# Control with Piecewise Constant Dynamics

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# Outline

1. The big picture: Control systems with discontinuous dynamics
   - Some previous studies
   - Reachable sets
   - Structured discontinuous systems (stratified domains and dynamics)

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1. The big picture: Control systems with discontinuous dynamics
   - Some previous studies
   - Reachable sets
   - Structured discontinuous systems (stratified domains and dynamics)
2. A more modest problem
   - Convex polytopes with constant dynamics
   - Minimal time problems with gauge functions
   - Johann Bernoulli's approach to solving the brachistochrone
   - The boundary of the reachable set

# I. Control systems with discontinuous dynamics

- **1696:** Johann Bernoulli's solution to brachistochrone
- **The 1960's and 70's:** Filippov; Hermes; Krasovskii, Hájek
- **Sweeping Process:** Moreau; Marques and Kunze, Colombo and Goncharov, Georgiev and Ribarska
- **Multiprocesses (1988):** Clarke and Vinter
- **Our book (1998):** Clarke, Stern, Ledyaev, and PW
- **One-sided Lipschitz:** Tz. Donchev, Farkhi, Rios and PW
- **Stratified Systems:** Bressan and Hong, Barnard and PW; Barles, Briani, and Trelat; Rao and Zidani; Hermosilla, Zidani, and PW.

# Assumptions

on given $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with Hamiltonian $H : \mathbb{R}^n \to \mathbb{R}$ given by $H(x,\zeta) := \sup_{v \in F(x)} \langle v, \zeta \rangle$.

**Always** (**S**tanding **H**ypotheses):

**(SH)**
$$\begin{cases} F(x) \text{ is nonempty, convex, and compact } \forall x \in \mathbb{R}^n; \\ \operatorname{gr} F := \{(x,v) : v \in F(x)\} \text{ is closed } (\Leftrightarrow x \mapsto H(x,\zeta) \text{ is usc}); \\ \exists c > 0 \text{ with } |H(x,\zeta)| \le c\,\|\zeta\|\,(1+|x|)\ \forall x, \zeta \in \mathbb{R}^n. \end{cases}$$

**Once** (**O**ne-**S**ided **L**ipschitz): There exists $c > 0$ with

**(OSL)**
$$H(x, x-y) - H(y, x-y) \le c\,\|x-y\|^2 \quad \forall x, y \in \mathbb{R}^n.$$

**Usually** (**L**ocally **L**ipschitz): For all compact $K \subseteq \mathbb{R}^n$, there exists $c > 0$

**(LL)**
$$|H(x,\zeta) - H(y,\zeta)| \le c\,\|\zeta\|\,\|x-y\|\ \forall x, y \in K, \zeta \in \mathbb{R}^n$$

# Reachable sets

Given a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, $T > 0$ and $x \in \mathbb{R}^n$, consider the Differential inclusion

$$(\mathbf{DI}) \qquad \begin{cases} x(t) \in F\big(x(t)\big) \text{ a.e. } t \in [0, T] \\ x(0) = x \end{cases}$$

The Reachable Set at time $T$ from a closed set $S$ is defined as

$$R^{(T)}(S) := \big\{x(T) : \ \exists x(\cdot) \text{ satisfying } (\mathbf{DI}) \text{ where } x \in S\big\}.$$

**Major issue is to characterize $R^{(T)}(S)$**

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**Major issue is to characterize $R^{(T)}(S)$**

**We'll characterize when a boundary point of $R^{(T)}(S)$ propagates**

# Characterizations of $R^{(T)}(x)$

1. $\boxed{\textbf{Constant systems: } F(x) = F\textbf{:}}$

$$R^{(T)}(S) = S + TF := \left\{x + Tv : x \in S,\ v \in F\right\}.$$

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$$R^{(T)}(S) = e^{AT}S + \int_0^T e^{(T-s)A}BU\,ds.$$

3. **Exponential Formula (PW '90):** Under **(LL)**,

$$R^{(T)}(S) = \lim_{k\to\infty}\left(I + \frac{T}{k}F\right)^k(x).$$

The power refers to multifunction composition:

$$(G_1 \circ G_2)(S) := \{z : \exists x \in S, \exists y \in G_2(x) \text{ with } z \in G_1(y)\}.$$

Then $G^2 := G \circ G$, etc.

4. $\boxed{\textbf{Semigroup characterization (PW '90):}}$ Under **(LL)**,

$T \mapsto R^{(T)}(\cdot)$ is the unique one-parameter multifunction semigroup (under set composition) whose infinitesimal generator is $F(\cdot)$. That is, $R^{(\cdot)}(\cdot)$ is the unique multifunction satisfying ($\forall$compact $S \subseteq \mathbb{R}^n$)

$$R^{(0)}(S) = S$$

$\forall T_1,\ T_2 > 0$, one has $R^{(T_1+T_2)}(S) = R^{(T_1)}\big(R^{(T_2)}(S)\big)$

$d_{\mathcal{H}}\left(\dfrac{R^{(h)}(x) - x}{h}, F(x)\right) \to 0$ uniformly over $x \in S$ .

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6. **(HJ) inequalities (Clarke '00):** For $S \subseteq \mathbb{R}^n$ closed, denote the graph of $T \rightrightarrows R^T(S)$ by $\mathcal{R} = \left\{(T, y) : T \geq 0,\ y \in R^{(T)}(x), x \in S\right\}$. Then under **(LL)**, the closed set $\mathcal{R}$ is characterized by the HJ property

$$\sigma + H(y, \zeta) = 0 \quad \forall (T, y) \in \mathcal{R}, (\sigma, \zeta) \in N^P_{\mathcal{R}}(T, y).$$

7. **Levels sets to minimal time function (PW & Y. Zhuang '98):**

For $S \subseteq \mathbb{R}^n$ closed and $x \in \mathbb{R}^n$, the least time $T$ for which $\exists$ a trajectory $x(\cdot)$ of (DI) satisfying $x(T) \in S$ is denoted by $T(x)$.
Then for $r > 0$, the sub-level set is the reachable set in reversed time:

$$\mathrm{lev}_{\leq r}\big(T(\cdot)\big) := \{x : \ T(x) \leq r\} = R_{-F}^{(\leq r)}(S).$$

Under **(LL)**, we showed $T(\cdot)$ is the unique lsc solution to the proximal HJ equation plus a boundary condition:

$$\begin{aligned} 1 - H(x, -\zeta) &= 0 \quad \forall x \notin S, \zeta \in \partial_P T(x); \\ 1 - H(x, -\zeta) &\geq 0 \quad \forall x \in S, \zeta \in \partial_P T(x). \end{aligned}$$

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8. **Approximate HJ equation (Donchev, Rios, & PW '12)**

Under **(OSL)**, $T(\cdot)$ is the unique lsc function satisfying

$$\left\{ \begin{array}{l} 1 - H(x, -\zeta) \leq 0 \\ 1 - \limsup\limits_{y \to_\zeta x} H(y, -\zeta) \geq 0 \end{array} \right\} \quad \forall x \notin S, \zeta \in \partial_P T(x);$$

$$1 - \limsup_{y \to_\zeta x} H(y, -\zeta) \geq 0 \quad \forall x \in S, \zeta \in \partial_P T(x).$$

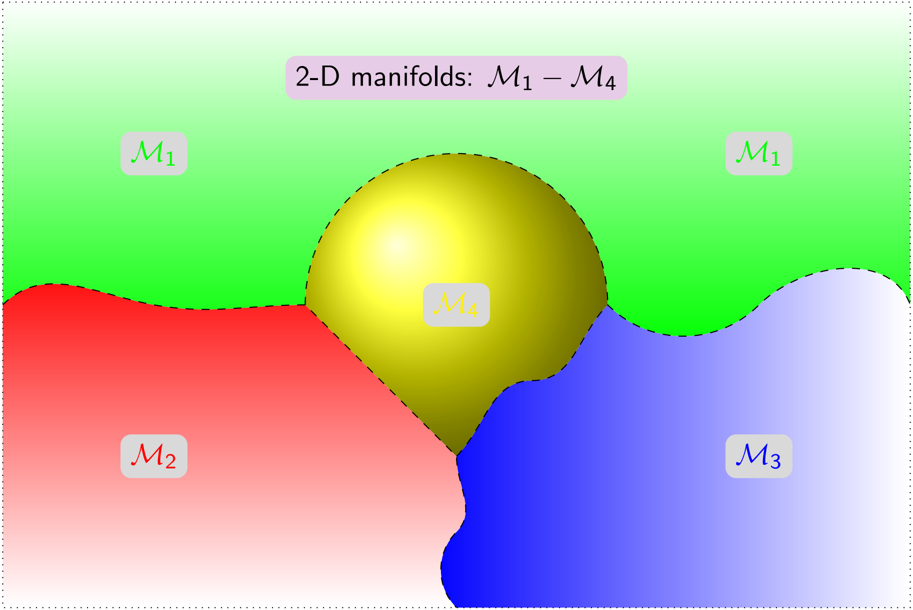
# General framework: Structured discontinuous systems

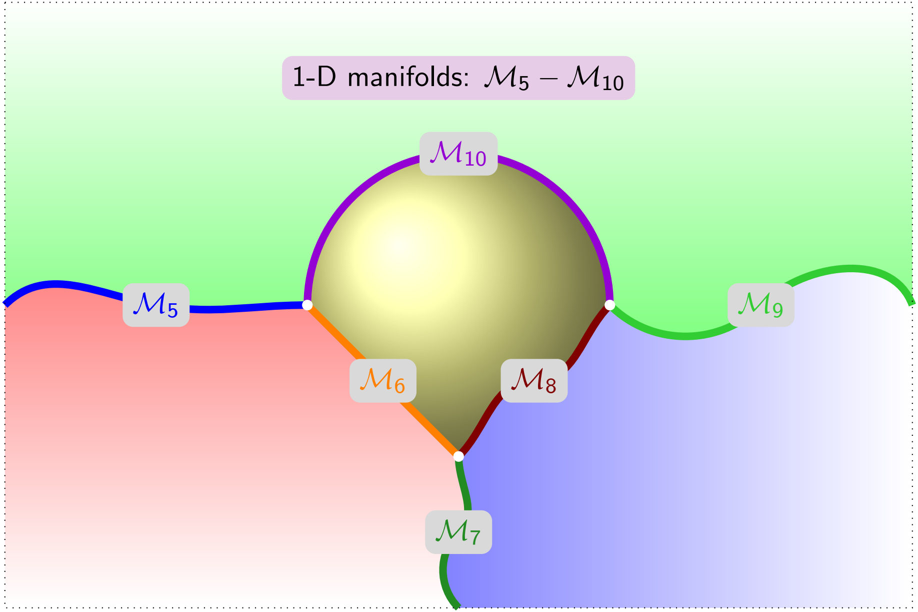
**Stratified domains:** The state space is partitioned into a finite collection $\{\mathcal{M}_1, \ldots, \mathcal{M}_M\}$ of smooth manifolds embedded in $\mathbb{R}^N$ such that

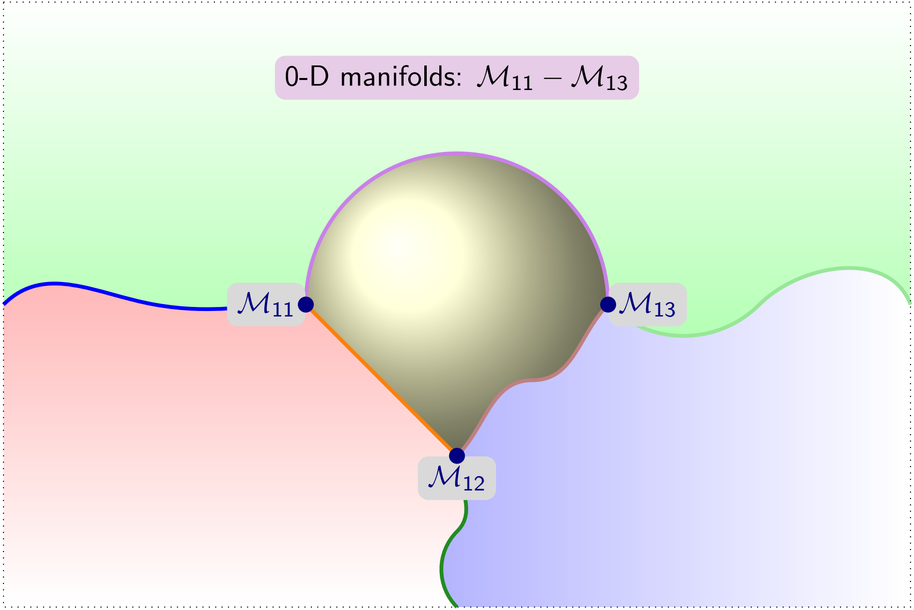
1. $\mathbb{R}^N = \bigcup_{i=1}^M \mathcal{M}_i$; $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ for all $i \neq j$.
2. If $\overline{\mathcal{M}}_i \cap \mathcal{M}_j \neq \emptyset$ , then $\mathcal{M}_j \subseteq \overline{\mathcal{M}}_i$.
3. Each $\overline{\mathcal{M}}_i$ is **proximally smooth** of radius $\delta > 0$;
4. Each $\overline{\mathcal{M}}_i$ is **relatively wedged**.

$\overline{\mathcal{M}}$ **Proximally smooth:** The distance function $d_{\overline{\mathcal{M}}}(x) := \inf_{y \in \overline{\mathcal{M}}} ||x - y||$ is differentiable on $\{\mathcal{M} + \delta\mathbb{B}\} \backslash \overline{\mathcal{M}}$. **One consequence:** The Clarke normal cone $\mathcal{N}_{\overline{\mathcal{M}}}(x)$ is the proximal one, and has closed graph.

$\overline{\mathcal{M}}$ **relatively wedged:** The dimension of the relative interior of the tangent cone $\mathcal{T}_{\overline{\mathcal{M}}}(x)$ is the dimension of $\mathcal{M}$ for all $x \in \overline{\mathcal{M}}$.


2-D manifolds: $\mathcal{M}_1 - \mathcal{M}_4$
$\mathcal{M}_1$
$\mathcal{M}_1$
$\mathcal{M}_4$
$\mathcal{M}_2$
$\mathcal{M}_3$


1-D manifolds: $\mathcal{M}_5 - \mathcal{M}_{10}$
$\mathcal{M}_{10}$
$\mathcal{M}_5$
$\mathcal{M}_9$
$\mathcal{M}_6$
$\mathcal{M}_8$
$\mathcal{M}_7$


0-D manifolds: $\mathcal{M}_{11} - \mathcal{M}_{13}$
$\mathcal{M}_{11}$
$\mathcal{M}_{13}$
$\mathcal{M}_{12}$

**Stratified dynamics:** Associated to each manifold $\mathcal{M}_i$ is a multifunction $F_i : \mathcal{M}_i \rightrightarrows \mathbb{R}^N$ for which $(\mathcal{M}_i, F_i)$ satisfies the Basic Assumptions **(BA)**:

$$\textbf{(BA)} \quad \begin{cases} \textbf{1)}\ \mathrm{gr}\ F(\cdot) := \{(x,v) : v \in F(x)\} \text{ is closed w.r.t. } \mathcal{M}, \\ \textbf{2)}\ \forall\, x \in \mathcal{M},\ F(x) \subseteq \mathcal{T}_{\mathcal{M}}(x) \text{ is nonempty, convex, and compact,} \\ \textbf{3)}\ \exists\ r > 0 \text{ so that } \max\{|v| : v \in F(x)\} \leq r(1+|x|), \textit{ and} \\ \textbf{4)}\ F(\cdot) \text{ is Lipschitz on bounded sets of } \mathcal{M}. \end{cases}$$

The **basic velocity** multifunction $F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ is defined by

$$F(x) = F_i(x) \quad \text{whenever} \quad x \in \mathcal{M}_i.$$

This multifunction induces a differential inclusion with no general existence theory or compactness of trajectories. The **Krasovskii regularization** $G : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$

$$G(x) = \bigcap_{\varepsilon > 0} \overline{\mathrm{co}} \bigcup \{F(y) : \|y - x\| < \varepsilon\} = \mathrm{co} \bigcup_{x \in \overline{\mathcal{M}_i}} \overline{F}_i(x).$$

rectifies that issue, but is generally discontinuous on $\mathbb{R}^N$.

**Proposed research program:**

**Extend and develop the elements of optimal control theory to structured discontinuous systems**

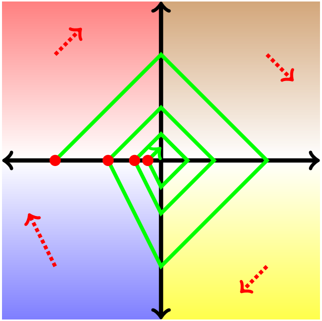
# Major difficulties:

1. Need a structural assumption to have well-posed optimization problems and to possibly describe how arcs can enter and cross the submanifolds.
2. No obvious way to characterize the reachable set multifunction.
3. If the submanifold path or index path is known, then the issue of deriving necessary conditions fits into the Clarke-Vinter multiprocess (1988) framework. But the optimal submanifold path is unknown and is part of the optimization; a bilevel-like problem emerges where the upper problem is discrete but whose solutions are of indeterminate length.

# A more modest problem

We simplify by assuming:

- Each $\mathcal{M}_i$ is a convex polytope: $\mathcal{M}_i = \{x \in \mathbb{R}^n : A_i x \leq b_i\}$
- The dynamics are piecewise constant: $F_i(x) = F_i$



**Pitfall:** The Zeno effect occurs when a trajectory makes infinitely many domain switches in finite time. For example, an infinite spiral can reach the origin in finite time. It is then difficult to capture the behavior at the origin with first order information.

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- The dynamics are piecewise constant: $F_i(x) = F_i$
- And even more special: Each $F_i$ belongs to $\mathcal{F}$, which is the collection

$$\{F \subseteq \mathbb{R}^n : \ F \text{ is compact, convex, bounded, and with } 0 \in \operatorname{int}(F)\}$$

- Default structural assumption: If $\mathcal{M}_j \subset \overline{\mathcal{M}}_i$, then $\operatorname{proj}_{\mathcal{M}_j}(F_i) \subseteq F_j$.

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The polar of $F \in \mathcal{F}$ is defined as

$$F^\circ = \{\zeta \in \mathbb{R}^n : \; \langle \zeta, v \rangle \leq 1 \quad \forall v \in F\},$$

and one has for any $F \subseteq \mathbb{R}^n$ that

$$\boxed{F \in \mathcal{F} \iff F^\circ \in \mathcal{F}}$$

# Gauge functions (aka Minkowski functionals)

Fix $F \in \mathcal{F}$. The gauge associated with $F$ is given by

$$\gamma_F(v) = \inf\{t : \; v \in t\,F\} \quad \left( = \sup\{\langle v, \zeta\rangle : \zeta \in F^\circ\} = (\mathcal{I}_{F^\circ})^*(v) \right)$$

Our assumptions imply $\gamma_F(\cdot)$ is finite and convex, and its relationship with $\gamma_{F^\circ}(\cdot)$ is summarized by: The following are equivalent for $v$, $\zeta \in \mathbb{R}^n$:

**(a)** $\langle v, \zeta\rangle = \gamma_F(v) \cdot \gamma_{F^\circ}(\zeta)$

**(b)** $\frac{\zeta}{\gamma_{F^\circ}(\zeta)} \in \partial\gamma_F(v)$

**(c)** $\zeta \in \mathcal{N}_F\left(\frac{v}{\gamma_F(v)}\right)$

**(d)** $\max\left\{\langle v', \zeta\rangle : \; v' \in F\right\} = \left\langle \frac{v}{\gamma_F(v)}, \zeta \right\rangle$

**(e)** $\frac{v}{\gamma_F(v)} \in \partial\gamma_{F^\circ}(\zeta)$

**(f)** $v \in \mathcal{N}_{F^\circ}\left(\frac{\zeta}{\gamma_{F^\circ}(\zeta)}\right)$

**(g)** $\max\left\{\langle v, \zeta'\rangle : \; \zeta' \in F^\circ\right\} = \left\langle v, \frac{\zeta}{\gamma_{F^\circ}(\zeta)} \right\rangle$

# The Minimal time problem

Suppose the target $S \subseteq \mathbb{R}^n$ is closed (and later, convex). Firstly, if there is just one dynamic velocity set $F \in \mathcal{F}$, the minimal time function reduces to

$$T(x) = \min_{y \in S} \left\{ \gamma_F \left( y - x \right) \right\}$$

G. Colombo & PW studied subgradient properties of $T(\cdot)$ in Hilbert space.

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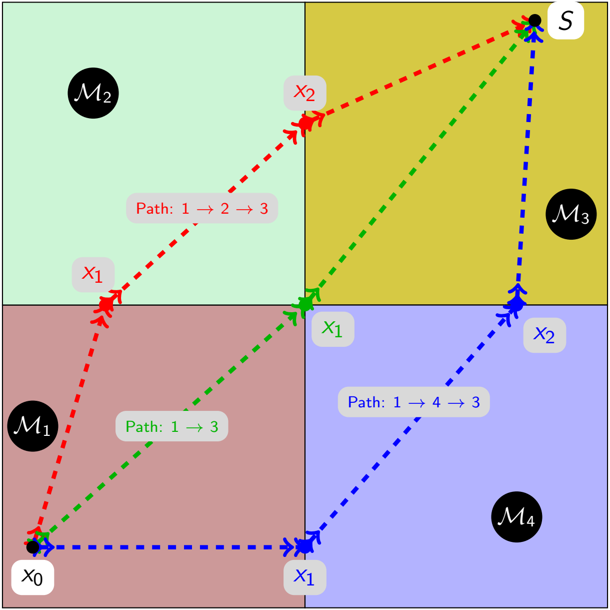
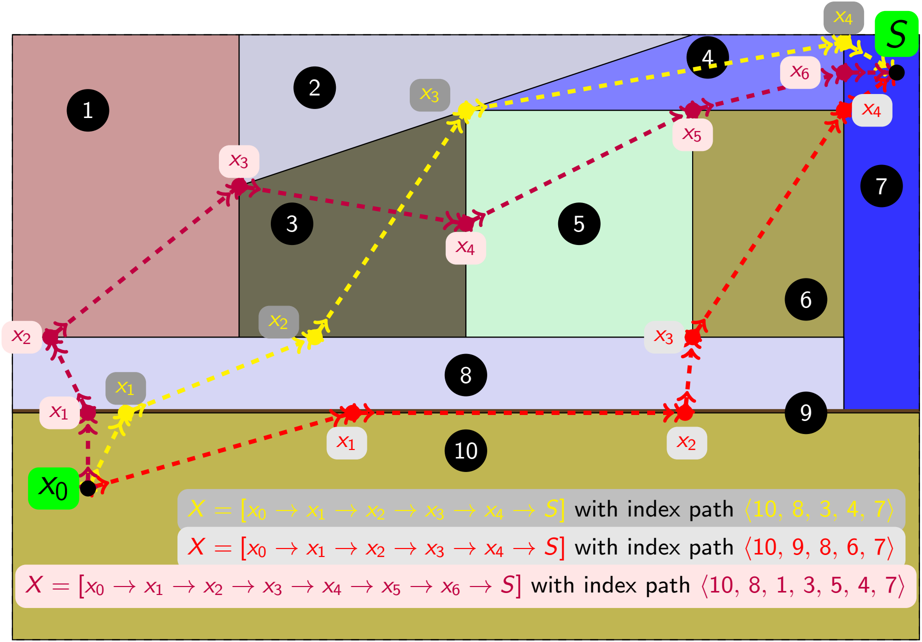
Consider the polytope case, and let $x_0 \in \mathbb{R}^n$ and $S \subseteq \mathbb{R}^n$ be closed and convex. A (velocity) index path is a (finite) collection of indices $\langle i_1, i_2, \ldots, i_k \rangle$ with the following property:

$$\exists x_1 \in \overline{\mathcal{M}}_{i_1} \text{ with } (x_0, x_1) \subseteq \mathcal{M}_{i_1}; \quad \exists x_2 \in \overline{\mathcal{M}}_{i_2} \text{ with } (x_1, x_2) \subset \mathcal{M}_{i_2};$$
$$\exists x_3 \in \overline{\mathcal{M}}_{i_3} \text{ with } (x_2, x_3) \subset \mathcal{M}_{i_3}; \quad \ldots \qquad \ldots$$
$$\exists x_k \in \overline{\mathcal{M}}_{i_k} \text{ with } (x_{k-1}, x_k) \subset \mathcal{M}_{i_k} \text{ and } x_k \in S.$$

The only feasible trajectories that need be considered are associated with a velocity index path, and is the linear interpolation of points $\{x_0, \ldots, x_k\}$). Denote this trajectory by $X = [x_0 \to x_1 \to \cdots \to x_k]$.



$S$
$\mathcal{M}_2$
$x_2$
Path: $1 \to 2 \to 3$
$\mathcal{M}_3$
$x_1$
$x_1$
$x_2$
Path: $1 \to 4 \to 3$
$\mathcal{M}_1$
Path: $1 \to 3$
$\mathcal{M}_4$
$x_0$
$x_1$




$X = [x_0 \to x_1 \to x_2 \to x_3 \to x_4 \to S]$ with index path $\langle 10, 8, 3, 4, 7 \rangle$

$X = [x_0 \to x_1 \to x_2 \to x_3 \to x_4 \to S]$ with index path $\langle 10, 9, 8, 6, 7 \rangle$

$X = [x_0 \to x_1 \to x_2 \to x_3 \to x_4 \to x_5 \to x_6 \to S]$ with index path $\langle 10, 8, 1, 3, 5, 4, 7 \rangle$

# Optimal time problem with a fixed velocity index path

Suppose a velocity index path $\langle i_1, \ldots, i_k \rangle$ is fixed and a feasible trajectory $X = [x_0 \to x_1 \to \cdots \to x_k]$ is given . The time traveled by $X$ is

$$J(X) = \sum_{j=1}^{k} \gamma_{F_{i_j}}\left(x_j - x_{j-1}\right).$$

Minimizing time over all feasible trajectories is the same as minimizing $J(\cdot)$ over the $x_j$'s $(1 \leq j \leq k)$ under the constraint

$$x_j \in \overline{\mathcal{M}}_{i_j} \cap \overline{\mathcal{M}}_{i_{j+1}} =: \Sigma_{i_j}$$

where $\mathcal{M}_{k+1} := S$ (which could be any convex set).

**This a convex, finite-dimensional optimization problem!**

We solve it using convex calculus.

# Necessary and sufficient conditions

With $\Upsilon_{\Sigma_{i_j}}(\cdot)$ as the indicator of $\Sigma_{i_j} = \overline{\mathcal{M}}_{i_j} \cap \overline{\mathcal{M}}_{i_{j+1}}$.The problem reduces to

$$\min_{\substack{x_j \in \mathbb{R}^n \\ 1 \le j \le k}} \quad \sum_{j=1}^{k} \left\{ \gamma_{F_{i_j}}\left(x_j - x_{j-1}\right) + \Upsilon_{\Sigma_{i_j}}(x_j) \right\}$$

Suppose for a fixed velocity index path that

$$X = [x_0 \to x_1 \to \cdots \to x_k]$$

is feasible. Then

$$0 \in \partial J(X)^{\mathrm{Tr}} = \left( \begin{array}{c} \partial_{x_1}\left(\gamma_{F_{i_1}}(x_1 - x_0) + \gamma_{F_{i_2}}(x_2 - x_1) + \Upsilon_{\Sigma_{i_1}}(x_1)\right) \\ \partial_{x_2}\left(\gamma_{F_{i_2}}(x_2 - x_1) + \gamma_{F_{i_3}}(x_3 - x_2) + \Upsilon_{\Sigma_{i_2}}(x_2)\right) \\ \vdots \\ \partial_{x_k}\left(\gamma_{F_{i_k}}(x_k - x_{k-1}) + \Upsilon_S(x_k)\right) \end{array} \right)$$

is both necessary and sufficient for $X$ to be optimal.

Since each gauge function is finite and convex, Rockafellar's sum rule applies and yields: for each $j$ $(1 \leq j \leq k)$, one has

$$\exists\, \zeta_j \in \partial_{x_j} \gamma_{F_{i_j}}(x_j - x_{j-1}) \text{ and } \exists\, \xi_j \in -\partial_{x_j} \gamma_{F_{i_{j+1}}}(x_{j+1} - x_j)$$

with

$$-\zeta_j + \xi_j \in N_{\Sigma_{i_j}}(x_j) \quad \text{(generalized Snell's Law)}$$

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By the properties of gauge functions,

$$\gamma_{F^\circ_{i_j}}(\zeta_j) = 1 = \gamma_{F^\circ_{i_{j+1}}}(-\xi_j)\,,$$

$$\max_{v \in F_{i_j}} \langle v, \zeta_j \rangle = \left\langle \frac{x_j - x_{j-1}}{\gamma_{F_j}(x_j - x_{j-1})}, \zeta_j \right\rangle, \text{ and}$$

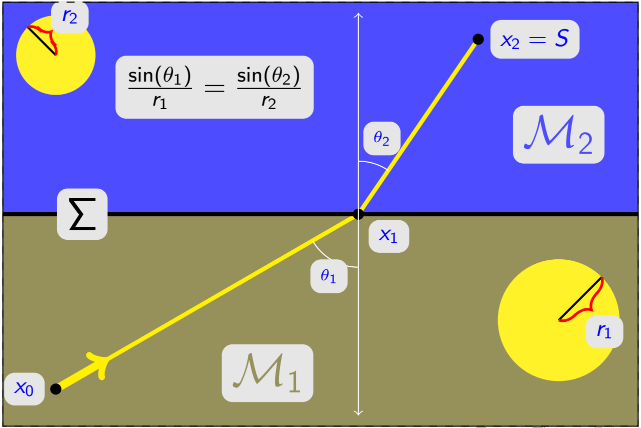
$$\max_{v \in F_{i_{j+1}}} \langle v, -\xi_j \rangle = \left\langle \frac{x_{j+1} - x_j}{\gamma_{F_{j+1}}(x_{j+1} - x_j)}, -\xi_j \right\rangle.$$

The last two statements constitute the Maximum Principle.

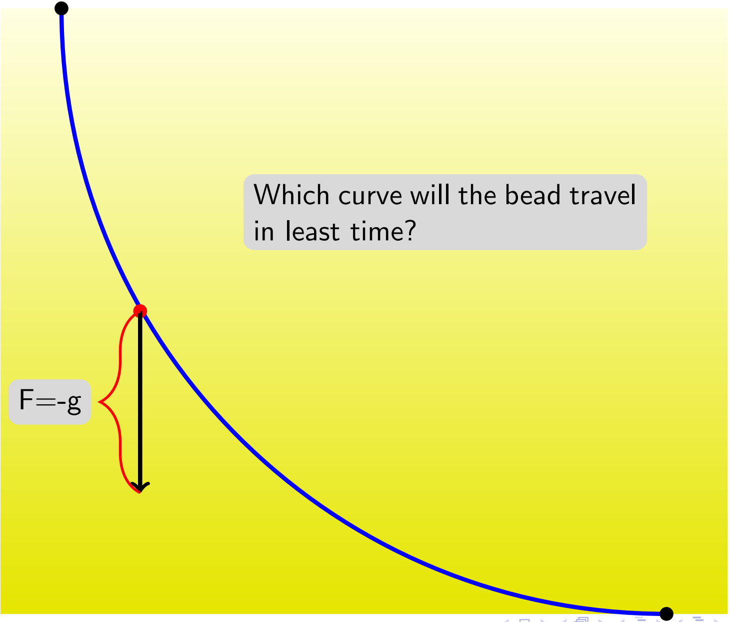
The conditions reduce to Snell's Law if $F_1 = r_1\overline{\mathbb{B}}$ and $F_2 = r_2\mathbb{B}$. In that case $F_1^\circ = \frac{1}{r_1}\overline{\mathbb{B}}$ and $F_2^\circ = \frac{1}{r_2}\overline{\mathbb{B}}$, and thus for some angles $\theta_1$, $\theta_2$, we have

$$\zeta = \frac{1}{r_1}\begin{pmatrix}\sin(\theta_1)\\ \cos(\theta_1)\end{pmatrix} \text{ and } -\xi = \frac{1}{r_2}\begin{pmatrix}\sin(\theta_2)\\ \cos(\theta_2)\end{pmatrix}$$

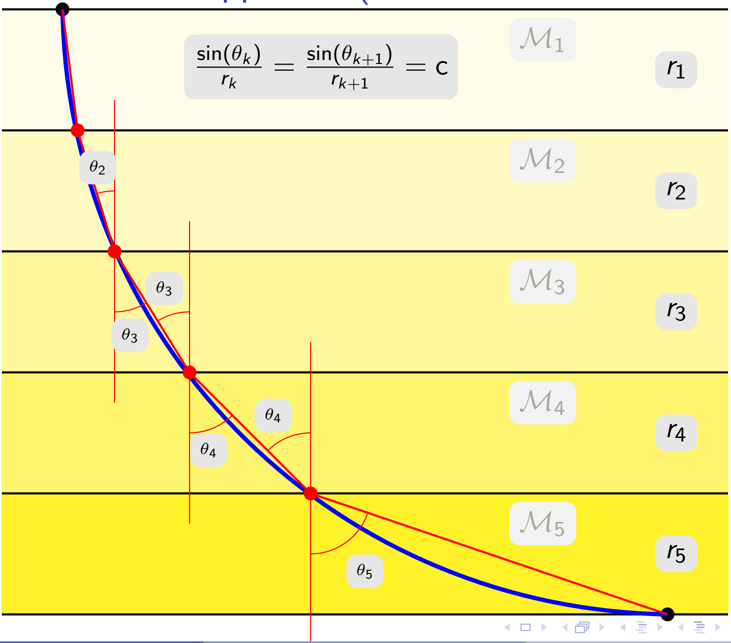
The condition $-\zeta + \xi \in N_\Sigma(x_1) = y-$axis is the classical Snell's Law.



# The Brachistochrone



# Johann Bernoulli's approach (local constant velocities)

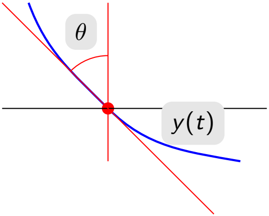


The condition

$$\boxed{\frac{\sin(\theta_1)}{r_1} = \frac{\sin(\theta_2)}{r_2} = \cdots = \frac{\sin(\theta_N)}{r_N} = \mathbf{c}}$$

"limits" to

$$r(y) \cdot c = \sin(\theta) = \frac{1}{\sqrt{1 + (y')^2}} \quad \Rightarrow \quad t = c \int \frac{r(y)}{\sqrt{1 - c^2\, r(y)^2}}\, dy$$

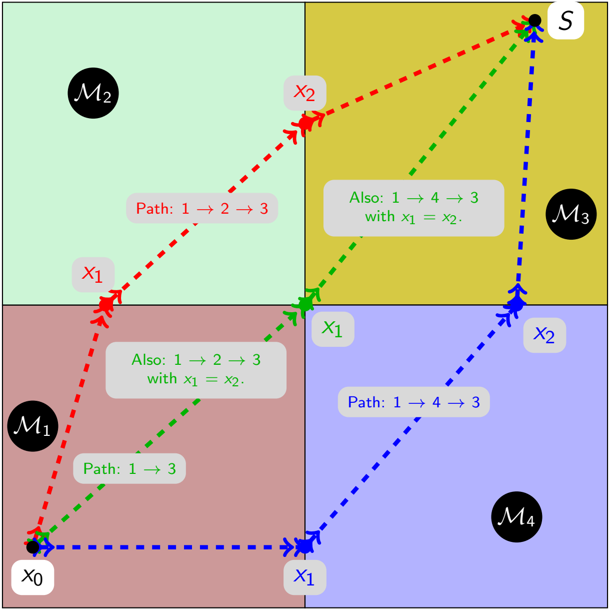



Brachistochrone has $r(y) = \sqrt{2\, g\, (y - y_0)}$.

# Observations

Again consider a polytope system where all manifolds $\mathcal{M}_i$ are convex polytopes with a constant velocity set $F_i \in \mathcal{F}$.

1. A point $x \in \mathbb{R}^n$ belongs to the boundary of $R^{(T)}(x_0)$ if and only if $T$ is the minimal time to reach $x$ from $x_0$.
2. The Zeno effect has no effect on $R^{(T)}(x_0)$ – this follows from the assumption $0 \in \operatorname{int} F_i \; \forall i$. Thus $x \in \mathbb{R}^n$ belongs to $R^{(T)}(x_0)$ if and only if there exists an index path with an associated trajectory $X = [x_0 \to x_1 \to \cdots \to x_k]$ with $J(X) \leq T$ and $x = x_k$.
3. If one knew all the possible index paths that produce boundary trajectories, then one can estimate the reachable set. For Bernoulli's discrete brachistochrone, the only such index path has successively higher indices. It gets *much* more complicated in general.
4. If a trajectory reaching a point $x \in \operatorname{bdry}\big(R^{(t)}(x_0) \cap \big(\operatorname{int}(\mathcal{M}_i)\big)$ is unique, then continuing along with the same velocity will keep it a boundary trajectory. If not, then ?? - analog to conjugate points??

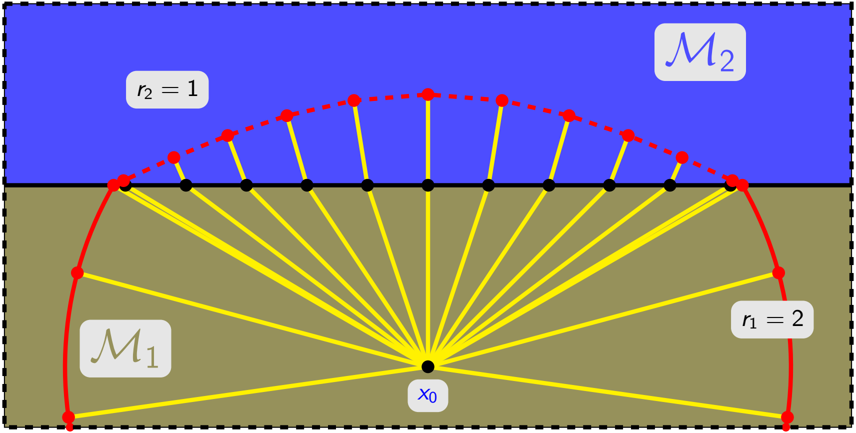


$S$
$\mathcal{M}_2$
$x_2$
Path: $1 \to 2 \to 3$
Also: $1 \to 4 \to 3$
with $x_1 = x_2$.
$\mathcal{M}_3$
$x_1$
$x_1$
$x_2$
Also: $1 \to 2 \to 3$
with $x_1 = x_2$.
$\mathcal{M}_1$
Path: $1 \to 4 \to 3$
Path: $1 \to 3$
$\mathcal{M}_4$
$x_0$
$x_1$

# The reachable set with two polytopes

Consider $\mathcal{M}_1$ , $\mathcal{M}_2$ as the upper, lower half-planes in $\mathbb{R}^2$, $\Sigma$ the x-axis, and $F_1 = r_1\mathbb{B}$, $F_2 = r_2\mathbb{B}$. We assume $x_0 \in \mathcal{M}_1$.

**Case 1:** $r_1 > r_2$: Only relevant index paths are $\langle 1 \rangle$ and $\langle 1, 2 \rangle$

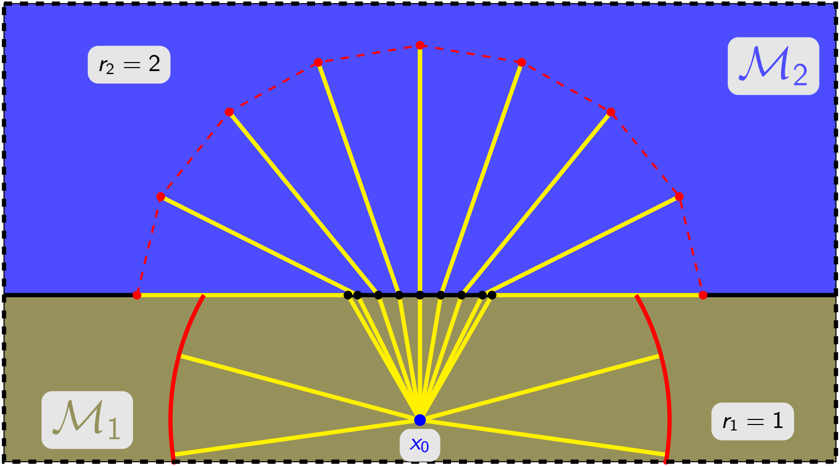



Examples of "optimal" trajectories with $T = 3$ in **Case 1**

**Case 2:** $r_1 < r_2$. Index paths are **(a)** $\langle 1 \rangle$, **(b)** $\langle 1, 2 \rangle$, and **(c)** $\langle 1, 2, 1 \rangle$.

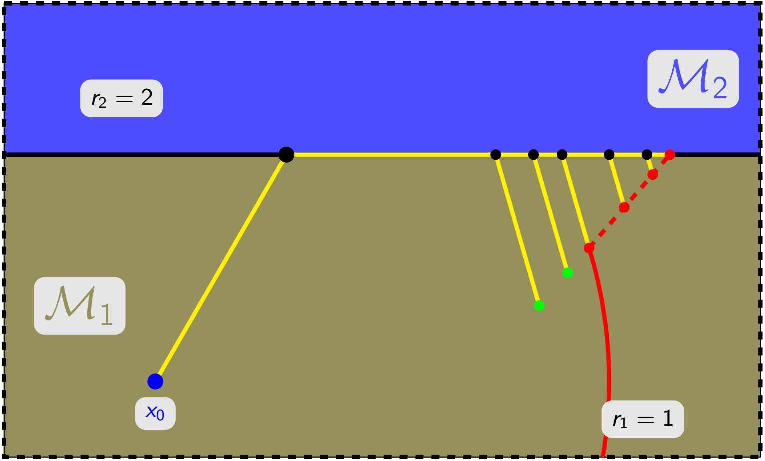
**Subcase 2(a):** Trajectory remains in $\mathcal{M}_1$ for all time.

**Subcase 2(b):** Trajectory hits $\overline{\mathcal{M}}_2$ and stays in $\overline{\mathcal{M}}_2$ for remaining time.




Examples of optimal trajectories of Subcases **2(a)** and **2(b)**

**Subcase 2(c):** The trajectory hits $\overline{\mathcal{M}}_2$ and stays on the interface but re-enters $\mathcal{M}_1$ in remaining time.




Examples of trajectories in **Subcase 2(c)**.

The green points lie in the interior of $R^{(6)}(x_0)$.

# Maintaining "boundariness" across an interface

Suppose $\bar{t} > 0$ and $x \in \operatorname{bdry}\big(R^{(\bar{t})}(x_0)\big) \cap \Sigma$ where $\Sigma \subseteq \operatorname{bdry}(\mathcal{M}_i)$.

**Question:**

How can a trajectory enter $\mathcal{M}_i$ and still maintain being on the boundary of $R^{(\bar{t}+h)}(x_0)$ for small $h > 0$?

**Answer:** $x$ comes from somehwere, so there is a polytope $\mathcal{M}_j$, a point $y \in \overline{\mathcal{M}}_j$, a time $t' > 0$, and a velocity $v \in F_j$ so that $x = y + t'v$.
In fact, $v = \frac{x-y}{\gamma_{F_j}(x-y)}$ and $t' = \gamma_{F_j}(x-y)$.
Let $\zeta \in N_{F_j}(v)$ with $\gamma_{F_j^\circ}(\zeta) = 1$. The answer is that $w \in F_i$ satisfies (for small $h > 0$)

$$x + hw \in \operatorname{bdry}\big(R^{(\bar{t}+h)}(x_0)\big) \bigcap \mathcal{M}_i$$

if and only if $\exists\, \xi \in N_{F_i}(w)$ with $\gamma_{F_i^\circ}(\xi) = 1$ satisfying

$$-\zeta + \xi \in N_\Sigma(x).$$

# Conclusions and future work

**1.** Characterizing the reachable set is challenging even when the dynamics are piecewise constant. It can be done easily if only a small number of index paths are relevant.

**2.** Once an index path is fixed, boundary trajectories can be constructed relative to that path using a generalized form of Snell's Law.

**3.** Choosing the right path for a specific choice of data seems quite challenging since it is a discrete problem.

**4.** Two distinct index paths may give the same least time for a given boundary point. I suspect this point could no longer be extended by a boundary trajectory. This seems analogous to conjugate points from the calculus of variations.

And finally,

And finally,

**Thank you**

# Franco and Giovanni

for the many years of friendship.