On the minimum time function for normal linear control systems

Giovanni Colombo

Abstract. The talk is based on joint papers with A. Marigonda, Nguyen T. Khai, Nguyen V. Luong and P. Wolenski.

Consider the linear control system $x' = Ax + bu$ in $\mathbb{R}^N$ with A and b satisfying the Kalman rank condition

$$\mathrm{rk}[b, Ab, \ldots, A^{N-1}b] = N,$$

where here for simplicity the control is assumed to be single-input, with $u \in [-1, 1]$. Results stating that

a) the minimum time function T to reach the origin is Hölder with exponent $1/N$ in the reachable set (which contains a neighborhood of 0);

b) the optimal control is unique, bang-bang with an upper bound on the number of switchings;

c) the reachable sets at every time are strictly convex

are classical from the early stages of control theory. Simple examples ($\ddot{x} = u$, $\ddot{x} + x = u$) show that T is **never** differentiable, and even Lipschitz, in any deleted neighborhood of the origin. Our work includes the following results:

1) a quantitative estimate on the modulus of strict convexity of reachable sets;

2) T is differentiable in an open set with full measure;

3) more precisely, T is differentiable out of a closed set which is a countable union of Lipschitz graphs of $N - 1$ variables;

4) T is twice differentiable out of a set with zero Lebesgue measure.

Methods of nonsmooth analysis and of geometric measure theory are used. Possible applications to robust numerical feedback control will also be presented (this is still at an early stage).