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DI INGEGNERIA
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UNIVERSITÀ
DEGLI STUDI
DI PADOVA

Non linear dynamics of Ring Lasers with amplitude dependent oscillations

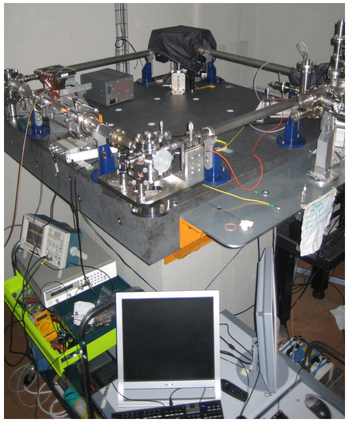
Davide Cuccato, DEI-INFN. September, 20th 2013. Control DEI

Ring Laser Gyroscopes

- *Large Size: (5-10 m) Geodesy, Astronomy, G.R. Tests.*

Sensibility of 10^{-9} rad/s @ 1 Hz

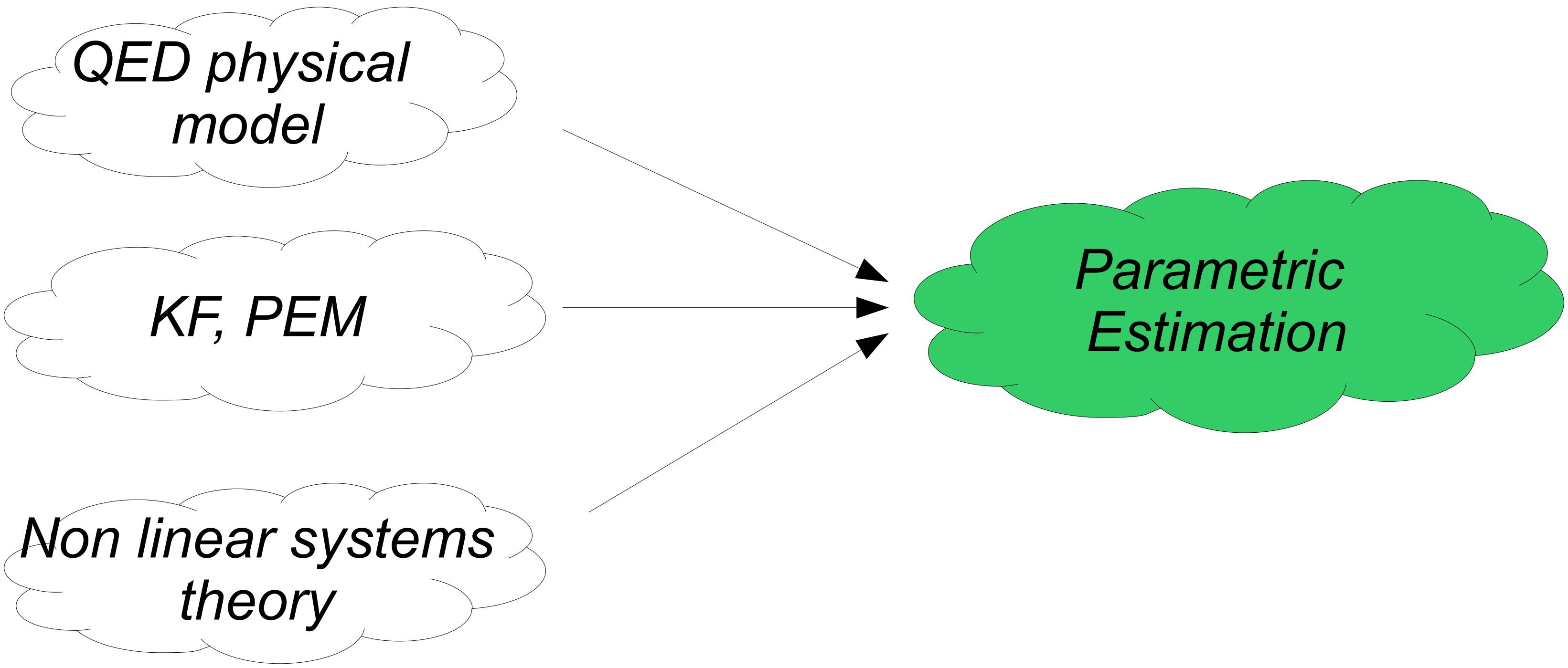
- *Medium Size: (1-5 m) Geophysics, Sismology, metrology.*



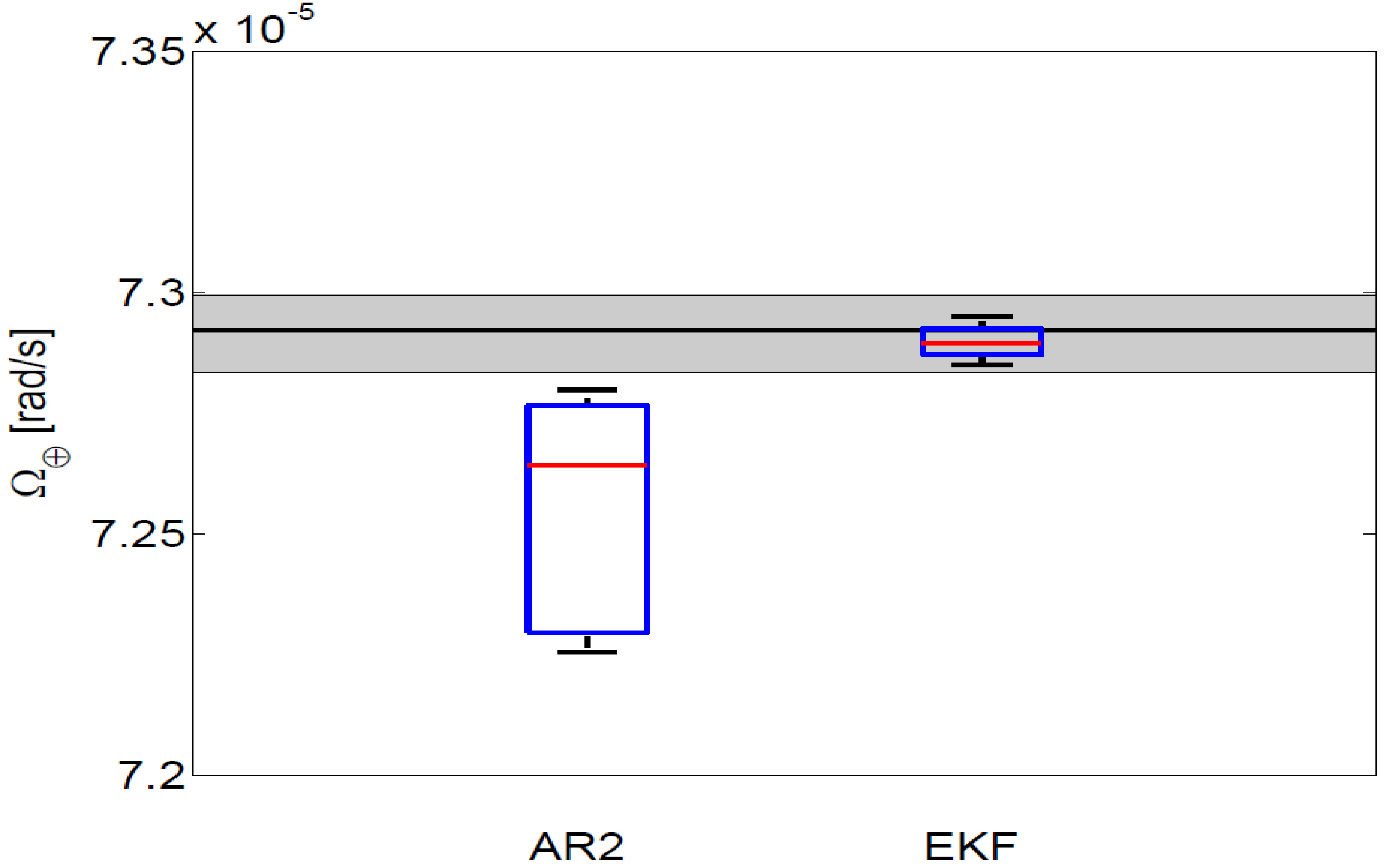
- *Small Size: (5-50 cm) Inertial Guidance.*

Earth rotation estimation

OUR GOAL: devise accurate estimation of the Earth rotation rate Exploiting Laser Physics



Earth rotation estimation



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The Landau-Stuart Oscillator

The simplest oscillator model incorporating amplitude and phase dynamic interaction

$$\dot{E} = \left[(\alpha + i\omega) - (\beta + i\gamma) |E|^2 \right] E$$

An isolated system near a Hopf bifurcation

Under the assumptions: $\alpha > 0, \quad \beta > 0$

The latter equation exhibits an attracting limit cycle actractor:

$$E = \sqrt{\frac{\alpha}{\beta}} e^{i\left(\omega - \frac{\gamma\alpha}{\beta}\right)t + \phi_0}$$

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RLG equations: non linear coupling of LSO

$$\dot{E}_1 = (\alpha_1 + i\omega_s) E_1 + r_2 e^{i\epsilon} E_2 - f_1(I_1, I_2) E_1$$
$$\dot{E}_2 = (\alpha_2 - i\omega_s) E_2 + r_1 e^{i\epsilon} E_1 - f_2(I_1, I_2) E_2$$

Where
$$\begin{cases} I_{1,2} = |E_{1,2}^2| \\ f_{1,2}(I_1, I_2) = \beta I_{1,2} + (\theta + i\tau) I_{2,1} \end{cases}$$

The oscillators are linearly coupled through $r_{1,2}$ ϵ

And non-linearly coupled through θ τ

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RLG equations: matrix form

$$\dot{\boldsymbol{E}} = \left(A - D_{\boldsymbol{E}^*} B \, D_{\boldsymbol{E}}\right) \boldsymbol{E} \,,$$

Where
$$\boldsymbol{E} = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \; D_{\boldsymbol{E}} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$A = \begin{pmatrix} \alpha_1 + i\,\omega_s & r_2 e^{i\epsilon} \\ r_1 e^{i\epsilon} & \alpha_2 - i\,\omega_s \end{pmatrix}, \; B = \begin{pmatrix} \beta & \theta + i\,\tau \\ \theta + i\,\tau & \beta \end{pmatrix}$$

Real coordinates for system study

$$
\begin{cases}
\dfrac{E_1^* E_2 + E_2^* E_1}{2} = x_1 \\
\dfrac{E_1^* E_2 - E_2^* E_1}{2i} = x_2 \\
\dfrac{E_1^* E_1 + E_2^* E_2}{2} = x_3 \\
\dfrac{E_1^* E_1 - E_2^* E_2}{2} = x_4
\end{cases}
$$

Oscillating Parts (x_1, x_2)

Absolute values (x_3, x_4)

Trajectiories stay on a $\mathbb{R}^4$ *cone:* $$x_1^2 + x_2^2 = x_3^2 - x_4^2$$

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Parameters positions

Accounting for asymmetry:

$$
\begin{cases}
\alpha = \alpha_1 + \alpha_2 \\
r = r_1 + r_2 \\
s = 2(\beta + \theta) \\
c = 2(\beta - \theta) \\
\delta_\alpha = \alpha_1 - \alpha_2 \\
\delta_r = r_1 - r_2
\end{cases}
$$

(α, r, s, c): *Usually* $\sim \omega$

(δ_α, δ_r): *Usually* $\ll \omega$

RLG equations: real form

$$\dot{\boldsymbol{x}} = \left(A - B\left(\boldsymbol{x} \right) \right) \boldsymbol{x}$$

$$A = \begin{pmatrix} \alpha & -\omega_s & r\cos\epsilon & \delta_r \sin\epsilon \\ \omega_s & \alpha & \delta_r \cos\epsilon & r\sin\epsilon \\ r\cos\epsilon & \delta_r \cos\epsilon & \alpha & \delta_\alpha \\ -\delta_r \sin\epsilon & -r\sin\epsilon & \delta_\alpha & \alpha \end{pmatrix} \qquad A = \begin{pmatrix} A_1 & R_2 \\ R_1 & A_2 \end{pmatrix}$$

$$B\left(\boldsymbol{x}\right) = \begin{pmatrix} s\,x_3 & 2\tau x_4 & 0 & 0 \\ -2\tau x_4 & s\,x_3 & 0 & 0 \\ 0 & 0 & s\,x_3 & c\,x_4 \\ 0 & 0 & c\,x_4 & s\,x_3 \end{pmatrix} \qquad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

Linear interaction matrix

Symmetric Case: $\delta_r = 0, \quad \delta_\alpha = 0$

Conservative Coupling: $\epsilon = \frac{\pi}{2} + k\pi$

$$A = \begin{pmatrix} \alpha & -\omega_s & 0 & 0 \\ \omega_s & \alpha & 0 & (-1)^k r \\ 0 & 0 & \alpha & 0 \\ 0 & (-1)^k r & 0 & \alpha \end{pmatrix}$$

$$\Lambda_A = \left\{ \alpha, \alpha, \pm i \sqrt{\omega^2 + r^2} \right\}$$

Dissipative Coupling: $\epsilon = 0 + k\pi$

$$A = \begin{pmatrix} \alpha & -\omega_s & (-1)^k r & 0 \\ \omega_s & \alpha & 0 & 0 \\ (-1)^k r & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$$

$$\Lambda_A = \left\{ \alpha, \alpha, \pm i \sqrt{\omega^2 - r^2} \right\}$$

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Linear interaction matrix

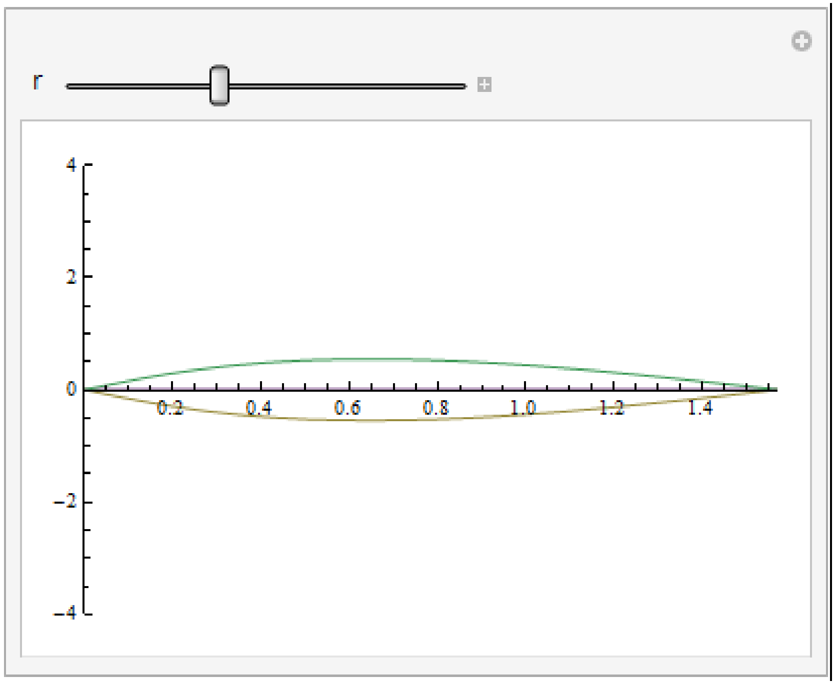
General Symmetric Case:

$$\Lambda(A)=\left\{\alpha \pm i\sqrt{\frac{\omega_s^2-r^2\cos 2\epsilon \pm \sqrt{\frac{r^4+\omega_s^4-2r^2\omega_s^2\cos 2\epsilon}{4}}}{2}}\right\}$$

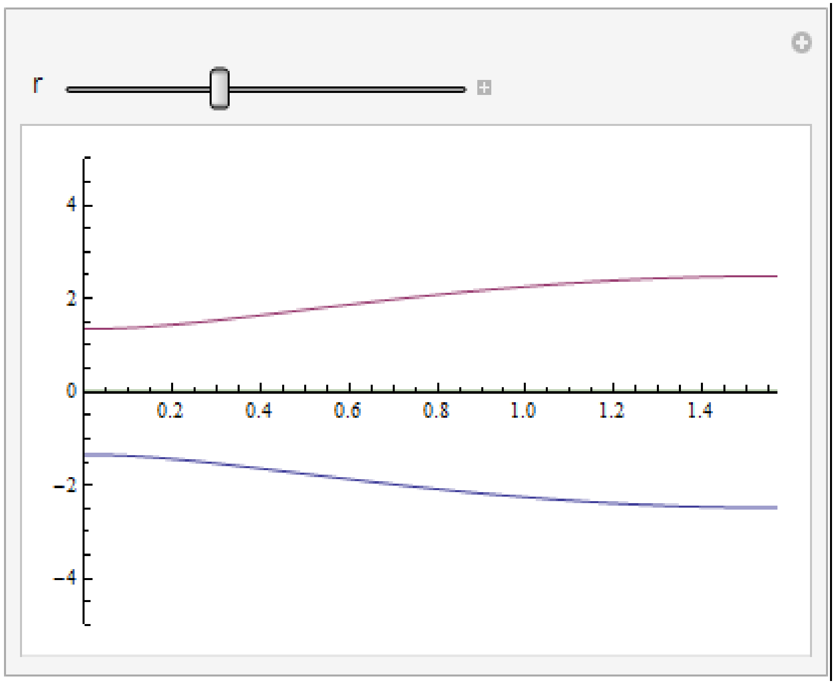
Linear interaction Matrix

General Symmetric Case: $\omega_s = 1,\ \ \alpha = 0,\ \ r \in [0 \div 1]$

$\Re[\Lambda_A]$



$\Im[\Lambda_A]$



Linear interaction Matrix

General Asymmetric Case:

$$\Lambda_A = \{\alpha \pm \sqrt{\frac{\Delta_1}{2}}\}$$

$$\Delta_1 = \sqrt{(r^2 - \delta_r^2)\cos 2\epsilon - (\omega_s^2 - \delta_\alpha^2) \pm \frac{\Delta_2}{4}}$$

$$\Delta_2 = \sqrt{(r^2 - \delta_r^2)^2 + (\delta_\alpha^2 + \omega_s^2)^2 - 2(r^2 - \delta_r^2)(\delta_\alpha^2 + \omega_s^2)\cos 2\epsilon - \delta_\alpha \omega_s \sin 2\epsilon}$$

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Linear interaction matrix

The coupling matrices $R_{1,2}$ *:*

$$R_1 R_2 = R_2 R_1$$

$$R_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R_1^T$$

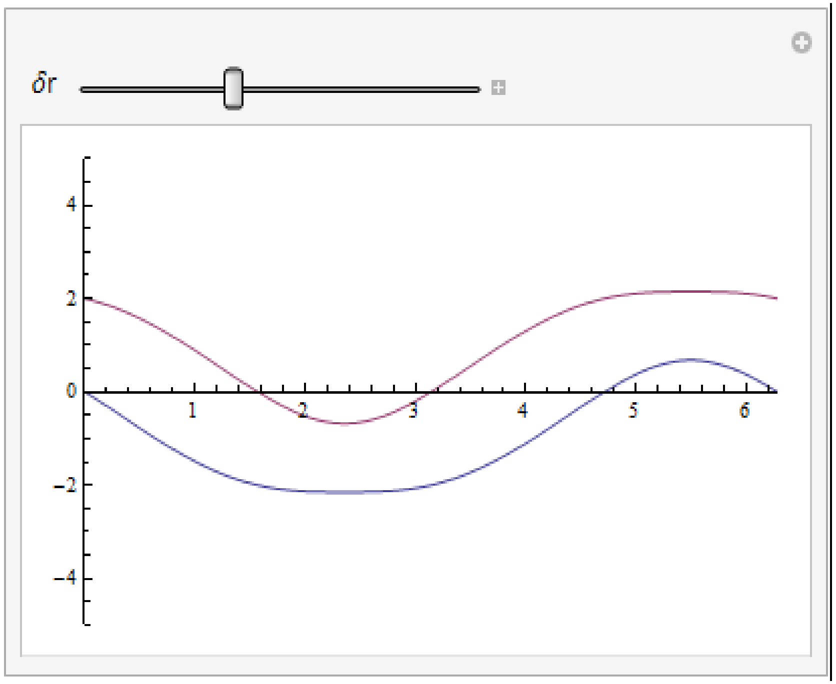
$$\Lambda_{R_1} = \left\{ r\left(\frac{\cos\epsilon - \sin\epsilon}{2}\right) \pm \frac{\sqrt{r^2 + (r^2 - 2\delta_r^2)\sin 2\epsilon}}{2} \right\}$$

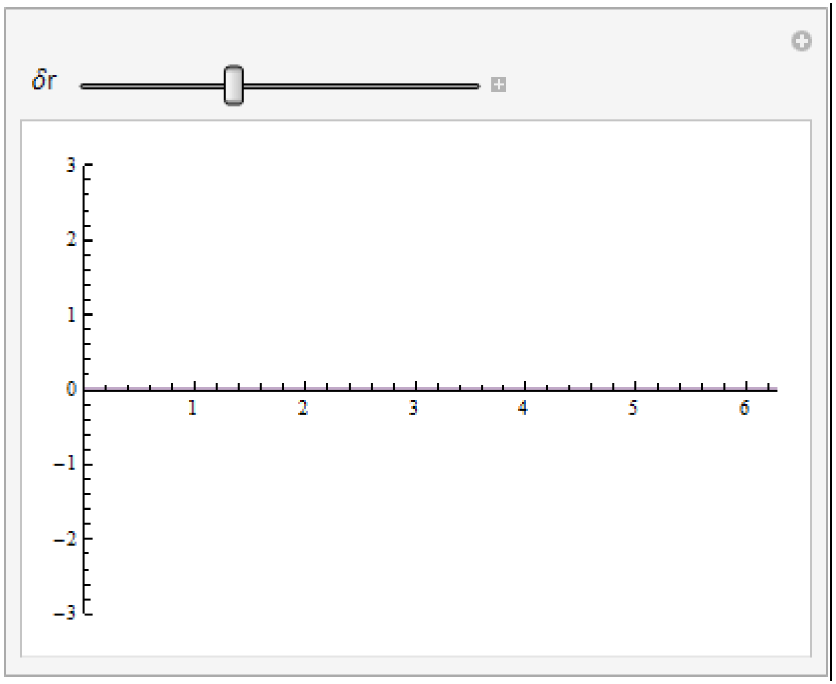
$$\Lambda_{R_2} = \left\{ r\left(\frac{\cos\epsilon + \sin\epsilon}{2}\right) \pm \frac{\sqrt{r^2 - (r^2 - 2\delta_r^2)\sin 2\epsilon}}{2} \right\}$$

Linear interaction matrix

The coupling matrices $R_{1,2}$ *:* $r = 1, \;\; \delta_r \in [-2 \div 2]$

$\Re\left[\Lambda_{R_1}\right]$ $\Im\left[\Lambda_{R_1}\right]$





Non-linear interaction matrix

Depends only on the last 2 components, i.e. on intensities

Deterministic solutions found if: $\tau = 0, \quad c = 0$

T.H. Chiba, Opt. Comm. 76 ***5,6*** *(1990).*

The auto functions of $A - B(X)$

have the same directions

of the eigenvectors of A !!!

$$B(\boldsymbol{x}) = s\, x_3\, I_4$$

In complex coordinates,
invariants are spheres of $\mathbb{R}^4$

Decoupled stability analysis

Study the stability of the fixed points of the system for perturbative treatment

$$\dot{\boldsymbol{x}} = \left(A_0 - B(\boldsymbol{x})\right)\boldsymbol{x}$$

$$A_0 = \begin{pmatrix} \alpha & -\omega_s & 0 & 0 \\ \omega_s & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \delta_\alpha \\ 0 & 0 & \delta_\alpha & \alpha \end{pmatrix}$$

Fixed points for $(x_{3,} x_4)$ *:*

$(0,0)$ ← *Laser switched off*

$\frac{\alpha - \delta_\alpha}{4}(1,-1)$

$\frac{\alpha + \delta_\alpha}{4}(1,1)$

← *1 intensity not lasing*

$(\frac{\alpha}{s}, \frac{\delta_\alpha}{c})$ ← *Limit Cycle!!!*

Perturbative Solution 0th order

Within the assumptions $(\omega_s > \alpha) \gg \delta_\alpha$

$\left(\frac{\alpha}{s}, \frac{\delta_\alpha}{c}\right)$ *is stable for* $(x_{3,} x_4)$

⇩

The RLG system exhibits a stable limit cycle:

$$\boldsymbol{x}^{(0)} = \left(\rho \sin(2\hat{\omega} t + \phi_0), \rho \sin(2\hat{\omega} t + \phi_1), \frac{\alpha}{s}, \frac{\delta_\alpha}{c}\right)^T$$

where $\hat{\omega} = \omega - \tau \delta_\alpha / c, \quad \rho = \sqrt{\alpha^2 / s^2 - \delta_\alpha^2 / c^2}$

G-PISA limit cycle

Perturbative Solution 1st order

We make use of:

$$\begin{cases} \boldsymbol{x} = \boldsymbol{x}^{(0)} + q\,\boldsymbol{x}^{(1)} \\ A = A_0 + q\,A_1 \end{cases}$$

where

$$A_1 = \begin{pmatrix} 0 & 0 & r\cos\epsilon & \delta_r\cos\epsilon \\ 0 & 0 & \delta_r\cos\epsilon & r\sin\epsilon \\ r\cos\epsilon & \delta_r\cos\epsilon & 0 & 0 \\ -\delta_r\cos\epsilon & -r\sin\epsilon & 0 & 0 \end{pmatrix}$$

… and perform the substitutions …

Perturbative Solution 1st order

The 1st order perturbative solution satisfies

$$\dot{\boldsymbol{x}}^{(1)} = P(t)\,\boldsymbol{x}^{(1)} + \boldsymbol{f}^{(1)}(t)$$

where

$$P(t) = \begin{pmatrix} 0 & -2\hat{\omega} & -s\,x_1^{(0)}(t) & 2\tau\,x_2^{(0)}(t) \\ 2\hat{\omega} & 0 & -s\,x_2^{(0)}(t) & -2\tau\,x_1^{(0)}(t) \\ 0 & 0 & -\alpha & -\delta_\alpha \\ 0 & 0 & -\delta_\alpha \dfrac{s}{c} & -\alpha \dfrac{c}{s} \end{pmatrix}, \quad \boldsymbol{f}^{(1)}(t) = \begin{pmatrix} 2\rho\left(\dfrac{\delta_r \delta_\alpha}{c} + \dfrac{r\,\alpha}{s}\right)\cos\epsilon \\ 2\rho\left(\dfrac{r\,\delta_\alpha}{c} + \dfrac{\delta_r\,\alpha}{s}\right)\sin\epsilon \\ x_1^{(0)}(t)\,r\cos\epsilon + x_2^{(0)}(t)\,\delta_r \sin\epsilon \\ -x_1^{(0)}(t)\,r\sin\epsilon - x_2^{(0)}(t)\,\delta_r \cos\epsilon \end{pmatrix}$$

Perturbative Solution 1st order

Iterating the procedure we show that the n^{th} order perturbative solution satisfies

$$\dot{\boldsymbol{x}}^{(n)} = P(t)\,\boldsymbol{x}^{(n)} + \boldsymbol{f}^{(n)}(t)$$

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Conclusions

- Study of the RLG non linear system to improve the performances of our algorithms for rotational frequency estimation.
- New coordinates for this study.
- Main properties of the equations discussed
- Experimental Regimes analysed in a new fashion.
- A perturbative procedure outlined.