A regularity analysis for linear minimum time problems

Joint works with

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Consider the controlled dynamics

$$\dot{x} = f(x) + g(x)u, \qquad x(0) = \xi \tag{1}$$

$\xi \in \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$, U compact and convex.

We are given a closed target S.

Let $\mathcal{T}(\xi)$ be the minimum time to reach the target S from ξ, i.e.,

$$\mathcal{T}(\xi) = \text{inf}\{t : y(t) \in S,\ y \text{ is a solution of (1)}\}.$$

It is immediately seen that $\mathcal{T}$ is practically never everywhere differentiable. The point is understanding where and why $\mathcal{T}$ is not differentiable and study the set of singularities (i.e., the nondifferentiability set), as well as identifying suitable regularity properties satisfied by $\mathcal{T}$.

The regularity of the minimum time function is in fact a widely studied topic under several viewpoints.

- Variational analysis: computing generalized gradients of $\mathcal{T}$ (C.-Wolenski, Mordukhovich, . . .), mainly for a *constant dynamics*, but with a general target
- Complete description of $\mathcal{T}$ as well as of optimal synthesis, under generic conditions, in 2 D with the origin as target (Boscain-Piccoli)
- Semiconcavity/semiconvexity of $\mathcal{T}$ (Cannarsa and Sinestrari (Calc. Var. (1995) and book by Birkhäuser (2004)) (mainly with "fat" target), which gives information on the structure of the singular set as well as on higher order a.e. differentiability and on representation of the generalized gradient, together with a reasonable feedback concept. (Semiconcavity/-convexity $\sim$ quadratic perturbation of concavity/convexity $\Rightarrow$ structured singularities.)

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- nonsmoothness of the target
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In particular, the two last facts give raise to different types of singularities (downward or upward kinks/cusps).

We have made some steps in order to clarify some of the above aspects, taking as the starting point the viewpoint of Cannarsa and Sinestrari.

In this talk I will focus on linear and (for simplicity) single input dynamics, under normality assumptions. Namely we consider the problem of reaching the origin in minimum time from ξ subject to the dynamics

$$\dot{x} = Ax + bu, \qquad |u| \leq 1, \quad x \in \mathbb{R}^N,$$

such that the Kalman rank condition holds

$$\mathrm{rk}\left[b, Ab, \ldots, A^{N-1}b\right] = N$$

(A is a $N \times N$ matrix, $b \in \mathbb{R}^N$).

This is a very classical topic, which was studied a lot in the '70. We try to bring some new idea and new result, considering it as a model problem for possible future developments.

What is known “since ever” on this problem:

– small time controllability holds (i.e., $\mathcal{T}$ is finite and ($\frac{1}{N}$-Hölder)-continuous in a (“big”) neighborhood of the origin);
– sublevels of $\mathcal{T}$ ($\mathcal{R}_T = \{x : \mathcal{T}(x) \leq T\}$) are strictly convex;
– the optimal control steering ξ to the origin is unique and bang-bang;
– (Brunovský) there exists a regular time optimal synthesis ($\sim$ a reasonable feedback): general proof, constructive in R^2;
– (Hájek) there exists a time $\epsilon > 0$ such that $\mathcal{R}_\epsilon$ contains an open dense set where $\mathcal{T}$ is analytic and where a time optimal feedback $u(x)$ is well defined (in the sense that the corresponding [nonlinear, discontinuos] ODE $\dot{x} = Ax + bu(x)$ has a solution).

From the above results one can understand in which sense our linear minimum time problem is a model one: it is focused on singularities due to switchings, no other singularities can occur.

Switchings are connected with higher order controllability assumptions and singularities of $\mathcal{T}$ may be of two types: non-Lipschitz* points (i.e., the graph has at least a horizontal normal) and kinks (i.e., the graph has multiple normals).

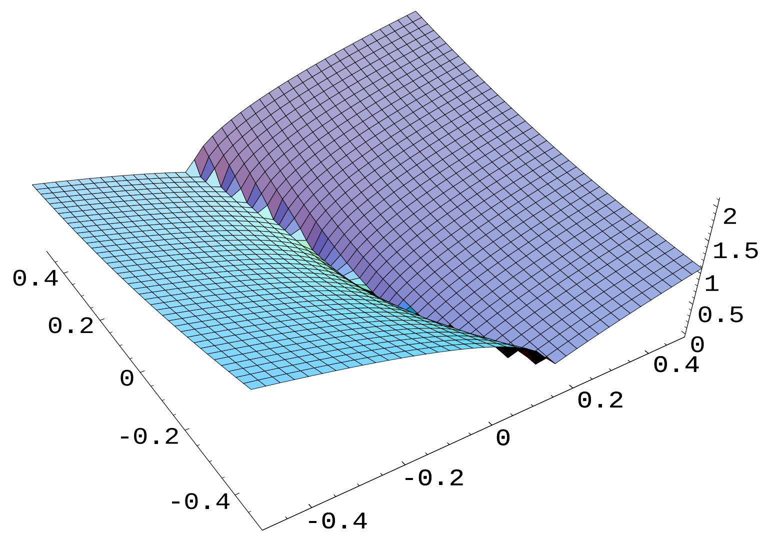
$^*f$ Lipschitz means that its difference quotients are uniformly bounded

Example 1 (rocket car). Consider the problem of reaching in minimum time the origin subject to the dynamics

$$\ddot{x} = u \in [-1, 1].$$

The minimum time function is:

$$\mathcal{T}(x,y) = \begin{cases} y + 2\sqrt{y^2/2 + x} & \text{for} \quad x \geq -y|y|/2, \\ -y + 2\sqrt{y^2/2 - x} & \text{for} \quad x < -y|y|/2 \end{cases}$$



Let $x \in \partial\mathcal{R}_T$ (i. e., $\mathcal{T}(x) = T$) and let ζ be normal to $\mathcal{R}_T$ at x. Pontryagin's Maximum Principle implies that the optimal control steering x to the origin is

$$u_x(t) = -\mathrm{sign}\Big(\langle \zeta, \mathrm{e}^{\mathrm{A}t}\mathrm{b}\rangle\Big).$$

In other words, the driving force of a minimum time (control affine) problem is the switching function

$$g_\zeta(t) = \langle \zeta, e^{At} b\rangle.$$

In our case, g is analytic and is not $\equiv 0$ (rank condition), whence every time optimal control is bang-bang and has finitely many switchings. Moreover, zeros of the switching function impose conditions on normals to reachable sets (linear conditions on normal vectors): actually the *normal cone* to reachable sets is characterized by such zeros. Actually Hájek's results are based on the fact that for small time all zeros of the switching function can be taken of first order and linearly independent.

The (minimized) Hamiltonian:

$$\begin{aligned} h(x,p) &= \min_{u\in\mathcal{U}}\Big(\langle f(x),p\rangle + \langle g(x)u,p\rangle\Big) \\ &= \langle Ax,p\rangle + \min_{|u|\le 1} u\langle b,p\rangle \\ &= \langle Ax,p\rangle - |\langle b,p\rangle|. \end{aligned}$$

Classical result (Bardi): the minimum time function is the unique bounded below (viscosity) solution of the boundary value problem

$$h(x,\nabla\mathcal{T}(x)) = -1,\ \mathcal{T}(0)=0,\ \lim_{x\to\partial\mathcal{R}}\mathcal{T}(x)=+\infty.$$

RESULTS

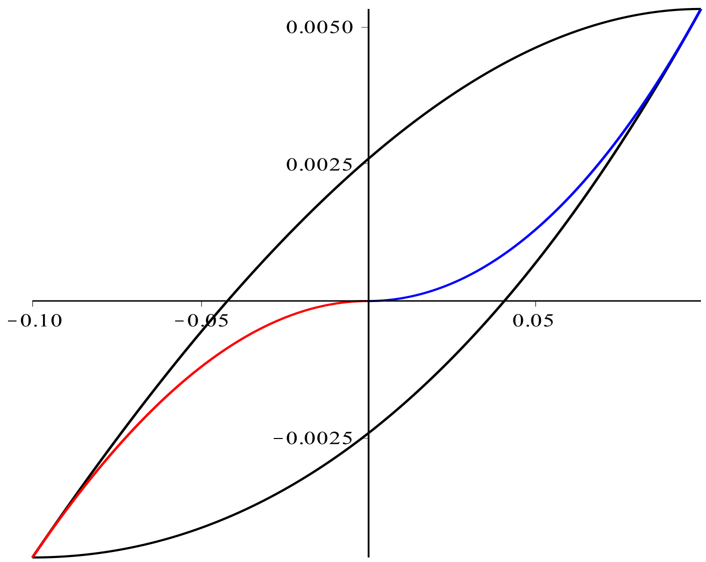
The Hamiltonian detects points around which $\mathcal{T}$ is not Lipschitz:

Theorem (Colombo, Nguyen T. Khai, Nguyen V. Luong). $\mathcal{T}$ is not Lipschitz around x if and only if there exists a nonzero normal ζ to $\mathcal{R}_{\mathcal{T}(x)}$ at x such that $h(x, \zeta) = 0$.
More precisely, ζ is normal to $\mathcal{R}_T$ at x if and only if $(\zeta, h(x, \zeta))$ is normal from below to the graph of $\mathcal{T}$ (= is normal to the epigraph of $\mathcal{T}$). In particular: corner singularities of $\mathcal{T}$ (kinks) are due only to kinks of reachable sets.
Finally, such normal cone is everywhere nontrivial (i.e., contains nonzero vectors) and $\mathcal{T}$ is a.e. (twice) differentiable.

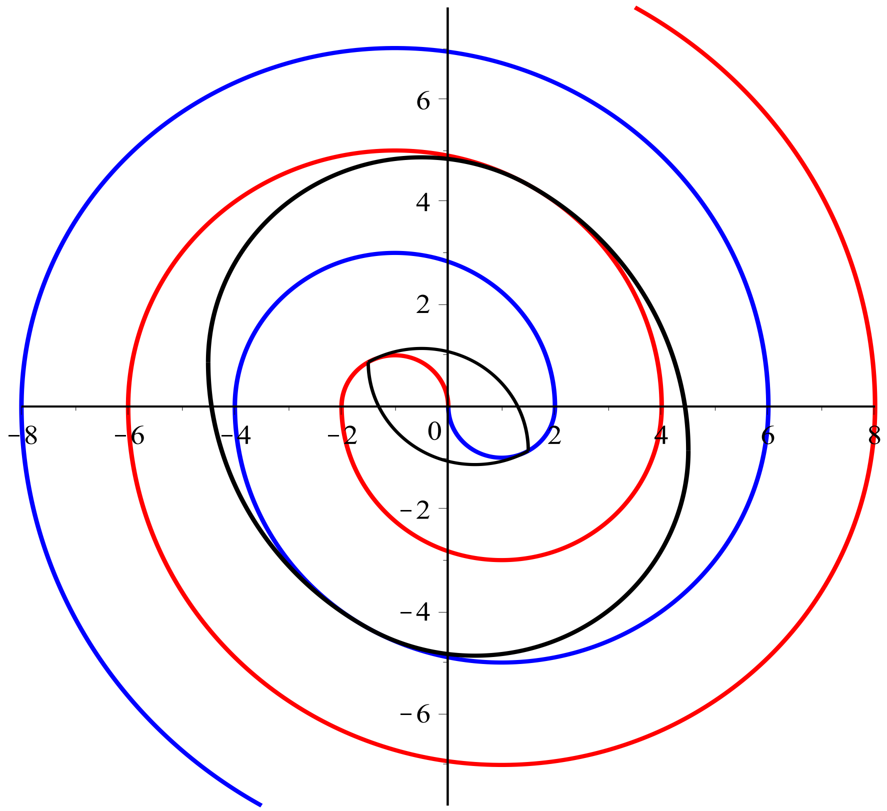
Since the Hamiltonian is constant along optimal trajectories, one can explicitly compute the non-Lipschitz set (that we call $\mathcal{S}$), through a connection with the location of zeros of the switching function: x is a non-Lipschitz point if and only if there exists a normal to $\mathcal{R}_T$ at x such that the corresponding switching function vanishes at $t = 0$.

Two examples.

The rocket car: $\ddot{x} = u, \ |u| \leq 1$.



The controlled harmonic oscillator: $\ddot{x} + x = u, \ |u| \leq 1$.



This is an interesting example: for $\mathcal{T} \geq \pi$ the graph of $\mathcal{T}$ is smooth, but $\mathcal{T}$ is nonsmooth (it has infinite partial derivatives).

In two dimensions there are essentially no other possibilities. In higher dimensions the situation is much more intricate, due to zeros of the switching function which may be of intermediate order (order k, $1 < k < N$) and linearly dependent ($e^{At_1}, \ldots, e^{At_j}$ linearly dependent). Apparently it is hopeless describing precisely what happens in connection with zeros of the switching function.

An intermediate result is showing that singularities occur “rarely”.

An example.

Set $\{q_n\} = \mathbb{Q} \cap [0, 1]$, and, for $x \in [0, 1]$,

$$g(x) = \sum_{q_n \le x} \frac{1}{2^n},$$
$$f(x) = \int_0^x g(t)\, dt.$$

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Can $\mathcal{T}$ be like f?

NO!

Theorem. (C., Nguyen V. Luong) Consider a normal linear time optimal problem in $\mathbb{R}^N$. Then there exists an open set Ω such that*

- $\mathcal{R} \setminus \Omega$ "has dimension $N-1$" (technically: it is countably $\mathcal{H}^{N-1}$-rectifiable);;
- $\mathcal{R} \setminus \Omega$ can be described precisely using the exponential matrix e^{At} and the switching function (technical);
- $\mathcal{T}$ is analytic in Ω;
- in Ω, $\nabla\mathcal{T}$ is a classical solution of the system of PDE's
$$A\,\nabla T(x) + \Big(Ax - \mathrm{sign}(\langle \nabla T(x), b\rangle)\,b\Big)\,\nabla^2 T(x) = 0.$$
Boundary conditions can also be specified.

*$\mathcal{R}$ denotes the set where $\mathcal{T}$ is finite

Technical remark: I'm not saying that the set of singularities is closed. I'm saying that it is contained in a (small) closed set.

One technical slide on proving $\mathcal{H}^{N-1}$-rectifiability for various exceptional sets.

A set E is $\mathcal{H}^k$-countably rectifiable if it is contained in the union of countably many images of Lipschitz functions of k real variables.

$$\mathcal{S} = \Big\{ x \in \mathbb{R}^N \ : \ \text{there exist}\, r > 0 \text{ and } \zeta \in \mathbb{S}^{N-1} \text{ such that} \\ x = \int_0^r e^{A(t-r)} b \, \text{sign}\left(\langle \zeta, e^{At} b\rangle\right) dt \\ \text{and } \langle \zeta, b\rangle = 0 \Big\}.$$

The set $\mathcal{S}$ is described using $N-1$ free parameters: $r \in \mathbb{R}$ and $\zeta \in \mathbb{S}^{N-1}$ subject to one linear condition, so $1 + N - 2$ parameters. The point is showing that one can split $\mathcal{S}$ into countably many Lipschitz graphs. Note that if $\bar{t}$ is a higher order zero of $g_{\bar{\zeta}}(\cdot)(= \langle \bar{\zeta}, e^{A\cdot} b\rangle)$ and ζ is close to $\bar{\zeta}$ then zeros of $g_{\zeta}(\cdot)$ around $\bar{t}$ are not a Lipschitz function of ζ.

Other sets of interest are parametrized by switching times. The point is proving that the set where the parametrization is not good (not a diffeomorphism) is $\mathcal{H}^{N-1}$-countably rectifiable. Of course, in general switchings are much more than $N-1$.