Lack of BV bounds in impulsive control systems

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Impulsive control system

- Consider

$$\begin{cases} \dot{x}(t) = g_0(x(t), u(t), v(t)) + \sum_{i=1}^{m} g_i(x(t), u(t))\, \dot{u}_i(t), \\ (x(0), u(0)) = (\bar{x}_0, \bar{u}_0) \end{cases} \quad (1)$$

where

- the control (u, v) ranges over a compact set $U \times V \subset \mathbb{R}^m \times \mathbb{R}^q$
- u is the *impulsive* control; v is the *ordinary* control
- (usual hypotheses on g_i: local Lipschitz continuity and linear growth in (x, u)...)

- Let $T > 0$ and $v \in L^1$. A classical, Carathéodory solution x of (1) in $[0, T]$ exists only for **$\mathbf{u} \in \mathbf{AC}$**.

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- If $\mathbf{u} \in \mathbf{BV}$, there are fairly equivalent concepts of generalized solutions $x \in BV$, which I will refer to as **graph completion solutions,** ([Rishel, Warga, Bressan, Rampazzo, Dal Maso, Motta, Sartori, Miller, Rubinovich, Vinter, Silva, Arutyunov, Karamzin, de Oliveira, Pereira, Guerra, Sarychev, Wolenski, Zabic', Mazzola,...])

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- If $\mathbf{u} \in \mathcal{L}^{\mathbf{1}}$ (set of pointwisely defined L^1 functions), there is a notion of solution for **commutative systems,** where the Lie brackets $[(\mathbf{e}_i, g_i), (\mathbf{e}_j, g_j)] = 0$ for all $i, j = 1, \ldots, m$ ($\mathbf{e}_i$, $\mathbf{e}_i$ vectors of the canonical basis in $\mathbb{R}^m$) [Bressan, Rampazzo, '91], [A.V. Sarychev, 91], [Dykhta, 94], or other notions when **the Lie Algebra is non trivial**. (**looping controls** in [Bressan, Rampazzo, '94], **limit solutions** in [Aronna, Rampazzo,'15], ...)

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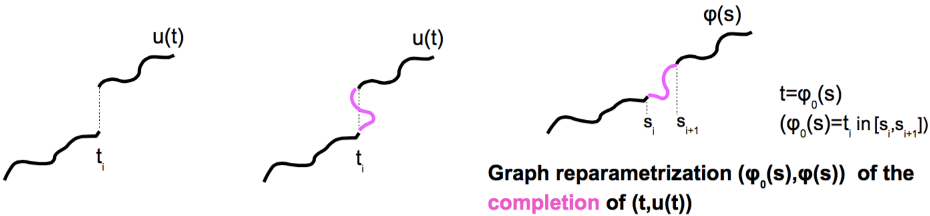
- For **commutative systems**, all these concepts of solution coincide

BV inputs u and graph completions

Let us first illustrate the graph completion approach for **u** $\in$ **BV**

(we assume that $U \subset \mathbb{R}^m$ has the Whitney property; e.g. let U be a compact, star-shaped set):



- Using the arc-length parametrization, (φ_0, φ) is 1-Lipschitz and

$$\varphi_0'(s) + |\varphi'(s)| = 1 \text{ for a.e. } s$$

- $\varphi_0(S) = T \implies \boxed{S = T + Var_{[0,S]}[\varphi].}$

- $\varphi_0^{-1} : [0, T] \to [0, S]$ is set-valued

The space-time system associated to the graph completion

- Let ξ be the solution of the ORDINARY space-time system:

$$\begin{cases} \xi'(s) = g_0(\xi, \varphi, v \circ \varphi_0)\varphi_0'(s) + \sum_{i=1}^m g_i(\xi, \varphi)\varphi_i'(s), \\ \xi(0) = \bar{x}_0. \end{cases} \tag{2}$$

(recall: $t = \varphi_0(s)$, time-change such that $t_i = \varphi_0(s)$ for $s \in [s_i, s_{i+1}]$)

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- If $\sigma : [0, T[\to [0, S[$ is a **selection** of φ_0^{-1}, called a **clock**, such that

$$(\varphi_0, \varphi)(\sigma(t)) = (t, u(t)) \ \text{ for every } t \in [0, T[, \sigma(0) = 0,$$

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- When $u \in AC$ and (φ_0, φ, S) is the arc-length parametrization of $(t, u(t))$,
  graph-completion solution $=$ **Carathéodory** solution.

Limit Solutions

INDEPENDENTLY FROM GRAPH COMPLETIONS THERE IS THE FOLLOWING

Definition 1 (Simple limit solution; Aronna, Rampazzo, ’15).

Let $(u, v) \in \mathcal{L}^1 \times L^1$ with $u(0) = \bar{u}_0$.

A map x is called a **simple limit solution** of (1), shortly ***S* limit solution**, if, there exists a sequence of controls $(u_k)_k \subset AC$ with $u_k(0) = \bar{u}_0$, pointwisely converging to u, and such that

(i) the sequence $(x_k)_k$ of the Carathéodory solutions to (1) corresponding to (u_k, v) is equibounded in $[0, T]$;

(ii) for any $t \in [0, T]$,

$$\lim_k x_k(t) = x(t).$$

BV Limit Solutions

Definition 2 (Aronna, Rampazzo, '15).

Let $(\mathbf{u}, \mathbf{v}) \in \mathbf{BV} \times \mathbf{L^1}$ with $u(0) = \bar{u}_0$. A map x is called a ***BV*** **simple limit solution** of (1) if

i) there exists $(u_k) \subset AC$, $u_k(0) = \bar{u}_0$, with equibounded variation, converging pointwisely to u;

ii) the corresponding solutions x_k to (1) converge pointwisely to x.

Relation between *BV* limit solutions and graph completions

Theorem 3 (Representation formula, Aronna, Rampazzo, '15).

Let $(\mathbf{u}, \mathbf{v}) \in \mathbf{BV} \times \mathbf{L}^1$ *with* $u(0) = \bar{u}_0$.

A map x *is a* **graph completion solution** *IF AND ONLY IF it is a BV* **simple limit solution**.

NEW RESULTS

Inputs $u \in BV_{loc}$

OUR GOAL IS TO INTRODUCE A NOTION OF GENERALIZED SOLUTION, the **BV_{loc} graph completion solution**, in an **intermediate situation,** for **BV_{loc} inputs u where**:

Definition 4 (BV_{loc} controls).

Let $T > 0$. We say that $\mathbf{u} \in \mathbf{BV_{loc}}$ if $u : [0, T] \to U$ and

$$Var_{[0,t]}(u) < +\infty \quad \text{for every } t < T\text{, but} \quad Var_{[0,T]}(u) \leq +\infty.$$

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AGREEING with the concept of **simple limit solution**

BUT with the advantages of **graph completion solutions**

Motivations

WHY BV_{loc} graph completion solutions instead of limit solutions?

1) **Because, they have explicit representation formula**:

- SUITABLE TO PROVE

  - **properness of the impulsive problem, HJ equations, approximations**
    ([Aronna, Motta, Rampazzo, '15], [Motta, Sartori,'15]; for $u \in BV$, e.g., [Motta, Rampazzo,'96], [Camilli, Falcone, '99], ...)

  - **optimality conditions** (for $u \in BV$: e.g., [Pereira, Silva, '00], ...)

2) **They may be the natural setting for**:

Controllability issues:
given a closed set $\mathcal{C} \subset \mathbb{R}^n \times U$, called target, select (x, u) such that $(x(T), u(T)) \in \mathcal{C}$ for u in BV or in BV_{loc};

Specific optimal control problems, as

$$\underset{(x,u,v)}{\text{Minimize}} \int_0^T [\ell_0(x(t), u(t), v(t)) + \ell_1(x(t), u(t)) \, |\dot{u}|] \, dt,$$
$$(x(T), u(T)) \in \mathcal{C}$$

WITH

Motivations

- **"target-weighted" weak coercivity**:

$$\ell_0 \geq 0, \qquad \ell_1(x,u) \geq c(\mathbf{d}((x,u),\mathcal{C}))$$

for some strictly increasing, continuous function $c : \mathbb{R}_+ \to \mathbb{R}_+$,

$\Longrightarrow$ only $\mathbf{u} \in \mathbf{BV}_{\mathbf{loc}}$ have **finite cost**

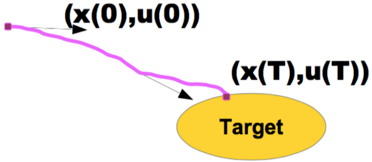
- GENERALIZATION of the well known **weak coercivity**:

$$\ell_0 \geq 0, \qquad \ell_1 \geq C_1 > 0$$

$\Longrightarrow$ only $\mathbf{u} \in \mathbf{BV}$ have **finite cost** (assumed in several applications).

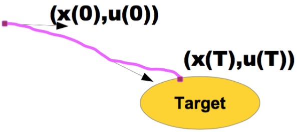
GIVEN a TARGET $\mathcal{C} \subset \mathbb{R}^n \times U$, WHAT DOES IT MEAN $(x(T), u(T)) \in \mathcal{C}$?

- **If u $\in$ AC and v $\in$ L^1:** $\exists!$ Carathéodory solution x of (1) in $[0, T]$:

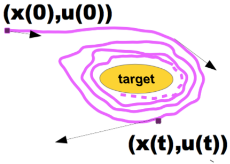


A trajectory-control pair (x, u, v) is **feasible** if and only if $(x(T), u(T)) \in \mathcal{C}$.

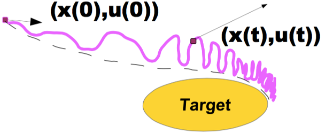
If u $\in$ AC$([0, t])$ for every $t < T$, but possibly $Var_{[0,T[}(u) = +\infty$ (u $\in$ AC$_{loc}$), and v $\in$ L^1: $\exists!$ Carathéodory solution x of (1) in $[0, T[$:



$\lim_{t\to T^-}(x(t), u(t)) \in \mathcal{C}$;



$\lim_{t\to T^-} \mathbf{d}((x(t), u(t)), \mathcal{C}) = 0$;



$\liminf_{t\to T^-} \mathbf{d}((x(t), u(t)), \mathcal{C}) = 0$.

WE WANT TO EXTEND the pairs (x, u) at $t = T$, so that WE CAN SAY that all all the previous (x, u) verify

$$(x(T), u(T)) \in \mathcal{C}\textbf{!}$$

Definition 5 (AC_{loc} solutions).

Given a control pair $(u, v) \in AC_{loc} \times L^1$, we introduce a **set-valued extension** of the Carathéodory solution x of (1) and of u to $t = T$:

$$(\mathbf{x}, \mathbf{u})_{\mathbf{set}}(\mathbf{T}) := \{\lim_j (x, u)(\tau_j), \quad (\tau_j)_j \text{ increasing and } \lim_j \tau_j = T\}.$$

We call (single-valued) AC_{loc} **trajectory-control pair** any (x, u, v) with

$$(x, u)(T) \in (x, u)_{set}(T).$$

According to this definition, given a target $\mathcal{C}$:

There exists $(x,u)(T) \in \mathcal{C} \iff \liminf_{t\to T^-} \mathbf{d}((x(t),u(t)),\mathcal{C}) = 0;$

TOWARDS THE DEFINITION of

BV_{loc} graph completion solution

("compatible" with an endpoint constraint $(x(T), u(T)) \in \mathcal{C}$)

BV_{loc} graph completion solutions

WE EXTEND the graph completion approach to $\mathbf{u} \in \mathbf{BV_{loc}}$, where it may happen that (using the arc-length parametrization)

$$S = T + Var_{[0,S]}[\varphi] = +\infty \quad (\Longleftrightarrow \; Var_{[0,S]}[\varphi] = +\infty).$$

GENERALIZED CONTROLS:

Definition 6 (BV_{loc} graph completions).

Given $\mathbf{u} \in \mathbf{BV_{loc}}$, we say that (φ_0, φ, S) with $S = +\infty$, is a **$\mathbf{BV_{loc}}$ graph completion of u** if

i) $\forall t \in [0, T[, \exists s \in [0, S[$ such that $(\varphi_0, \varphi)(s) = (t, u(t))$;

ii) moreover,

$$\lim_{s \to +\infty} \varphi_0(s) = T, \quad \lim_j \varphi(s_j) = u(T) \quad \text{for some } s_j \nearrow +\infty.$$

BV_{loc} graph completion solutions

Recall the space-time system introduced before.
Let ξ be the solution of the ORDINARY space-time system:

$$\begin{cases} \xi'(s) = g_0(\xi, \varphi, v \circ \varphi_0)\varphi_0'(s) + \sum_{i=1}^{m} g_i(\xi, \varphi)\varphi_i'(s), \\ \xi(0) = \bar{x}_0. \end{cases} \tag{3}$$

BV_{loc} graph completion solutions

GENERALIZED SOLUTIONS:

Definition 7 (BV_{loc} graph completion solutions).

Given a BV_{loc} graph completion $(\varphi_0, \varphi, +\infty)$ of $\mathbf{u} \in \mathbf{BV_{loc}}$ and a clock $\sigma : [0, T[\to [0, +\infty[$ **selection** of φ_0^{-1} and a control $\mathbf{v} \in \mathbf{L^1}$, let ξ be the solution of the space-time system (3).

We call **(single-valued)** ***BV_{loc}* graph completion solution** to (1), the map

$$x(t) := \xi \circ \sigma(t) \qquad \text{for } t \in [0, T[$$

extended to $t = T$ by considering $(x(T), u(T)) \in (\xi, \varphi)_{set}(+\infty)$, where

$$(\xi, \varphi)_{\mathbf{set}}(+\infty) := \{\lim_j (\xi, \varphi)(s_j) : \ s_j \nearrow +\infty \text{ s. t. } \lim_j \varphi(s_j) = u(T)\}.$$

BV_{loc} graph completion solutions

When $u \in AC_{loc}$ and $(\varphi_0, \varphi, +\infty)$ is the arc-length parametrization of $(t, u(t))$,

BV_{loc} graph completion solution $=$ AC_{loc} graph completion solution.

When $u \in BV_{loc}$, however, we DON'T require that

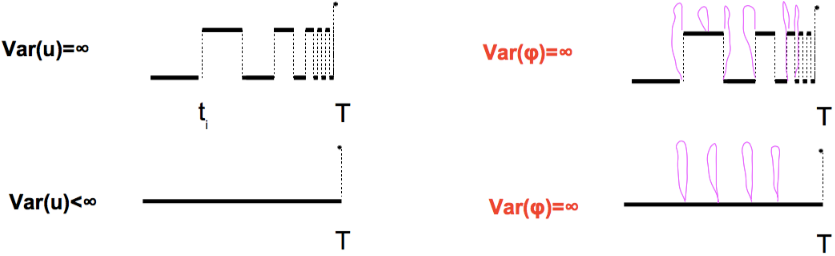
$$(x,u)(T) = \lim_{t_j \to T} (x,u)(t_j) \qquad \text{for some } t_j \nearrow T$$

but only the WEAKER assumption:

$$(x,u)(T) = \lim_{s_j \to +\infty} (\xi,\varphi)(s_j) \qquad \text{for some } s_j \nearrow +\infty$$

BV_{loc} graph completion solutions

This last condition DOES NOT IMPLY regularity of (x, u) at $t = T$.
E.g., considering just u:



$$u(T) = \lim_{s_j \to +\infty} \varphi(s_j) \qquad \text{for some } s_j \nearrow +\infty$$

The EXISTENCE of graph completions is NOT OBVIOUS and depends on the FORM of U:

Definition 8 (Whitney property).

A compact set $U \subset \mathbb{R}^m$ has the **Whitney property** if there is some $C \geq 1$ such that for all $u_1, u_2 \in U$, there exists $\tilde{u} \in AC([0,1], U)$ verifying

$$\tilde{u}(0) = u_1, \quad \tilde{u}(1) = u_2, \quad Var_{[0,1]}[\tilde{u}] \leq C|u_1 - u_2|.$$

For instance, compact, star-shaped sets verify the Whitney property.

Existence of BV_{loc} graph completion solutions

Theorem 9.

If U has the Whitney property, for any $u \in BV_{loc}$ there exists a graph-completion $(\varphi_0, \varphi, +\infty)$.

Comparison with limit solutions

RECALL the definition already introduced above:

Definition 10 (Simple limit solution; Aronna, Rampazzo, '15).

Let $(u, v) \in \mathcal{L}^1 \times L^1$ with $u(0) = \bar{u}_0$.

1. A map x is called a **simple limit solution** of (1), shortly ***S* limit solution**, if, there exists a sequence of controls $(u_k)_k \subset AC$ such that $u_k(0) = \bar{u}_0$ and,
   - (i) the sequence $(x_k)_k$ of the Carathéodory solutions to (1) corresponding to (u_k, v) is equibounded in $[0, T]$;
   - (ii) for any $t \in [0, T]$, $\lim_k (x_k, u_k)(t) = (x, u)(t)$.
2. An S limit solution x is called ***BVS* limit solution** of (1) if the approximating inputs u_k have equibounded variation.

Main theorems

Theorem 11 (Motta, Sartori, '18).

Let $(\mathbf{u}, \mathbf{v}) \in \mathbf{BV_{loc}} \times \mathbf{L^1}$ *with* $u(0) = \bar{u}_0$. *Then any* BV_{loc} *graph completion solution* x *of* (1) *is an* S *limit solution of* (1).

- The proof IS NOT a routine adaptation of an analogous result for *BV* inputs and solutions due to Aronna and Rampazzo, since **we loose any compactness**. Indeed, $x = \xi \circ \sigma$ may correspond to a graph completion (φ_0, φ, S) with $S = +\infty$ and a clock $\sigma : [0, T[\to [0, +\infty[$.

Main theorems

Steps of the proof

- x is a BV_{loc} GRAPH COMPLETION SOLUTION
  It is associated to $(\varphi_0, \varphi, \psi)$, to ξ solution of the space-time system and to a clock $\sigma : [0, T[\to \mathbb{R}_+$ such that $x(t) := \xi \circ \sigma(t)$ and $(\varphi_0, \varphi)(\sigma(t)) = (t, u(t))$.
- Find a sequence $\sigma_h \to \sigma$ in $[0, T]$ such that $\varphi_{0_h} := \sigma_h^{-1} \to \varphi_0$ in $[0, +\infty[$, φ_{0_h} Lipschitz.
- Define $u_h = \varphi \circ \sigma_h$. Modify the non (BV) controls u_h so that their variation is equibounded in $[0, t]$ for $t < T$.
- For a suitable subsequence of the modified u_h, the corresponding trajectories x_h converge pointwisely to x. Hence x is S LIMIT SOLUTION.
- It is a BV_{loc} SIMPLE LIMIT SOLUTION.

Main theorems

VICE-VERSA:

BV_{loc} graph completion solutions are **SPECIAL** simple limit solutions:

Definition 12 ($BV_{loc}S$ limit solution; Motta, Sartori, '16).

An S limit solution x is called a BV_{loc} **simple limit solution** of (1), shortly a $BV_{loc}S$ **limit solution**, if the approximating inputs u_k:

i) have equibounded variation in $[0, t]$ for every $t < T$;

ii) have "equiuniformity" at T $^{(*)}$

$^{(*)}$: $\exists\, \tilde\varepsilon(j) \underset{j}{\to} 0$, $\tilde s_j \nearrow +\infty$ and $k_j \geq j$ such that, for τ_k^j implicitly defined by $\tau_k^j + Var_{[0,\tau_k^j]}(u_k) = \tilde s_j$

$$|(x_k, u_k)(\tau_k^j) - (x_k, u_k)(T)| \leq \tilde\varepsilon(j) \qquad \text{for every } k > k_j,$$

The $BV_{loc}S$ limit solution are the right subset to prove the vice-versa of our theorem.

Theorem 13 (Motta, Sartori, '18).

Let $(\mathbf{u}, \mathbf{v}) \in \mathbf{BV_{loc}} \times \mathbf{L^1}$ *with* $u(0) = \bar{u}_0$.
Then any $BV_{loc}S$ limit solution x of (1) is a BV_{loc} graph completion solution of (1).

Main theorems

Steps of the proof

- x is a BV_{loc} SIMPLE LIMIT SOLUTION. The approximating inputs u_k have equibounded variation in $[0, t]$ for every $t < T$ and have "equiuniformity" at T.
- Define $\sigma_k := t + Var_{[0,t]}(u_k)$, $\varphi_{0_k} = \sigma_k^{-1}$, $\varphi_k := u_k \circ \varphi_{0_k}$.
- There exists a subsequence of $(\varphi_{0_k}, \varphi_k)$ and of σ_k converging locally uniformly to a (φ_0, φ) and to σ, resp.. Let ξ_k be the corresponding solution of the space-time system.
- (φ_0, φ) is a BV_{loc} graph completion, $(\varphi_0, \varphi) \circ \sigma = (t, u(t))$ and $x(t) = \lim x_k(t) = \lim \xi_k \circ \sigma_k$.
  $\Longrightarrow$ x is a BV_{loc} graph completion on $[0, T[$.
- Use the "equiuniformity " to show that x is a BV_{loc} GRAPH COMPLETION ON THE WHOLE $[0, T]$.

Main theorems

BV_{loc} simple limit solutions $\iff$ BV_{loc} graph completion solutions

Theorem 14 (Motta, Sartori, '18).

Let $(\mathbf{u}, \mathbf{v}) \in \mathbf{BV_{loc}} \times \mathbf{L^1}$ *with* $u(0) = \bar{u}_0$*. Then x is a* BV_{loc} *graph completion solution of* (1) **if and only if** *x is a* $BV_{loc}S$ *limit solution of* (1)*.*

This generalizes the equivalence between usual graph completion solutions and *BVS* limit solutions proved in [Aronna, Rampazzo, '15]

Example 15.

Let us consider the control system

$$\dot{x} = g_1(x)\dot{u}_1 + g_2(x)\dot{u}_2, \qquad (S)$$

with $x \in \mathbb{R}^3$, $u \in \mathbb{R}^2$ and $|u| \leq 1$,
with initial and terminal conditions

$$(x,u)(0) = ((1,0,1),(1,0)), \qquad (x,u)(T) = ((1,0,0),(1,0))$$

where

$$g_1(x) := \begin{pmatrix} 1 \\ 0 \\ x_3 x_2 \end{pmatrix}, \quad g_2(x) := \begin{pmatrix} 0 \\ 1 \\ -x_3 x_1 \end{pmatrix}.$$

EXAMPLE

- For any $u \in AC$ verifying $u(0) = (1,0)$, the corresponding Carathéodory solution x with $x(0) = (1,0,1)$ is

$$(x_1, x_2, x_3)(t) = \left(u_1(t), u_2(t), e^{-\int_0^t (-u_2\dot{u}_1 + u_1\dot{u}_2)(s)\,ds}\right) \quad \forall t \in [0, T].$$

In particular, since $|\int_0^t (-u_2\dot{u}_1 + u_1\dot{u}_2)(s)\,ds| \leq Var_{[0,T]}(u)$,

$$x_3(T) \geq e^{-Var_{[0,T]}(u)} > 0$$

and no solutions verifying $x_3(T) = 0$ exist.

EXAMPLE

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$$(x_1, x_2, x_3)(t) = \left(u_1(t), u_2(t), e^{-\int_0^t (-u_2\dot{u}_1 + u_1\dot{u}_2)(s)\,ds}\right) \quad \forall t \in [0, T].$$

In particular, since $|\int_0^t (-u_2\dot{u}_1 + u_1\dot{u}_2)(s)\,ds| \leq Var_{[0,T]}(u)$,

$$x_3(T) \geq e^{-Var_{[0,T]}(u)} > 0$$

and no solutions verifying $x_3(T) = 0$ exist.

- Consider $u \in AC_{loc}[0, T[$ given by

$$u(t) := \left(\cos\left(\frac{1}{T-t} - \frac{1}{T}\right), \sin\left(\frac{1}{T-t} - \frac{1}{T}\right)\right), \quad \text{for } t \in [0, T[. \qquad (1)$$

The corresponding solution is $(x_1, x_2, x_3)(t) = \left(u_1(t), u_2(t), e^{-\frac{t}{T(T-t)}}\right)$

EXAMPLE

so that $\lim_{t\to T} x_3(t) = 0$ and the (extended) AC_{loc} solution:

$$(x,u)(T) := \lim_k (x,u)(t_k) = ((1,0,0),(1,0)) \qquad \text{where } t_k := \frac{2k\pi T^2}{1+2k\pi T}$$

satisfies the terminal constraint.
The extended map x is a BV_{loc}S limit solution.
Indeed, for every k, set

$$t_k := \frac{2k\pi T^2}{1+2k\pi T}, \quad u_k(t) := u(t)\chi_{[0,t_k]}(t) + u(t_k)\chi_{]t_k,T]}(t),$$

where u is as in (1), giving $u(t_k) = (\cos(2k\pi), \sin(2k\pi)) = (1,0)$.

x is the pointwise limit of x_k, corresponding to $u_k \in \mathrm{AC}(T)$.

and

$$|(x_k,u_k)(t_j) - (x_k,u_k)(T)| = |x(t_j) - x(t_k)| \le e^{-\frac{t}{T(T-t_j)}} \to 0,$$

EXAMPLE

MINIMIZATION PROBLEM FOR THIS SYSTEM

Payoff

$$J(u) := \int_0^T [|1 - u_1(t)| + |u_2(t)| + |x_3(t)||\dot{u}(t)|]\, dt$$

with terminal constraint

$$(x, u)(T) \in \mathbb{C} := (U \times \{0\}) \times U.$$

We have $\inf_{u \in AC(T)} J(u) = +\infty$. In AC_{loc} the terminal constraint is equivalent to

$$(x, u)(T) \in \mathbb{C} \quad \underset{\text{In } AC_{loc}}{\Longleftrightarrow} \quad \liminf_{t \to T^-} d\big((x(t), u(t)), \mathbb{C}\big) = 0.$$

EXAMPLE

Hence, for every k, implementing the control

$$u_k(t) := (1,0)\chi_{[0,T-(1/k)]} + \left(\cos\left(\frac{1}{T-t} - k\right), \sin\left(\frac{1}{T-t} - k\right)\right)\chi_{[T-(1/k),T[}$$

we get the solution

$$x_k(t) = (1,0,1)\chi_{[0,T-(1/k)]} + \left(u_{1_k}(t), u_{2_k}(t), e^{k-\frac{1}{T-t}}\right)\chi_{[T-(1/k),T[},$$

with (x_k, u_k) verifying the constraints and $1 \leq J(u_k) \leq 1 + \frac{3}{k}$, so that $\lim_k J(u_k) = 1$.

EXAMPLE

The extended cost is

$$\mathcal{J}(\varphi_0,\varphi,S) := \int_0^S [(|1-\varphi_1(s)| + |\varphi_2(s)|)\varphi_0'(s) + |\xi_3(s)||\varphi'(s)|]\, ds,$$

where $S \leq +\infty$ and $\lim_{s\to S}\varphi_0(s) = T$.
The infimum is a minimum on the set of BV_{loc} graph completions, obtained for

$$(\varphi_0,\varphi)(s) := (s,1,0)\chi_{[0,T[}(s) + (T,(\cos(s-T),\sin(s-T))\chi_{[T,+\infty[}(s)$$

and the corresponding trajectory

$$\xi(s) = (1,0,1)\chi_{[0,T[}(s) + (\cos(s-T),\sin(s-T),e^{-s+T})\chi_{[T,+\infty[}(s).$$

We have

$$\mathcal{J}(\varphi_0,\varphi,+\infty) = 1.$$

EXAMPLE

Q. IS THIS THE MINIMUM ON THE SET OF S LIMIT SOLUTIONS?
A. YES!!
Add to the system the variable

$$\dot{x}_4 = |1 - u_1(t)| + |u_2(t)| + |x_3(t)||\dot{u}(t)|, \qquad x_4(0) = 0$$

In the class of S limit solutions, the problem is equivalent to minimize $x_4(T)$. For every sequence $(x_k, u_k)_k$ of equibounded, absolutely continuous maps defining an S limit solution verifying the terminal constraint, one has $\lim_k Var_{[0,T]}(u_k) = +\infty$ and

$$x_{4_k}(T) = J(u_k) \geq \int_0^T e^{-\int_0^t |\dot{u}_k|\, dr} |\dot{u}_k|\, dt = 1 - e^{-Var_{[0,T]}(u_k)} \to 1 \quad \text{as } k \to +\infty.$$

Actually, WE PROVE that the minimum value is obtained in the subset of $BV_{loc}S$ limit solutions.

An extended notion of limit solution

FOLLOWING [Aronna, Rampazzo, '15],

for any control $(\mathbf{u}, \mathbf{v})$ we define (simple) limit solutions x using approximating inputs

$$(u_k, \mathbf{v})$$

where the "ordinary" control v is fixed.

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for any control $(\mathbf{u}, \mathbf{v})$ we define (simple) limit solutions x using approximating inputs

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where the "ordinary" control v is fixed.

WHAT ABOUT CONSIDERING

$$(u_k, \mathbf{v_k}), \quad v_k \to v \text{ in } L^1?$$

RECALL the control system

$$\dot{x} = g_0(x, u, v) + \sum_{i=1}^{m} g_i(x, u)\, \dot{u}_i(t), \quad x(0) = \bar{x}_0,$$

where only the the DRIFT depends on v

RECALL the control system

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where only the the DRIFT depends on v

NEVERTHELESS, it may happen that

$$\mathbf{x_k \text{ corresponding to } (u_k, v)} \longrightarrow \mathbf{x}$$

and

$$\mathbf{x_k \text{ corresponding to } (u_k, v_k)} \longrightarrow \mathbf{x}$$

WHERE

- $\mathbf{x} \neq \mathbf{x}$;
- $\mathbf{x}$ **IS NOT a simple limit solution**

Example 16.

For $t \in [0, 2\pi]$, let us consider the control system

$$\dot{x} = g_0(x) + g_1(x)\dot{u}_1 + g_2(x)\dot{u}_2, \qquad -1 \leq v \leq 1, \quad |u| \leq 1, \qquad (4)$$

with initial condition $(x, u)(0) = ((0, 0, 1, 0), (0, 0))$, and

$$g_0(x) := \begin{pmatrix} 0 \\ 0 \\ 0 \\ v \end{pmatrix}, \quad g_1(x) := \begin{pmatrix} 1 \\ 0 \\ x_3 x_2 \\ -x_4 x_2 \end{pmatrix}, \quad g_2(x) := \begin{pmatrix} 0 \\ 1 \\ -x_3 x_1 \\ x_4 x_1 \end{pmatrix}.$$

Let $(u, v) \equiv (0, 0)$.

- For every k, set

$$u_k(t) := \frac{1}{\sqrt[3]{k}} \big(\cos(kt) - 1, \sin(kt)\big) \chi_{[2\pi/k, 2\pi]}(t), \quad \text{for } t \in [0, 2\pi],$$

$$v_k := k\, e^{-2\pi \sqrt[3]{k}} \chi_{[0, 2\pi/k]}.$$

- The solution x_k corresponding to (u_k, v), has $x_{4_k} \equiv 0$ and converges to the simple limit solution $x := (0, 0, 1, 0)\, \chi_{\{t=0\}}$.
  In fact, $x_4 \equiv 0$ for **any** simple limit solution.

- The solution $\tilde{x}_k$ corresponding to (u_k, v_k) has

$$\tilde{x}_{4_k} = k\, e^{-2\pi \sqrt[3]{k}}\, t \chi_{[0, 2\pi/k[} + 2\pi\, e^{\sqrt[3]{k}\left(t - 2\pi - \frac{\sin(kt)}{k} - \frac{2\pi}{k}\right)} \chi_{[2\pi/k, 2\pi]}$$

and converges to a map $x \neq x$, since

$$\mathbf{x}_4(2\pi) = 2\pi \neq 0 = \mathbf{x}_4(2\pi).$$

Thus **x IS NOT a simple limit solution!**

This suggest to **EXTEND the notion of limit solution**, by considering **approximating inputs**

(u_k, v_k) **with** $v_k \to v$ **in** L^1 **instead of** (u_k, v)

Remark: **EXTENDED** and **USUAL** limit solutions **coincide** in all existing results!

Remark: **EXTENDED** and **USUAL** limit solutions **coincide** in all existing results!

IN PARTICULAR, this is true for *BVS* and *$BV_{loc}S$* solutions:

Theorem 17 (M., Sartori, '18).

Let $(\mathbf{u}, \mathbf{v}) \in \mathbf{BV_{loc}} \times \mathbf{L^1}$ *be such that* $u(0) = \bar{u}_0$. *Then a map x is an* **extended** *BVS [resp.* **extended** *$BV_{loc}S$] limit solution* **if and only if** *it is a BVS [resp. $BV_{loc}S$] limit solution.*

LAST, LAST, LAST

Notice that we for a system

$$\dot{x}(t) = g_0(x(t), u(t), v(t)) + \sum_{i=1}^{m} g_i(x(t), u(t), v(t))\, \dot{u}_i(t),$$

we can show that in the BV case extended limit solutions coincide with graph completion solutions.

Some references

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Thank you for your attention!

HAPPY BIRTHDAY to GIOVANNI and FRANCO!!

The EXISTENCE of graph completions is NOT OBVIOUS and depends on the FORM of U:

Definition 18 (Whitney property).

A compact set $U \subset \mathbb{R}^m$ has the **Whitney property** if there is some $C \geq 1$ such that for all $u_1, u_2 \in U$, there exists $\tilde{u} \in AC([0,1], U)$ verifying

$$\tilde{u}(0) = u_1, \quad \tilde{u}(1) = u_2, \quad Var_{[0,1]}[\tilde{u}] \leq C|u_1 - u_2|.$$

For instance, compact, star-shaped sets verify the Whitney property.

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For instance, compact, star-shaped sets verify the Whitney property.

Theorem 19.

If U has the Whitney property, for any $u \in BV_{loc}$ there exists a graph-completion $(\varphi_0, \varphi, +\infty)$.

This result generalizes [Aronna, Rampazzo, '15] for BV inputs

- BOTH CONDITIONS i) **and ii)** in the definition of $BV_{loc}S$ limit solution are **necessary** for its CONSISTENCY with AC_{loc} solutions:

- BOTH CONDITIONS i) **and ii)** in the definition of $BV_{loc}S$ limit solution are **necessary** for its CONSISTENCY with AC_{loc} solutions:

Example 20.

Consider the AC_{loc} control

$$u(t) = \left(1 - \cos\left(\frac{1}{T-t} - \frac{1}{T}\right), \sin\left(\frac{1}{T-t} - \frac{1}{T}\right)\right), \text{ for } t \in [0, T[.$$

If $t_k := \frac{2k\pi T^2}{1+2k\pi T}$ and $\bar{t}_k := \frac{T^2(2k+1)\pi}{1+T(2k+1)\pi}$, so that $t_k < \bar{t}_k$, $t_k, \bar{t}_k \nearrow T$, and

$$u(t_k) = (0,0), \qquad u(\bar{t}_k) = (2,0),$$

the approximating inputs

$$u_k(t) := u(t)\chi_{[0,t_k]}(t) + 3u(t)\chi_{[t_k,\bar{t}_k]}(t) + (6,0)\chi_{[\bar{t}_k,T]}(t),$$

are in AC, have equibounded variation and converge to u in $[0,t]$ for any $t < T$,

BUT

$$\lim_k u_k(T) = (6,0) \notin u_{set}(T) \subset [0,2] \times [-1,1].$$

Theorem 21.

If $T_{AC_{loc}}$ is continuous on $\mathcal{C}$, then

$$T_{AC_{loc}} = T_{BV_{loc}\,g.c.} = T_{BV_{loc}S\,l.s.} = T_{l.s.} \quad (\leq T_{BV\,g.c.} = T_{BVS\,l.s.} < T_{AC}).$$

If T_{AC} is continuous on $\mathcal{C}$, **all** *these minimum times coincide.*

For every $\varepsilon > 0$ and $(\bar{x}_0, \bar{u}_0))$, let us define the $\varepsilon-$*penalized value function*

$$T_\varepsilon(\bar{x}_0, \bar{u}_0) := \inf_{(u,v)\in AC\times L^1} \int_0^{t_{(u,v)}} (1 + \varepsilon|\dot{u}(s)|)\, ds,$$

where

$$t_{(u,v)} := \inf\{t > 0 : \ (x(t), u(t)) \in \mathbb{C}\}.$$

Theorem 22.

For every $\varepsilon > 0$, let T_ε be continuous on $\mathcal{C}$. Then $T_{\mathcal{AC}}$ is continuous on $\mathcal{C}$ and

$$\lim_{\varepsilon\to 0^+} T_\varepsilon = T_{\mathcal{AC}}.$$

If moreover $T_{\mathcal{AC}}$ is continuous in its whole domain, the above limit is locally uniform.