Growth Model for Tree Stems and Vines

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Outline of the Talk

- Motivations
- The model
- Numerical Simulations
- Existence of Solutions
- Uniqueness
- Conclusions: Open Problems and Future Directions
Motivations

- There are many geometric structures in nature that still have to be studied...

- Plenty of them cannot be found in the Mathematical literature yet....
Main (Philosophical) Questions

- How does Nature control growth?

- What are the simplest mathematical models which can capture the heart of the matter?
Motivations

Today we will discuss a growth model for tree stems and vines...
The model takes into account:

1. the elongation due to cell growth,
2. the upward bending, as a response to gravity,
3. an additional bending, in case of a vine clinging to branches of other plants,
4. the reaction produced by obstacles, such as rocks, trunks or branches of other trees.
The Model: Some Notations

We assume that:

- $t_0$ is the initial time;
- an initial stem $\bar{P}(s)$ (curve in $\mathbb{R}^3$) is given for $s \in [0, t_0]$;
- for $t \geq t_0$ the stem starts to grow, bend, curl, cling etc...
- $P(t, s)$ is the position at time $t$ of the cell born at time $s$;
- The domain of $P(\cdot, \cdot)$ is $\mathcal{D} := \{(t, s) : t \geq t_0, \quad 0 \leq s \leq t\}$;
- A new cell is generated at the tip of the stem $P(t, t)$. 
The Model: Linear Elongation

- For the sake of simplicity: assume that the length of the stem at time $t$ is:

$$\ell(t) = \int_0^t ds = t$$

(in other words: the rate of growth of the stem is constant $= 1$).

- $s \mapsto P(t, s)$ is a curve (parametrized by $s$) of length $t$.

- $k(t, s)$ is the unit tangent vector to the stem at the point $P(t, s)$:

$$k(t, s) = \frac{P_s(t, s)}{|P_s(t, s)|}$$
The change in the position of points on the stem, in response to gravity, is described by

\[
\frac{\partial}{\partial t} P(t, s) = \int_0^s \kappa e^{-\beta(t-\sigma)} (k(t, \sigma) \times e_3) \times (P(t, s) - P(t, \sigma)) \, d\sigma = F_1(t, s).
\]

Here:

- \( \kappa > 0 \) is a constant, measuring the strength of the response;
- \( e^{-\beta(t-s)} \) is a **stiffness factor** (older parts of the stem are more rigid and they bend more slowly).
\( \omega(t, \sigma) = k(t, \sigma) \times e_3 \) is an angular velocity at the point \( P(t, \sigma) \). Notice that \( \omega \) affects all the upper portion of the stem.
The bending of the vine around the obstacle $\Omega$ can be described by

$$\frac{\partial}{\partial t} P(t, s) = \int_0^s e^{-\beta(t-\sigma)} \left( \nabla \psi(P(t, \sigma)) \times k(t, \sigma) \right) \times \nabla \psi(P(t, s)) - P(t, \sigma) \right) d\sigma \doteq F_2(t, s).$$

where

$$\psi(x) \doteq \eta(d(x, \Omega)) \quad x \in \mathbb{R}^3 \setminus \Omega,$$

for $\eta$ a smooth function measuring the sensitivity of the vine to cling to external obstacles.
For given $\delta_0 > 0$, on the left we find a good choice of $\eta$;

$F_2$ is a term which bends the stem toward the obstacle, at points which are sufficiently close (i.e. $< \delta_0$).
The Model: Avoiding Obstacles when $P(t, s) \in \Omega$

$$\omega(\sigma) = \text{angular velocity producing a bending at the point } P(\sigma).$$

$$\tilde{P}(s) - P(s) = \int_{0}^{s} \omega(\sigma) \times \left( P(s) - P(\sigma) \right) d\sigma$$
For each $t$, look for $\tilde{\omega}$ minimizing the elastic energy:

$$J(\omega) \equiv \int_0^t e^{\beta(t-s)} |\omega(s)|^2 \, ds.$$ 

over some unilateral linear constraints.

This produces a $v(t, s)$ such that

$$v(t, s) = \int_0^t \tilde{\omega}(t, s) \times (P(t, s) - P(t, \sigma)) \, d\sigma$$
Numerical Simulations: Avoiding Obstacles

(a) Center $O = (1.2, 1.5)$

(b) Center $O = (1.25, 1.5)$
Numerical Simulations: Clinging to Obstacles

(a) $\delta_0 = 0.05$, $\eta(r) = 7(1 - e^{-r})$.

(b) $\delta_0 = 0.05$, $\eta(r) = 4(1 - e^{-r})$.

(c) $\delta_0 = 0.05$, $\eta(r) = 3(1 - e^{-r})$. 
The Model: Summary of the Equations

\[(*)\quad P_t(t,s) = F_1(t,s) + F_2(t,s) + \mathbf{v}(t,s), \quad (t,s) \in \mathcal{D}\]

where

\[\mathcal{D} \doteq \{(t,s); \quad t \geq t_0, \quad s \in [0,t]\}\],

coupled with the conditions

\[P(t_0,s) = \overline{P}(s), \quad s \in [0,t_0],\]

\[P_{ss}(t,s) \bigg|_{s=t} = 0, \quad t > t_0,\]

and the constraint

\[P(t,s) \notin \Omega \quad \text{for all} \quad (t,s) \in \mathcal{D}.\]
The main equation (*) can be reformulated as a differential inclusion:

\[
\frac{d}{dt} P(t, \cdot) \in \Psi(P(t, \cdot)) + \Gamma(P(t, \cdot)), \quad P(t, \cdot) \in H^2([0, T]; \mathbb{R}^3).
\]

\(\Gamma(P(t, \cdot))\) is a (discontinuous) cone containing \(v(t, s)\).

A related model is the Perturbed Sweeping process

\[
\frac{d}{dt} P(t, s) \in \Psi(P(t, \cdot)) - N_{\Omega}(P(t, \cdot)), \quad P(t, \cdot) \in H^2([0, T]; \mathbb{R}^3).
\]

However,

\(\Gamma(P(t, \cdot)) \neq -N_{\Omega}(P(t, \cdot))\)

Here, \(N_{\Omega}\) is the normal cone to \(\Omega\).
The model is NOT defined on a FIXED domain.

To overcome this problem, we call solution of the model a function $P(\cdot, \cdot)$ s.t.:

(i) $t \mapsto P(t, \cdot)$ is Lipschitz continuous from $[t_0, T]$ into $H^2([0, T]; \mathbb{R}^3)$.

(ii) $P(\cdot, \cdot)$ satisfies the equation of the model.

(iii) $P(t, \cdot)$ is prolonged on $[0, T]$ using the relation

$$P(t, s) = P(t, t) + (s - t)P_s(t, t)$$

for all $t \in [t_0, T], s \in [t, T]$

for every $t \in [t_0, T]$, requiring that the constraint is satisfied just on $[0, t]$. 

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The solution, defined on \( \{(t, s) : 0 \leq s \leq t\} \), is extended for \( s \in [t, T] \).

According to \((iii)\), such an extension may end up into \( \Omega \).

This trick is carried out in order to work on the **fixed** domain \([t_0, T] \times [0, T]\).
Existence of a Solution

Theorem: (A. Bressan, M. P., W. Shen)

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with $C^2$ boundary. At time $t_0$, consider the initial data $s \mapsto \tilde{P}(t_0, s)$ is in $H^2([0, t_0]; \mathbb{R}^3)$ and satisfies

$$
\tilde{P}(t_0, 0) = 0 \notin \partial \Omega, \quad \tilde{P}(t_0, s) \notin \Omega \quad \text{for all} \quad s \in [0, t_0].
$$

Then the solution to $(\ast)$ exists as long as the BREAKDOWN condition $(B)$ is NOT reached.
(B) The tip of the stem touches the obstacle perpendicularly, namely

$$\bar{P}(t_0) \in \partial\Omega, \quad \bar{P}_s(t_0) = -\mathbf{n}(\bar{P}(t_0)).$$

Moreover,

$$\bar{P}_{ss}(s) = 0 \quad \text{for all } s \in (0, t) \text{ such that } \bar{P}(s) \notin \partial\Omega.$$
- When $P(t,s) \notin \partial \Omega$, then Existence and Uniqueness of the solution is standard! ($F_1$ and $F_2$ are smooth and $\Gamma = \{0\}$).

- When the stem touches the obstacle, the dynamics becomes discontinuous.

- If (B) occurs, the cone of reactions $\Gamma$ becomes tangent to the obstacle.
What happens when the stem touches $\Omega$...

Suppose that $\gamma$ is in $\Omega$. Call

$$\gamma_\omega(s) = \gamma(s) + \int_0^s \omega(\sigma) \times (\gamma(s) - \gamma(\sigma)) \, d\sigma$$

$\gamma_\omega$ is the rotated curve, $\omega \in \mathbb{R}^3$.

**Goal**: Find the “best” $\omega$ which pushes the stem out from $\Omega$!

This leads to:

minimize: $J(\omega) = \int_0^t e^{\beta(t-s)}|\omega(s)|^2 \, ds$,

subject to: $\gamma_\omega(s) \notin \Omega$ for all $s \in [0, t]$. 
What happens when the stem touches $\Omega$...

This leads to the study of a related optimal control problem for which, if condition (B) does NOT hold:

- we can prove a “controllability” result: exists $\omega \in \mathbb{R}^3$ bounded s.t.
  \[
  \left\langle \int_0^s \omega(\sigma) \times (\gamma(s) - \gamma(\sigma)) \, d\sigma, \nabla \Phi(\gamma(s)) \right\rangle \geq 1,
  \]

- the necessary conditions hold true in normal form, leading to the expression:
  \[
  \bar{\omega}(s) = - \int_s^t \int_{[\sigma,t]} e^{-\beta(t-s)} \nabla \Phi(\gamma(\sigma')) d\mu(\sigma') \times \gamma'(\sigma) \, d\sigma,
  \]

(Representation of an optimal angular velocity $\bar{\omega}$)

$\Phi(\cdot)$ is the signed distance from $\Omega$. 
The effect of the Push-out operator (an “integral rotation matrix”) is to apply a rotation able to move the stem outside the obstacle.
Idea of the proof

We construct a sequence of approximate solutions. For $\epsilon > 0$, define $t_k = t_0 + k\epsilon$.

- Suppose that a solution exists in $[0, t_{k-1}]$. An approximate solution $P(t_{k-}, s)$ is then constructed on $[t_{k-1}, t_k]$ in a suitable manner. Such a solution may lie inside $\Omega$.
- Find an optimal angular velocity $\bar{\omega}_k$. This is a solution of an optimal control problem with state constraint. Necessary Conditions imply the existence of a state constraint multiplier $\mu_k$.
- Apply a rotation matrix to the curve $s \mapsto P(t_{k-}, s)$, with optimal angular velocity $\bar{\omega}_k$.
- KEY ESTIMATE: $\|\mu_k\|_{T.V.} \leq C\epsilon$ ("controllability" condition)
- Compactness arguments lead to the existence of a solution.
In the sweeping process (or classic ODE) literature, uniqueness follows from the inequality

\[
\frac{d}{dt} \| \gamma_1(\cdot, t) - \gamma_2(\cdot, t) \|_{H^2([0, T])} \leq C \| \gamma_1(\cdot, t) - \gamma_2(\cdot, t) \|_{H^2([0, T])},
\]

which follows from the monotonicity property of the normal cone \(-N_{\Omega}(\cdot)\).

Here, another approach is required!

The key idea is the following: given \(k_1, k_2\) unit tangent vectors of two curves \(\gamma_1, \gamma_2\), we estimate the evolution w.r.t. \(t\) of the rotation vector between \(k_1\) and \(k_2\).
Consider $\gamma_1$ and a rotated curve $\gamma_2$. A bending determined by an angular velocity $\omega$ is reflected in a rotation of the tangent vectors.
Given $k_1$, $k_2$, $w$ and an initial time $\tau$ such that

$$k_2(\tau) = R[w(\tau)]k_1(\tau)$$

and assume that

$$k_{i,t}(t) = \omega_i(t) \times k_i(t), \quad i = 1, 2,$$

for some $\omega_1$, $\omega_2$ angular velocities. Then, for all $t \in [\tau, T]$,

$$\left| \frac{d}{dt} w(t) - (\omega_2(t) - \omega_1(t)) \right| \leq C \cdot \left( |\omega_1(t)| + |\omega_2(t)| \right) |w(t)|.$$

An integral version of the above estimate leads to uniqueness of the solution...
Uniqueness of the Solution

**Theorem:** (A. Bressan, M. P.)

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with $C^2$ boundary. At time $t_0$, consider the initial data $s \mapsto \bar{P}(t_0, s)$ is in $H^2([0, t_0]; \mathbb{R}^3)$ and satisfies

$$
\bar{P}(t_0, 0) = 0 \notin \partial\Omega, \quad \bar{P}(t_0, s) \notin \Omega \quad \text{for all} \quad s \in [0, t_0].
$$

Then the solution to the model is unique as long as the BREAKDOWN condition (B) is NOT reached.
Related Problem (F. Ancona, A. Bressan, O. Glass, W. Shen)

Stabilizing growth in vertical direction.

If the initial datum is in a tube of radius $\delta$, then the stem remains in a tube of radius $\epsilon$. 
Related Problem: Numerical Simulations

\[ \beta = 2.0 \quad \beta = 0.8 \quad \beta = 0.1 \]

- Stability is always achieved.
- Decreasing the stiffness \( \beta \) increases the oscillations.
A completely new model for the growth of tree stems and vines has been presented.

Main Results: Well-posedness and Characterization of the solution.

What's next? Modeling the stem growth is just the first step...

Deriving a model that explains **Phototropism** as a “competitive behaviour” among stems will be the next step.
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Happy Birthday
Giovanni and Franco
and thanks for your contributions
and your friendship!!