Well Posedness of Optimal Control Problems with Linear Growth in the Control

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Joint work with Monica Motta and Franco Rampazzo

Conference: Optimization, State Constraints and Opt. Control

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Happy Birthday Franco and Giovanni!

Outline of the Talk

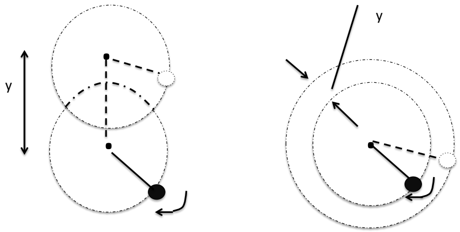
- *Optimal control problems with linear growth in the control*
- *The extended problem*
- *Conditions for existence of an infimum gap*
- *Special cases*
- *Open problems*
- *Concluding remarks*

Motivation

Consider a dynamical system, in which we seek to modify the motion by assigning some coordinates associated with independent constraints.

Framework: Assume state space to be product of manifolds $\mathcal{X} \times \mathcal{U}$. We assign time variation of $t \to u(t) \in \mathcal{U}$, by means of frictionless constraints.

Structural Control of Mechanical System



(a): pendulum with moving pivot (b): pendulum with variable length

Dynamical Equations

- How should chose $u(t)$ to stabilize $x(t)$?
- Calculate optimal controls.

$\mathcal{X}$ coordinates are governed by equations

$$\frac{dx}{dt}(t) = f(x(t)) + \sum_{j=1}^{m} g^j(x(t)) \frac{du}{dt}(t)$$

(for variable length pendulum)

- time derivative of directly controlled state appear!
- What is appropriate class of control functions (BV, $W^{1,1}$, . . .)?

Optimal Control Problems With Linear Growth

We focus on related control problems arising in:

- Control of Mechanical Systems (Bressan, Rampazzo, Moreau)
- Economics/Management Science (Bensoussan)
- 'Midcourse Guidance of Space Vehicles' (Rishel)

- Existence of $W^{1,1}$ minimizers NOT guaranteed

  (Classical 'superlinear growth' hyp. for existence of $W^{1,1}$ minimizers are violated)
- Extensive literature providing existence/optimaity conditions, when we allow larger classes of state trajectories e.g. (BV arcs)

Calculus of Variations: Murray, Rockafellar

Optimal Control: Arutyunov, Bressan, Motta, Pereira, Rampazzo, ,Silva, Vinter, Warga, . . .

The Optimal Control Problem

Consider

$$
(P) \begin{cases}
\text{Minimize } h(x(1)) \\
\text{over } (x,u)(\cdot) \in W^{1,1} \text{ satisfying} \\
\dfrac{dx}{dt}(t) = f(x(t)) + \sum_{j=1}^{m} g_j(x(t)) \dfrac{du^j}{dt}(t) \quad \text{a.e. } t \in [0,1] \\
\dfrac{du}{dt} \in V \text{ a.e. } t \in [0,1], \\
Var(u) := \int_0^1 \left| \dfrac{du}{dt}(t) \right| dt \le K, \\
x(0) = x_0, \quad x(1) \in C.
\end{cases}
$$

Data: $h : \mathbb{R}^n \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g_j : \mathbb{R}^n \to \mathbb{R}^n$, $j = 1, \ldots, m$,

$x_0 \in \mathbb{R}^n$, $C \subset \mathbb{R}^n$ *(closed set)*, V *(closed convex cone).*

$\{u^j(t), j = 1, \ldots, m\}$ are 'control-like variables' (via derivatives).

Strict Sense Processes

A *strict sense process* $(x^0(.), x(.), u(.))$ is a collection of absolutely continuous functions that satisfy

$$\begin{cases} \dfrac{dx^0}{dt}(t) = 1 \\ \dfrac{dx}{dt}(t) \ = \ f(x(t)) + \displaystyle\sum_{j=1}^{m} g_j(x(t))\dfrac{du^j}{dt}(t) \quad \text{a.e. } t \in [0, t], \\ \dfrac{du}{dt}(t) \in V \quad \text{a.e. } t \in [0, t]. \end{cases}$$

It is feasible if

$$x(0) = x_0, x(1) \in C, x^0(0) = 0 \text{ and } \mathrm{Var}(u(.)) \leq K\,.$$

$x(^0(.)$ is 'extra' variable that records time

$$x^0(t) = t\,.$$

Reparameterization

Fix an absolutely continuous control $u(.)$. Then the corresponding states $(x^0, x(.))$ are obtained by solving

$$\begin{cases} \dfrac{dx}{dt}(s) = f(x(t)) + \sum_{j=1}^{m} g_j(x(t)) \dfrac{du^j}{dt}(t) \quad \text{a.e. } t \in [0,1], \\ \dfrac{dx^0}{dt}(t) = 1 \\ \dfrac{du}{dt}(t) \in V \quad \text{a.e. } t \in [0, S]. \end{cases}$$

To be feasible, they must satisfy

$$x^0(0) = 0, x^0(1) = 1, x(0) = 0, x(1) \in C \text{ and } \mathrm{Var}(u(.)) \leq K\,.$$

How do we interpret these equations when $u(.)$ has bounded variation?

Reparameterization, Cont.

Change of independent variable (depending on $u(.)$):

$$\sigma(t) = \int_{[0,t]} (1 + |\frac{du}{dt}(t')|) dt' \quad \left(S = \int_{[0,1]} (1 + \frac{du}{dt}(t')) dt' \right)$$

Then reparameterized state trajectories $(y^0(.), y(.))$

$$(y^0(s), y(s)) := (x^0(\sigma^{-1}(s), x^0(\sigma^{-1}(s)), \; 0 \leq s \leq S$$

satisfy

$$\begin{cases} \frac{dy}{ds}(s) = f(y(s))(1 - |w(s)|) + \sum_{j=1}^{m} g_j(x(s)) w^j(s) \quad \text{a.e. } s \in [0, S], \\ \frac{dy^0}{ds}(s) = 1 - |w(s)| \end{cases}$$

in which

$$w^j(s) = \frac{du^j}{dt}(\sigma^{-1}(s)) \Big/ \left(1 + |\frac{du}{dt}(\sigma^{-1}(s))| \right), \; 0 \leq s \leq S\,.$$

Reparameterization, Cont.

We find: $(y^0(.), y(.), w(.)) \in W^{1,1} \times W^{1,1} \times L^\infty$ arises from a feasible strict sense process

if and only if

$$\begin{cases} \dfrac{dy}{ds}(s) \;=\; f(y(s))(1-|w(s)|) + \displaystyle\sum_{j=1}^{m} g_j(x(s))w^j(s) \quad \text{a.e. } s \in [0,S], \\ \dfrac{dy^0}{ds}(s) = 1 - |w(s)| \end{cases}$$

and

$$y(0) = x_0, y(S) \in C, y^0(0) = 0, y^0(S) = 1,\ \int_{[0,S]} |w(s)|ds \le K$$

$$w(s) \in V \cap \overset{\circ}{B} \qquad (\overset{\circ}{B} := \text{open unit ball in } \mathbb{R}^m)\,.$$

Reformulation of Optimal Control Problem

Equivalent formulation of the optimal control problem:

$$
(P') \begin{cases}
\text{Minimize } h(y(S)) \\
\text{over } S > 0,\ (y^0, y, w)(\cdot) \in W^{1,1} \times W^{1,1} \times L^\infty \ \text{ satisfying} \\
\dfrac{dy}{ds}(s) \;=\; f(y(s))(1 - |w(s)|) + \displaystyle\sum_{j=1}^{m} g_j(y(s)) w^j(s) \ \text{ a.e. } s \in [0, S]\,, \\
\dfrac{dy^0}{ds}(s) \;=\; (1 - |w(s)|) \ \text{ a.e. } s \in [0, S]\,, \\
y(0) = x_0, y(S) \in C, y^0(0) = 0, y^0(S) = 1\,, \\
\int_{[0,S]} |w(s)| ds \le K\,, \\
w(s) \in V \cap \overset{\circ}{B} \ \text{ a.e. } s \in [0, S]\,.
\end{cases}
$$

'Given feasible process (S, y^0, y, w), there is a feasible strict sense process (x^0, x, u) with same cost'

Extension

Velocity set for P_l

$$F(t,y) \in \{(f(y),1)(1-|w|) + (\sum_{j=1}^{m} g_j(y(s)),0)w^j \mid w(s) \in V \cap \overset{\circ}{B}\}$$

is convex but not closed.

Does reformulated problem have a minimizer?

No: because, because velocity set is not closed!

To guarantee existence of minimizers,

replace $V \cap \overset{\circ}{B}$ by $V \cap \bar{B}$

($\bar{B}$ is closed unit ball.)

This gives extended optimal control problem:

Extended Optimal Control Problem

$$
(P')\begin{cases}
\text{Minimize } h(y(S)) \\
\text{over } S > 0,\ (y^0, y, w)(\cdot) \in W^{1,1} \times W^{1,1} \times L^{\infty} \ \text{ satisfying} \\
\dfrac{dy}{ds}(s) \ = \ f(y(s))(1-|w(s)|) + \displaystyle\sum_{j=1}^{m} g_j(y(s))w^j(s) \ \text{ a.e. } s \in [0,S]\,, \\
\dfrac{dy^0}{ds}(s) \ = \ (1-|w(s)|) \ \text{ a.e. } s \in [0,S]\,, \\
y(0) = x_0, y(S) \in C, y^0(0) = 0, y^0(S) \in C\,, \\
\int_{[0,S]} |w(s)| ds \leq K\,, \\
w(s) \in V \cap \bar{B} \ \text{ a.e. } s \in [0,S]\,.
\end{cases}
$$

($V \cap \bar{B}$ has replaced $V \cap \mathring{B}$)

Existence of minimizers is guaranteed, if the set of feasible extended sense process is non-empty!

Extended Sense Feasible Processes

We say (S, y^0, y, w) is an extended sense process if:

$$\begin{cases} \dfrac{dy}{ds}(s) \;=\; f(y(s))(1 - |w(s)|) + \displaystyle\sum_{j=1}^{m} g_j(x(s))w^j(s) \quad \text{a.e. } s \in [0, S], \\ \dfrac{dy^0}{ds}(s) = 1 - |w(s)| \quad \text{a.e. } s \in [0, S]\,, \\ w(s) \in V \cap B \quad \text{a.e. } s \in [0, S] \end{cases}$$

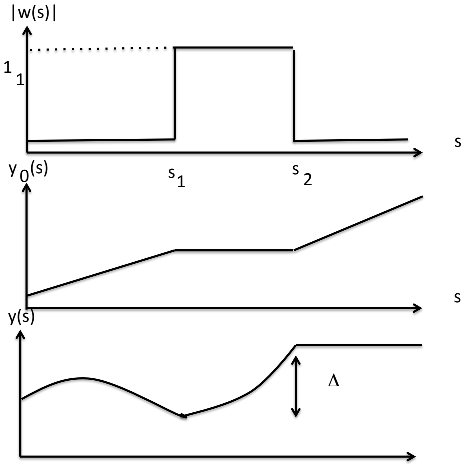
Feasible if : $y(0) = x_0, y(S) \in C, y^0(0) = 0, y(S) \in C, \int_{[0,S]} |w(s)|ds \leq K.$

(S, y^0, y, w) is a (feasible) embedded strict sense process if:

$$w(s) \;\in\; V \cap \overset{\circ}{B} \;\text{ a.e. } s \in [0, S].$$

(Then $y^0(.), y(.)$ is a reparameterization of a classical (absolutely continuous) state trajectory $(x^0(t) \equiv t, x(.))$.)

Structure of extended state trajectories



Jump: $x(t^+) - x(t) = \Delta$, where $t = \sigma^{-1}(s_1)$

Lemma

Given any extended sense process $(\bar{S}, \bar{y}^0, \bar{y}, \bar{w})$, and $\delta > 0$, we can find a embedded strict sense process $(\bar{S}, \bar{y}^0, \bar{y}, \bar{w})$ such that

$$\|(y^0(\cdot), y(\cdot)) - (\bar{y}^0(\cdot), \bar{y}(\cdot))\|_{L^\infty(0,\bar{S}\wedge S)} + |S - \bar{S}| \leq \delta\,.$$

Proof

Replace $w(s) \in V \cap B$ by $w'(s) = (1-\epsilon)w(s) \in V \cap B$ for suitably small $\epsilon > 0$. Then

$$V \cap \overset{\circ}{B} \text{ a.e. } s \in [0, S].$$

But, you cannot approximate feasible extended sense processes by
feasible embedded strict sense processes!

Significance of Extended Optimal Control Problem

- In typical applications to mechanical control and mid-course guidance, extended sense processes are idealizations, with controls that cannot be implemented.
- The optimal control problem posed over strict sense feasible processes may fail to have a solution.

So: seek process with cost 'close' to the infimum cost ('sub-optimal control').

We can find sub-optimal controls via the extended problem:

Step 1: Obtain a solution $(\bar{S}, \bar{y}^0, \bar{y}, \bar{w})$ to the extended problem, by analytic or computational means.

Step 2: Approximate $(\bar{S}, \bar{y}^0, \bar{y}, \bar{w})$ by a strict sense process.

BUT

This procedure works only if we can approximate feasible extended sense processes by feasible embedded strict sense processes.

Infimum Gap

Write

$\inf(P) :=$ infimum cost for (P) and $\inf(P_e) :=$ infimum cost for (P_e)

We have

$$\inf(P_e) \quad \leq \quad \inf(P)$$

(because (P_e) has a larger domain).

Question:

When does $\inf(P_e) = \inf(P)$? (no infimum gap)

If yes, we can construct sub-optimal strict sense controls . .

We seek conditions for 'no infimum gap'

Link with Multiplier Abnormality

There is a relation between infimum gaps and abnormality of Lagrange multipliers. (Identified by Warga, 1971)

Consider the finite dimensional optimization problem:

$$(P) \begin{cases} \text{Minimize } g(x) \\ \text{s.t. } \; h(x) < 0 \;\text{ and }\; x \in C \end{cases}$$

and its extension :

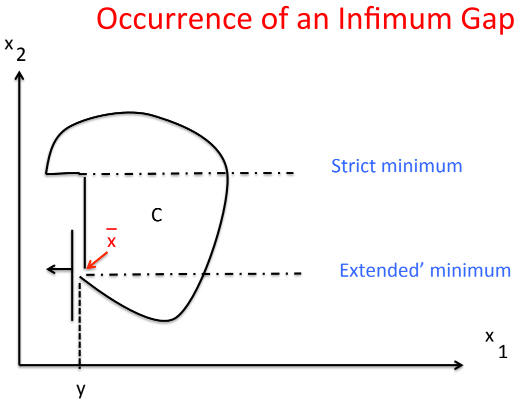
$$(P_e) \begin{cases} \text{Minimize } g(x) \\ \text{s.t. } \; h(x) \leq 0 \;\text{ and }\; x \in C \end{cases}$$

(i.e. replace $\{x|h(x) < 0\}$ by closed set $\{x|h(x) \leq 0\}$)

(Data: C^1 functions $h(.) : \mathbb{R}^n \to \mathbb{R}$, $g(.) : \mathbb{R}^n \to \mathbb{R}$,

closed set $C \subset \mathbb{R}^n$)

The Infimum Gap



Min (x_2 | x_1 < y , x ε C)

Finite Dimensional Case

Take $g(.) : \mathbb{R}^n \to \mathbb{R}$, $h(.) : \mathbb{R}^n \to \mathbb{R}$ and $C \subset \mathbb{R}^n$ (closed)

Fact: Suppose there exists $\bar{x} \in \mathbb{R}^n$ such that, for some $\gamma > 0$,

$h(\bar{x}) \leq 0$ and $\bar{x} \in C$

$g(\bar{x}) \; = \; \inf\{g(x) \,|\, h(x) \leq 0 \text{ and } x \in C\} < \; \inf\{g(x) \,|\, h(x) < 0 \text{ and } x \in C\} - \gamma \,.$

(inf cost is reduced, when we 'extend' domain $\{x|h(x) < 0\}$ to $\{x|h(x) \leq 0\}$)

Then

$$0 \;\in\; 0 \,.\, \nabla g(\bar{x}) + \nabla h(\bar{x}) + N_{\overline{C}}(\bar{x})$$

i.e. infimum gap implies abnormal multiplier rule

(Corresponds to 'horizontal' normals to hyperplanes of support . .)

Proof

Clearly $h(\bar{x}) = 0$. (Otherwise there can be no infimum gap)

Take $\epsilon_i \to 0$ and consider:

(P_i) Minimize $\left\{ J_i(x) \;:=\; \left(h(x) + \epsilon_i^2\right) \vee d_C(x) \,|\, x \in \mathbb{R}^n \right\}$.

For each i, $\bar{x}$ is an ϵ minimizer. So, by Ekeland's Thm., $\exists\, x_i$ such that:

1) $\bar{x}$ minimizes $\to J_i(x) + \epsilon_i |x - x_i|$, and

2) $|x - x_i| \;\leq\; \epsilon_i$

Then, $g(x_i) \leq g(\bar{x}) + \gamma$, for i sufficiently large. So

$$\left(h(x_i) + \epsilon_i^2\right) \vee d_C(x_i) > 0, \text{ for } i \text{ sufficiently large.}$$

'Stationarity at x_i and Max Rule of subdiff. calculus gives: $\exists \lambda_i \in [0, 1]$ s.t.

$$0 \;\in\; \lambda_i \nabla h(x_i) + (1 - \lambda_i)\partial d_C(x_i) \cap \{\xi \,|\, |\xi| = 1\} + \epsilon_i \bar{\mathbb{B}}$$

For subsequence $\lambda_i \to \lambda \in [0, 1]$.

Pass to limit: $0 \;\in\; \lambda_i \nabla h(\bar{x}_i) + (1 - \lambda)\partial d_C(\bar{x}) \cap \{\xi \,|\, |\xi| = 1\}$.

Implies $\lambda > 0$. Divide by λ, use $\partial d_C(\bar{x}) \subset N_C(\bar{x}) \implies 0 \in \nabla h(\bar{x}) + N_C(\bar{x})$.

Necessary Conditions for the Extended Problem

Maximum Principle for Extended Sense Miimizers

Take $(\bar{S}, \bar{y}^0(\cdot), \bar{y}(\cdot), \bar{\varphi}^0(\cdot), \bar{\varphi}(\cdot))$ minimizer for (P_e) .

Then there exist $(p_0(\cdot), p(\cdot)) \in W^{1,1}$ and real numbers , π, λ, with $\pi \leq 0$ and $\lambda \geq 0$, such that

$$(p(\cdot), \lambda) \neq (0, 0) ,$$

$$\frac{dp}{ds}(s) = -p(s) \cdot \left(\frac{\partial f}{\partial x}(\bar{y}(s))(1 - \bar{w}(s)) + \sum_{j=1}^{m} \frac{\partial g_j}{\partial x}(\bar{y}(s))\bar{w}^j(s) \right) \quad \text{a.e. } s \in [0, \bar{S}] ,$$

$$p(s) \cdot \left(f(\bar{y}(s))(1 - \bar{w}(s)) + \sum_{j=1}^{m} g_j(\bar{y}(s))\bar{w}^j(s) \right) + p_0(1 - |\bar{w}(s)|) + \pi\, |\bar{w}(s)| =$$

$$\max_{w \in V \cap B} \left\{ p(s) \cdot \left(f(\bar{y}(s)))(1 - |w|) + \sum_{j=1}^{m} g_j(\bar{y}(s))w^j \right) + p_0\, (1 - |w|) + \pi |w| \right\} = 0$$

$$-p(\bar{S}) \in \lambda \nabla h(\bar{y}(\bar{S})) + N_C(y(\bar{S})) . \qquad (\pi = 0 \text{ if } Var(\bar{\varphi}) < K)$$

Definition. A feasible extended sense process $(\bar{S}, y^0(.), y(.))$ is a normal extremal if for all possible choices of multipliers $(p^0, p(.), \lambda, \pi)$ in Max. Principle

$$\lambda \neq 0 .$$

Conditions for 'No Infimum Gap'

Theorem (Motta Rampazzo Vinter, 2017)

Assume

- There exists an extended sense minimizer that is a normal extremal

Then

$$\inf(P) = \inf(P_e) \quad \text{(No Infimum Gap!)}$$

All extended sense minimizers are extremals. But if at least one of them which is normal, then an infimum gap cannot occur.

Proof.

Similar to proof in finite dim. optimization (above)

Related to earlier work: 'When does relaxation reduce the minimum cost?' in classical control. (Warga, Palladino + Vnter)

Other Conditions and Special Cases

Proposition (Motta Rampazzo) Assume that, for any $x \in C$ and non-zero n-vector $\zeta \in N_C(x)$,

(i) We can find $w \in V$ such that
$\zeta \cdot \sum_{j=1}^{m} g_j(x)\, w^j \;<\; 0.$ (Fast 1-controllability condition)
or
$\zeta \cdot\; f(x) \;<\; 0.$ (Slow drift-controllability condition)

(ii): For some minimizer $(\bar{S}, \bar{y}^0, \bar{y}, \bar{w})$, $\int_{[0,\bar{S}]} |w(s)| ds < K$.

Then there is no infimum gap. $\square$

'Normality-Type' conditions covers these cases, because (i) and (ii) are suff. conditions for normality.

But note:

Proposition (MRV) Assume that

- $f(.) \equiv 0$ (no drift)

Then there is no infimum gap.

Also: 'normality-type condition' is not sufficient for 'no infimum gap'.

Examples

Examples are available distinguishing conditions

- Fast 1-controllability excludes infimum gap
- Normality excludes infimum gap, but 1-controllability does not.
- There is no infimum gap, but normality condition is not satisfied.

Open questions

- Identify special cases when ‘normality-type’ condition is directly verifiable
- Broaden study of problem (P_e), to allow larger classes of extended sense processes
- Develop normality-type sufficient conditions, that exclude infimum gaps for other kinds of impulse control problems (state constraints, non-smooth data, etc.).

Happy Birthdays

Franco and Giovanni

Thank you organizers,

for a wonderful workshop!