Growth Model for Tree Stems and Vines

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Outline of the Talk

- Motivations
- The model
- Numerical Simulations
- Existence of Solutions
- Uniqueness
- Conclusions: Open Problems and Future Directions

Motivations

- There are many geometric structures in nature that still have to be studied...

- Plenty of them cannot be found in the Mathematical literature yet....

Main (Philosophical) Questions

- How does Nature control growth?
- What are the simplest mathematical models which can capture the heart of the matter?

Motivations

Today we will discuss a growth model for tree stems and vines...

The Model

The model takes into account:

(1) the elongation due to cell growth,

(2) the upward bending, as a response to gravity,

(3) an additional bending, in case of a vine clinging to branches of other plants,

(4) the reaction produced by obstacles, such as rocks, trunks or branches of other trees.

The Model: Some Notations

We assume that:

- t_0 is the initial time;
- an initial stem $\bar{P}(s)$ (**curve in** $\mathbb{R}^3$) is given for $s \in [0, t_0]$;
- for $t \geq t_0$ the stem starts to grow, bend, curl, cling etc...
- $P(t, s)$ is the position at time t of the cell born at time s;
- The domain of $P(\cdot, \cdot)$ is $\mathcal{D} := \left\{(t, s) \;:\; t \geq t_0, \quad 0 \leq s \leq t\right\}$;
- A new cell is generated at the **tip** of the stem $P(t, t)$.

The Model: Linear Elongation

- For the sake of simplicity: assume that the length of the stem at time t is:

$$\ell(t) = \int_0^t ds = t$$

(in other words: the rate of growth of the stem is **constant=1**).

- $s \mapsto P(t,s)$ is a curve (parametrized by s) of length t.

- $\mathbf{k}(t,s)$ is the unit tangent vector to the stem at the point $P(t,s)$:

$$\mathbf{k}(t,s) = \frac{P_s(t,s)}{|P_s(t,s)|}$$

The Model: Response to Gravity

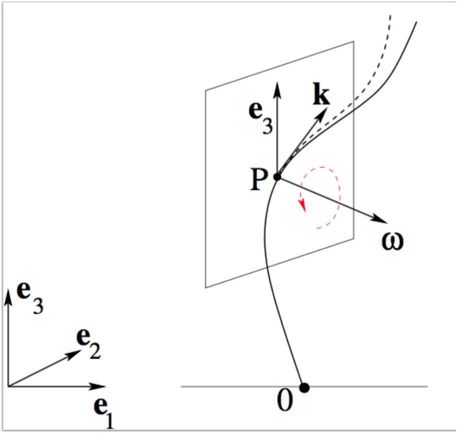
The change in the position of points on the stem, in response to gravity, is described by

$$\frac{\partial}{\partial t}P(t,s) = \int_0^s \kappa\, e^{-\beta(t-\sigma)} \big(\mathbf{k}(t,\sigma)\times\mathbf{e}_3\big)\times\big(P(t,s)-P(t,\sigma)\big)\, d\sigma \ \doteq\ F_1(t,s)\,.$$

Here:

- $\kappa > 0$ is a constant, measuring the strength of the response;
- $e^{-\beta(t-s)}$ is a **stiffness factor** (older parts of the stem are more rigid and they bend more slowly).

The Model: Response to Gravity



- $\omega(t,\sigma) = \mathbf{k}(t,\sigma) \times \mathbf{e}_3$ is an angular velocity at the point $P(t,\sigma)$. Notice that ω affects all the upper portion of the stem.

The Model: Clinging to Obstacles

The bending of the vine around the obstacle Ω can be described by

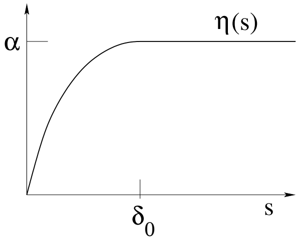
$$\frac{\partial}{\partial t}P(t,s) = \int_0^s e^{-\beta(t-\sigma)}\Big(\nabla\psi(P(t,\sigma)) \times \mathbf{k}(t,\sigma)\Big)\times \\ \times\Big(P(t,s) - P(t,\sigma)\Big)\,d\sigma \ \doteq \ F_2(t,s)\,.$$

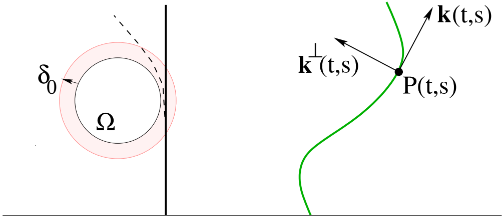
where

$$\psi(x) \ \doteq \ \eta\big(d(x,\Omega)\big) \qquad x \in \mathbb{R}^3 \setminus \Omega,$$

for η a smooth function measuring the sensitivity of the vine to cling to external obstacles.

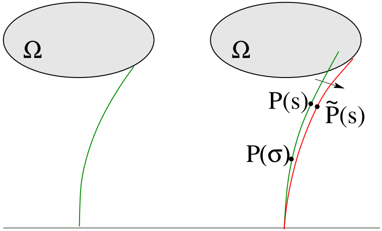
The Model: Clinging to Obstacles $P(s) \in \Omega$





- For given $\delta_0 > 0$, on the left we find a good choice of η;
- F_2 is a term which bends the stem toward the obstacle, at points which are sufficiently close (i.e. $< \delta_0$).

The Model: Avoiding Obstacles when $P(t, s) \in \Omega$



$\omega(\sigma)$ = angular velocity producing a bending at the point $P(\sigma)$.

$$\tilde{P}(s) - P(s) = \int_0^s \omega(\sigma) \times \Big(P(s) - P(\sigma) \Big) d\sigma$$

The Model: Avoiding Obstacles

For each t, look for $\bar{\omega}$ minimizing the elastic energy:

$$J(\omega) \doteq \int_0^t e^{\beta(t-s)} |\omega(s)|^2 \, ds.$$

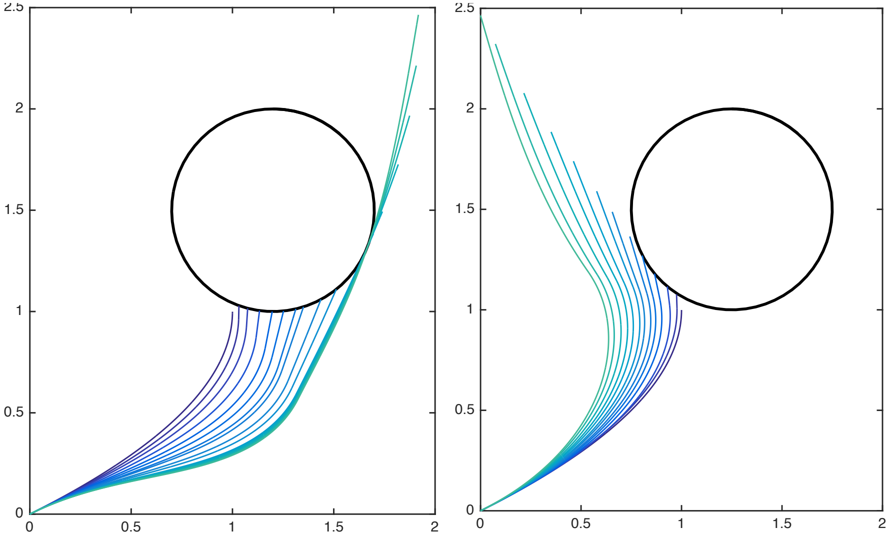
over some **unilateral** linear constraints.

This produce a $\mathbf{v}(t,s)$ such that

$$\mathbf{v}(t,s) = \int_0^t \bar{\omega}(t,s) \times (P(t,s) - P(t,\sigma)) \, d\sigma$$

Numerical Simulations: Avoiding Obstacles

(a) Center $O = (1.2, 1.5)$ **(b)** Center $O = (1.25, 1.5)$

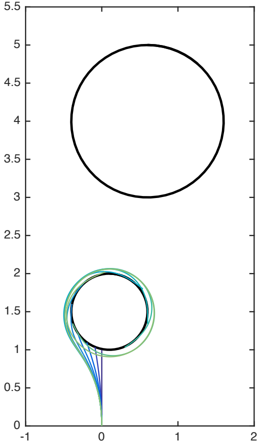


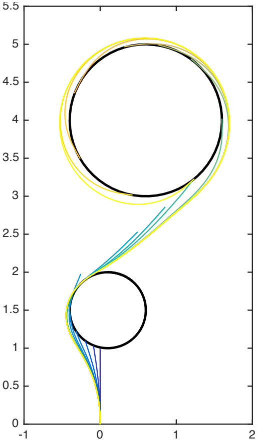
Numerical Simulations: Clinging to Obstacles

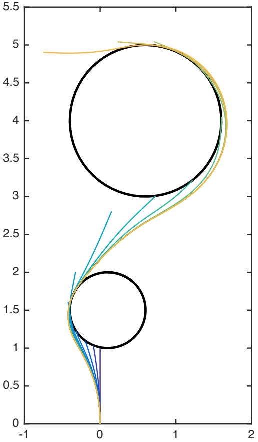
(a) $\delta_0 = 0.05$, $\eta(r) = 7(1 - e^{-r})$.

(b) $\delta_0 = 0.05$, $\eta(r) = 4(1 - e^{-r})$.

(c) $\delta_0 = 0.05$, $\eta(r) = 3(1 - e^{-r})$.







The Model: Summary of the Equations

$$(*) \qquad P_t(t,s) \;=\; F_1(t,s) + F_2(t,s) + \mathbf{v}(t,s), \quad (t,s) \in \mathcal{D}$$

where

$$\mathcal{D} \;\doteq\; \big\{(t,s)\,; \quad t \geq t_0\,, \quad s \in [0,t]\big\},$$

coupled with the conditions

$$P(t_0,s) \;=\; \overline{P}(s), \qquad\qquad s \in [0,t_0],$$

$$P_{ss}(t,s)\Big|_{s=t} \;=\; 0, \qquad\qquad t > t_0\,,$$

and the constraint

$$P(t,s) \;\notin\; \Omega \qquad \text{for all} \quad (t,s) \in \mathcal{D}.$$

Related Models

The main equation $(*)$ can be reformulated as a **differential inclusion**:

$$\frac{d}{dt}P(t,\cdot) \in \Psi(P(t,\cdot)) + \Gamma(P(t,\cdot)), \qquad P(t,\cdot) \in H^2([0,T];\mathbb{R}^3).$$

$\Gamma(P(t,\cdot))$ is a (**discontinuous**) cone containing $\mathbf{v}(t,s)$.

A related model is the **Perturbed Sweeping process**

$$\frac{d}{dt}P(t,s) \in \Psi(P(t,\cdot)) - N_\Omega(P(t,\cdot)), \qquad P(t,\cdot) \in H^2([0,T];\mathbb{R}^3).$$

However,

$$\Gamma(P(t,\cdot)) \neq -N_\Omega(P(t,\cdot))\,!!$$

Here, N_Ω is the normal cone to Ω.

The Model: Definition of Solution

The model is NOT defined on a FIXED domain.

To overcome this problem, we call solution of the model a function $P(\cdot,\cdot)$ s.t.:

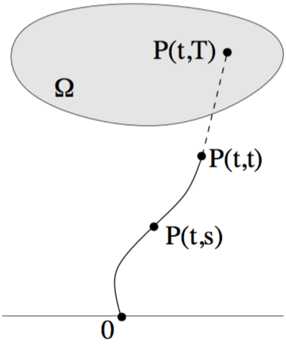
(i) $t \mapsto P(t,\cdot)$ is Lipschitz continuous from $[t_0, T]$ into $H^2([0, T]; \mathbb{R}^3)$.

(ii) $P(\cdot,\cdot)$ satisfies the equation of the model.

(iii) $P(t,\cdot)$ is prolonged on $[0, T]$ using the relation

$$P(t,s) = P(t,t) + (s-t)P_s(t,t) \qquad \text{for all } \; t \in [t_0, T], s \in [t, T]$$

for every $t \in [t_0, T]$, requiring that the constraint is satisfied just on $[0, t]$.



- The solution, defined on $\{(t,s) : 0 \leq s \leq t\}$, is extended for $s \in [t, T]$.
- According to (*iii*), such an extension may end up into Ω.
- This trick is carried out in order to work on the **fixed** domain $[t_0, T] \times [0, T]$.

Existence of a Solution

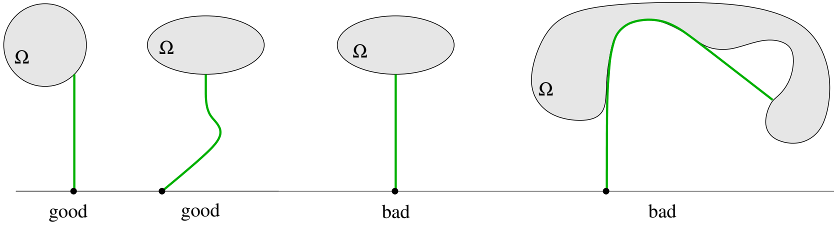
Theorem: (A. Bressan, M. P., W. Shen)

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with $\mathcal{C}^2$ boundary. At time t_0, consider the initial data $s \mapsto \bar{P}(t_0, s)$ is in $H^2([0, t_0]; \mathbb{R}^3)$ and satisfies

$$\bar{P}(t_0, 0) = 0 \notin \partial\Omega, \qquad \bar{P}(t_0, s) \notin \Omega \qquad \text{for all} \quad s \in [0, t_0].$$

Then the solution to $(*)$ exists as long as the BREAKDOWN condition **(B)** is NOT reached.

Breakdown Condition



(B) The tip of the stem touches the obstacle perpendicularly, namely

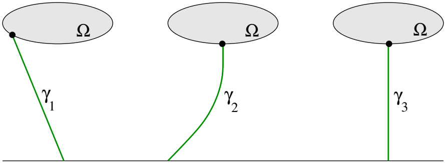
$$\bar{P}(t_0) \in \partial\Omega\,, \qquad \bar{P}_s(t_0) = -\mathbf{n}(\bar{P}(t_0)).$$

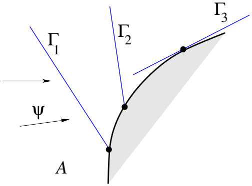
Moreover,

$$\bar{P}_{ss}(s) = 0 \qquad \text{for all } s \in (0,t) \text{ such that } \bar{P}(s) \notin \partial\Omega\,.$$

Comments on Existence Theorem

- When $P(t,s) \notin \partial\Omega$, then Existence and Uniqueness of the solution is standard! (F_1 and F_2 are smooth and $\Gamma = \{0\}$).
- When the stem touches the obstacle, the dynamics becomes discontinuous.
- if **(B)** occurs, the cone of reactions Γ becomes tangent to the obstacle.





What happens when the stem touches Ω...

Suppose that γ is in Ω. Call

$$\gamma_\omega(s) \doteq \gamma(s) + \int_0^s \omega(\sigma) \times \big(\gamma(s) - \gamma(\sigma)\big)\, d\sigma$$

γ_ω is the rotated curve, $\omega \in \mathbb{R}^3$.

Goal: Find the "best" ω which pushes the stem out from Ω!

This leads to:

$$\begin{aligned} \text{minimize:} \quad & J(\omega) \doteq \int_0^t e^{\beta(t-s)} |\omega(s)|^2\, ds, \\ \text{subject to:} \quad & \gamma_\omega(s) \notin \Omega \qquad \text{for all } \; s \in [0, t]. \end{aligned}$$

What happens when the stem touches Ω...

This leads to the study of a related optimal control problem for which, if condition **(B)** does NOT hold:

- we can prove a "controllability" result: exists $\omega \in \mathbb{R}^3$ bounded s.t.

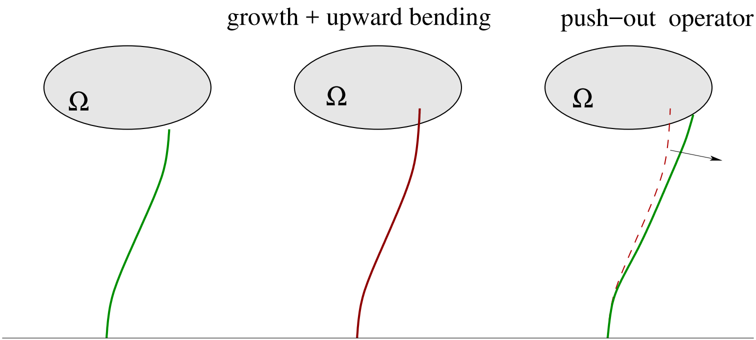
$$\left\langle \int_0^s \omega(\sigma) \times \big(\gamma(s) - \gamma(\sigma)\big)\, d\sigma \,,\ \nabla\Phi(\gamma(s)) \right\rangle \ \geq\ 1,$$

- the necessary conditions hold true in normal form, leading to the expression:

$$\bar{\omega}(s) \ =\ -\int_s^t \int_{[\sigma,t]} e^{-\beta(t-s)} \nabla\Phi(\gamma_{\bar{\omega}}(s'))d\mu(s') \times \gamma'(\sigma) d\sigma,$$

(Representation of an optimal angular velocity $\bar{\omega}$)

$\Phi(\cdot)$ is the signed distance from Ω.



The effect of the Push-out operator (an “integral rotation matrix”) is to apply a rotation able to move the stem outside the obstacle.

Idea of the proof

We construct a sequence of approximate solutions. For $\epsilon > 0$, define $t_k = t_0 + k\epsilon$.

- Suppose that a solution exists in $[0, t_{k-1}]$. An approximate solution $P(t_k-, s)$ is then constructed on $[t_{k-1}, t_k]$ in a suitable manner. Such a solution may lie inside Ω.
- Find an optimal angular velocity $\bar{\omega}_k$. This is a solution of an optimal control problem with state constraint. Necessary Conditions imply the existence of a state constraint multiplier μ_k.
- Apply a rotation matrix to the curve $s \mapsto P(t_k-, s)$, with optimal angular velocity $\bar{\omega}_k$.
- KEY ESTIMATE: $||\mu_k||_{T.V.} \leq C\epsilon$ ("controllability" condition)
- Compactness arguments lead to the existence of a solution.

Uniqueness of the solution: Discussion

In the sweeping process (or classic ODE) literature, uniqueness follows from the inequality

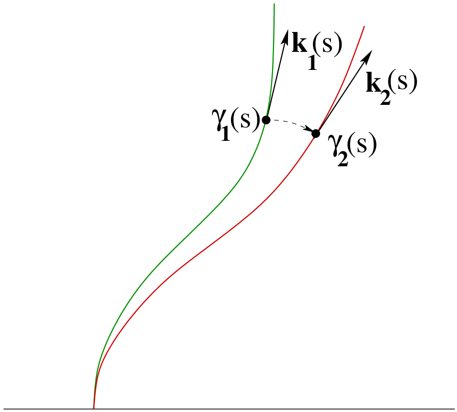
$$\frac{d}{dt}||\gamma_1(\cdot,t)-\gamma_2(\cdot,t)||_{H^2([0,T])} \leq C||\gamma_1(\cdot,t)-\gamma_2(\cdot,t)||_{H^2([0,T])},$$

which follows from the monotonicity property of the normal cone $-N_\Omega(\cdot)$.

Here, another approach is required!

The key idea is the following: given $\mathbf{k}_1, \mathbf{k}_2$ unit tangent vectors of two curves γ_1, γ_2, we estimate the evolution w.r.t. t of the rotation vector between $\mathbf{k}_1$ and $\mathbf{k}_2$.

Uniqueness of the solution: Geometric Intuition



Consider γ_1 and a rotated curve γ_2. A bending determined by an angular velocity ω is reflected in a rotation of the tangent vectors.

Uniqueness of the solution: Key Idea

Given $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{w}$ and an initial time τ such that

$$\mathbf{k}_2(\tau) = R[\mathbf{w}(\tau)]\mathbf{k}_1(\tau)$$

and assume that

$$\mathbf{k}_{i,t}(t) \; = \; \omega_i(t) \times \mathbf{k}_i(t), \qquad\qquad i = 1, 2,$$

for some ω_1, ω_2 angular velocities. Then, for all $t \in [\tau, T]$,

$$\left| \frac{d}{dt}\mathbf{w}(t) - (\omega_2(t) - \omega_1(t)) \right| \; \leq \; C \cdot \Big(|\omega_1(t)| + |\omega_2(t)| \Big) \, |\mathbf{w}(t)|.$$

An integral version of the above estimate leads to uniqueness of the solution...

Uniqueness of the Solution

Theorem: (A. Bressan, M. P.)

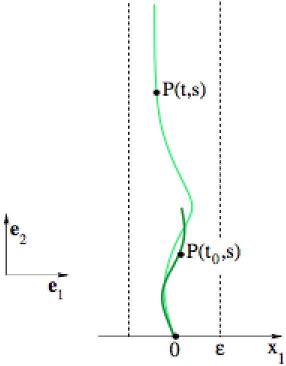
Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with $\mathcal{C}^2$ boundary. At time t_0, consider the initial data $s \mapsto \bar{P}(t_0, s)$ is in $H^2([0, t_0]; \mathbb{R}^3)$ and satisfies

$$\bar{P}(t_0, 0) = 0 \notin \partial\Omega, \qquad \bar{P}(t_0, s) \notin \Omega \qquad \text{for all} \quad s \in [0, t_0].$$

Then the solution to the model is unique as long as the BREAKDOWN condition **(B)** is NOT reached.

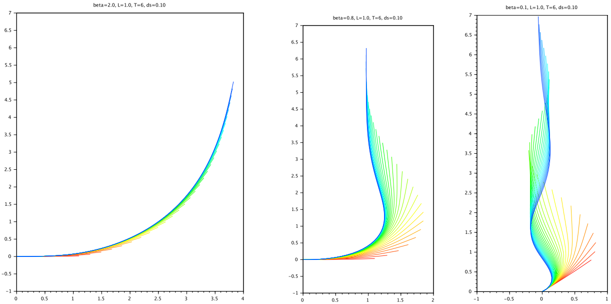
Related Problem (F. Ancona, A. Bressan, O. Glass, W. Shen)

Stabilizing growth in vertical direction.



If the initial datum is in a tube of radius δ, then the stem remains in a tube of radius ϵ.

Related Problem: Numerical Simulations



$\beta = 2.0$ $\beta = 0.8$ $\beta = 0.1$

- Stability is always achieved.
- Decreasing the stiffness β increases the oscillations.

Conclusions

- A completely new model for the growth of tree stems and vines has been presented.
- Main Results: Well-posedness and Characterization of the solution.
- What's next? Modeling the stem growth is just the first step...
- Deriving a model that explains **Phototropism** as a "competitive behaviour" among stems will be the next step.

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Happy Birthrday

Giovanni and Franco

and thanks for your contributions

and your friendship!!