y a longtemps déjà, l'existence des solutions périodiques du problème des trois corps; le résultat laissait cependant encore à désirer; car, si l'existence de chaque sorte de solution était établie pour les petites valeurs des masses, on ne voyait pas ce qui devait arriver pour des valeurs plus grandes, quelles étaient celles de ces solutions qui subsistaient et dans quel ordre elles disparaissaient. En réfléchissant à cette question, je me suis assuré que la réponse devait dépendre de l'exactitude ou de la fausseté d'un certain théorème de géométrie dont l'énoncé est très simple, du moins dans le cas du problème restreint et des problèmes de Dynamique où il n'y a que deux degrés de liberté.

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J'ai donc été amené à rechercher si ce théorème est vrai ou faux, mais j'ai rencontré des difficultés auxquelles je ne m'attendais pas. J'ai été obligé d'envisager séparément un très grand nombre de cas particuliers; mais les cas possibles sont trop nombreux pour que j'aie pu les étudier tous. J'ai reconnu l'exactitude du théorème dans tous ceux que j'ai traités. Pendant deux ans, je me suis efforcé sans succès, soit de trouver une démonstration générale, soit de découvrir un exemple où le théorème soit en défaut.

Ma conviction qu'il est toujours vrai s'affermissait de jour en jour, mais je restais incapable de l'asseoir sur des fondements solides.

Il semble que dans ces conditions, je devrais m'abstenir de toute publication tant que je n'aurai pas résolu la question; mais après les inutiles efforts que j'ai faits pendant de longs mois, il m'a paru que le plus sage était de laisser le problème mûrir, en m'en reposant durant quelques années; cela serait très bien si j'étais sûr de pouvoir le reprendre un jour; mais à mon âge je ne puis en répondre. D'un autre côté, l'importance du sujet est trop grande (et je chercherai plus loin à la faire comprendre) et l'ensemble des résultats obtenus trop considérable déjà, pour que je me résigne à les laisser définitivement infructueux. Je puis espérer que les géomètres qui s'intéresseront à ce pro-

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"Théorème de géométrie"

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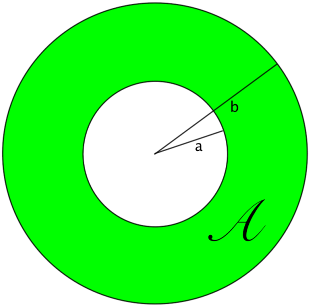

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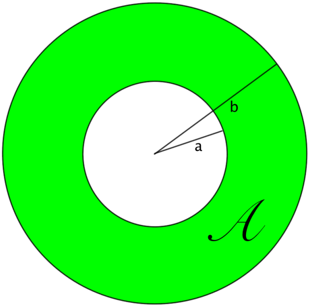

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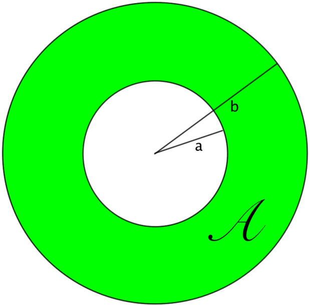

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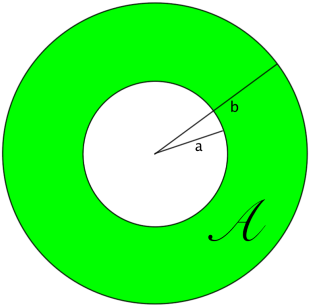

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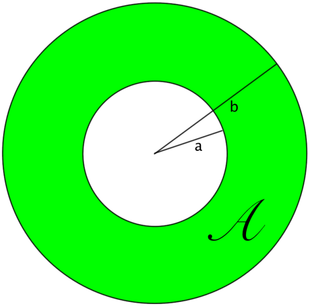

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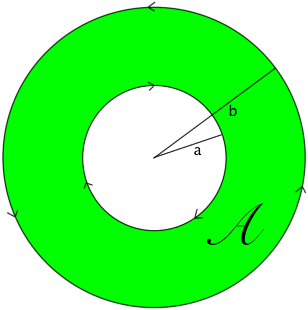

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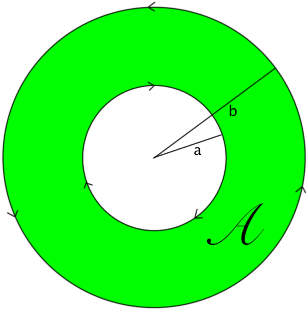

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Then, $\mathcal{P}$ has two fixed points.

An equivalent formulation

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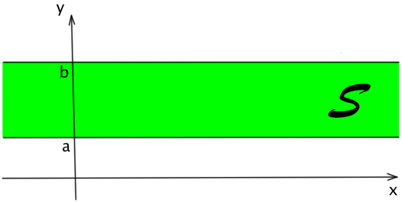

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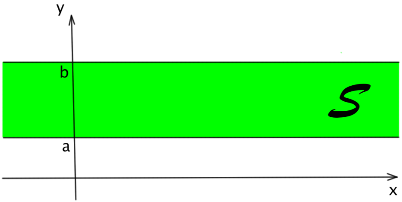

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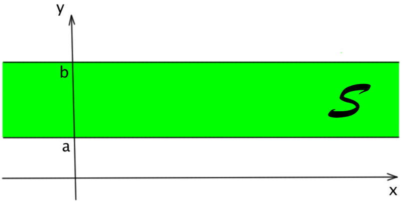

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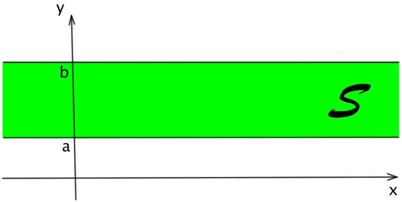

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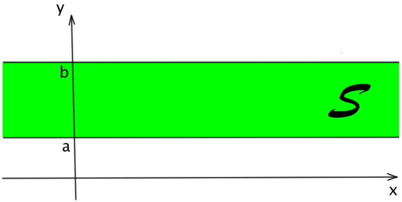

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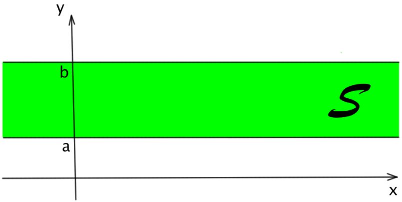

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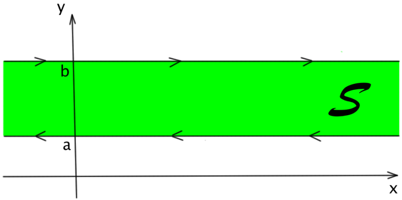

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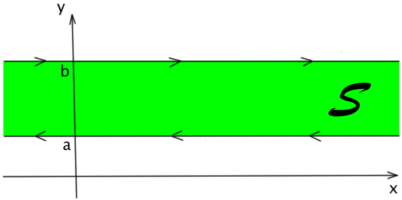

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Then, $\mathcal{P}$ has two geometrically distinct fixed points.

George David Birkhoff (1884 – 1944)

The Poincaré – Birkhoff theorem

In 1913 – 1925, Birkhoff proved Poincaré's "théorème de géométrie", so that it now carries the name

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Applications to the existence of periodic solutions were provided by:

Bonheure, Boscaggin, Butler, Del Pino, T. Ding, Fabry, Garrione, Hartman, Manásevich, Mawhin, Omari, Sfecci, Smets, Torres, Wang, Zanini, Zanolin, ...

Periodic solutions of a Hamiltonian system

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Two “simple” examples: the pendulum equation

$$\ddot{x} + \sin x = e(t),$$

and the superlinear equation

$$\ddot{x} + x^3 = e(t),$$

where $e(t)$ is a T-periodic forcing.

Periodic solutions as fixed points of the Poincaré map

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the "arrival point" (x_T, y_T) of the solution at time $t = T$.

Good and bad news

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The Poincaré map $\mathcal{P}$ is an area preserving homeomorphism.

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Bad news:

It is very difficult to find an invariant annulus for $\mathcal{P}$.

Generalizing the Poincaré – Birkhoff theorem (in the framework of Hamiltonian systems)

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Twist condition: the solutions $(x(t),y(t))$ with "starting point" $(x(0),y(0))$ on $\partial\mathcal{S}$ are defined on $[0,T]$ and satisfy

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Two remarks

1. Writing $\mathcal{S} = \mathbb{R} \times \overline{\mathcal{D}}$, with

$$\mathcal{D} = \,]a, b[\,,$$

and defining the "outer normal function" $\nu : \partial\mathcal{D} \to \mathbb{R}$ as

$$\nu(a) = -1\,, \qquad \nu(b) = +1\,,$$

the twist condition

$$(\star) \qquad (x(0), y(0)) \in \partial\mathcal{S} \quad \Rightarrow \quad x(T) - x(0) \begin{cases} < 0\,, & \text{if } y(0) = a\,, \\ > 0\,, & \text{if } y(0) = b\,, \end{cases}$$

can be written as

$$(\star) \qquad (x(0), y(0)) \in \partial\mathcal{S} \quad \Rightarrow \quad [x(T) - x(0)] \cdot \nu(y(0)) > 0\,.$$

2. The Poincaré map could be multivalued.

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Note: Arnold proposed some conjectures in the sixties.
Some of them are still open.

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$$\dot{x} = \frac{\partial H}{\partial y}(t, x, y), \qquad \dot{y} = -\frac{\partial H}{\partial x}(t, x, y),$$

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Then, there are $N + 1$ geometrically distinct T-periodic solutions.

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The periodicity in $x_1, \ldots, x_N$ permits to define the action functional on the product of a Hilbert space E and the N-torus $\mathbb{T}^N$:

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I. The “indefinite twist” condition:
for a regular symmetric $N \times N$ matrix $\mathbb{A}$,

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II. The “avoiding rays” condition:

$$(\star'') \quad (x(0), y(0)) \in \partial\mathcal{S} \quad \Rightarrow \quad x(T) - x(0) \notin \{-\lambda\nu(y(0)) : \lambda \geq 0\}\,.$$

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A. Boscaggin, A. Fonda and M. Garrione,
An infinite-dimensional version of the Poincaré–Birkhoff theorem on the Hilbert cube, preprint 2017

Buon compleanno!!!