The speculator: a case study of the dynamic programming equation of an infinite horizon stochastic problem

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The problem

A major speculator maximizes his profit from selling a stock F_0 of foreign currency under a floating exchange rate z by choosing the selling rate u. More generally F other assets like shares etc...

The model

(*A. Černý: Currency Crises: Introduction of Spot Speculators*, Int. J. Financ. Econ. **4**(1999)): For given F_0, z_0 maximize $J(F_0, z_0, u)$ over u, where

$$J(F_0, z_0, u) = E \int_0^{T:F_T=0} e^{-\rho t} u_t (z_t - \eta u_t) dt \tag{1}$$

subject to

$$\frac{dF}{dt} = rF - u \tag{2}$$

$$dz = \lambda z dt + \sigma z dW \tag{3}$$

$r, \rho, \eta, \sigma > 0$, W the standard Wiener process.

Ito formula $\longrightarrow$ DP (HJB) equation for the *value function*

$$V(F, z) = \sup_u J(F, z, u) :$$

$$\rho V = \sup_u \{u(z - \eta u) + D_F V(rF - u) + D_z V \lambda z + \frac{1}{2}\sigma z^2 D_{zz}^2 V\};$$

supremum achieved by $u = (z - D_F V)/\eta$:

$$\frac{1}{2}\sigma^2 z^2 D_{zz} V = -\lambda z D_z V - rFD_F + \rho V - \frac{(z - D_F V)^2}{4\eta} \text{ for } F > 0 \tag{4}$$

$$V(0, z) = 0. \tag{5}$$

If $\lambda + r < \rho$ (discount prevails) then

$$V(F, z) \leq Fz.$$

Indeed, for arbitrary $T > 0$

$$E \int_0^T e^{-\rho t} u_t (z_t - \eta u_t) dt \leq E \int_0^T e^{-\rho t} (rF_t - dF_t/dt) z_t dt$$

$$= F_0 z_0 - EF_T z_T + E \int_0^T e^{-\rho t} F_t z_t [(-\rho + r + \lambda) dt + \sigma dW_t] dt$$

$$\leq F_0 z_0 + E \int_0^T e^{-\rho t} (-\rho + r + \lambda) F_t z_t dt + \sigma E \int_0^T e^{-\rho t} F_t z_t dW_t dt$$

$$\leq F_0 z_0$$

because $F_t \geq 0, \; z_t > 0$ and

$$E \int_0^T \phi_t dW_t = 0$$

for ϕ continuous square integrable.

Reduction of HJB to ODE equation

Homogeneity $V(\kappa F, \kappa z) = \kappa^2 V(F, z) \implies$

$$V(F, z) = (\frac{z}{\eta})^2 V(\eta \frac{F}{z}, \eta) = \frac{z^2}{\eta} y(x),\ x = \eta F/z,\ y(x) = \frac{x^2}{\eta} V(x, \eta);$$

$y(x)$ satisfies

$$x^2 y'' = axy' + by - c(y' - 1)^2, \qquad x > 0 \tag{6}$$
$$y(0) = 0, \tag{7}$$

with $c > 0$; $\rho > \lambda + r > 0 \iff a + b > 0$.

Questions: Existence, uniqueness (of solutions relevant for original problem), verification of optimality

PB, A. Černý, M. Winkler 2013:

- If $a + b > 0$, $\forall x_0 > 0,\ 0 < y_0 < x_0\ \exists$ solution on $[0, x_0]$ such that $y(x_0) = y_0$,
- If $a + b = 0$ then $y(x) = x$ is a solution on $[0, \infty)$,
- No solution for $a + b < 0$.

Results - Continuation

- each solution satisfies

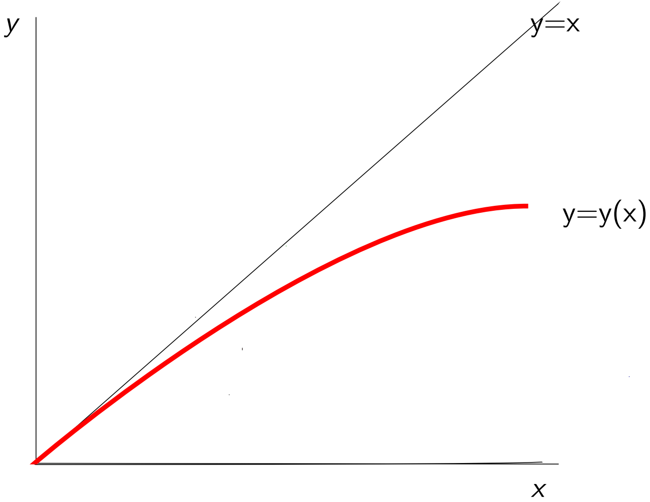
$$y'(0) = 1, \ \lim_{x \to 0} \frac{y'(x) - 1}{\sqrt{x}} = -\sqrt{\frac{a+b}{c}}$$

- there is a unique solution such that $0 \le y(x) \le x$ on $[0, \infty)$, satisfies $y' > 0, \ y'' < 0, \ y''' > 0$,

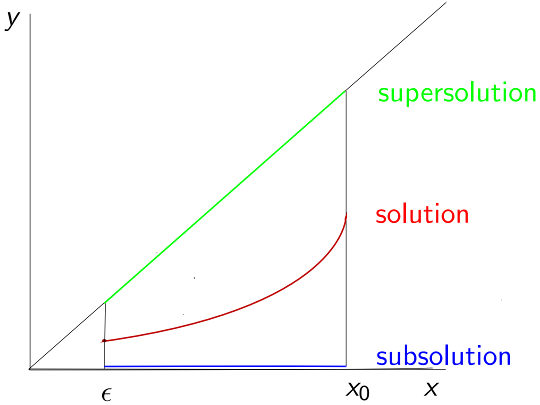
$$\lim_{x \to \infty} y'(x) = 0, \ y(x)/x < (1 + y'(x))/2$$

- this solution represents the value function of the problem

Note $y(x) \le x \iff V(F, x) \le Fx$


y
y=x
y=y(x)
x

- Existence for $a+b>0$ on $[\epsilon, x_0],\ 0<\epsilon<x_0$ follows from Nagumo (1938) because $\overline{y}(x)=x,\ \underline{y}(x)$ are super- resp. subsolution. For $\epsilon=0, x_0=\infty$ it follows by principle of nested intervals.
- Nonexistence for $a+b<0$: Transformation $x(t)=-t^{-1}$ of $0<x<\infty$ to $(-\infty<t<0)$ plus comparison of solutions


y
supersolution
solution
subsolution
ϵ
x0
x

Proposition *const* $\neq y$ *solving*

$$y'' = f(x)y' + g(x,y)(y - y^*) \text{ on } (0,\infty)$$

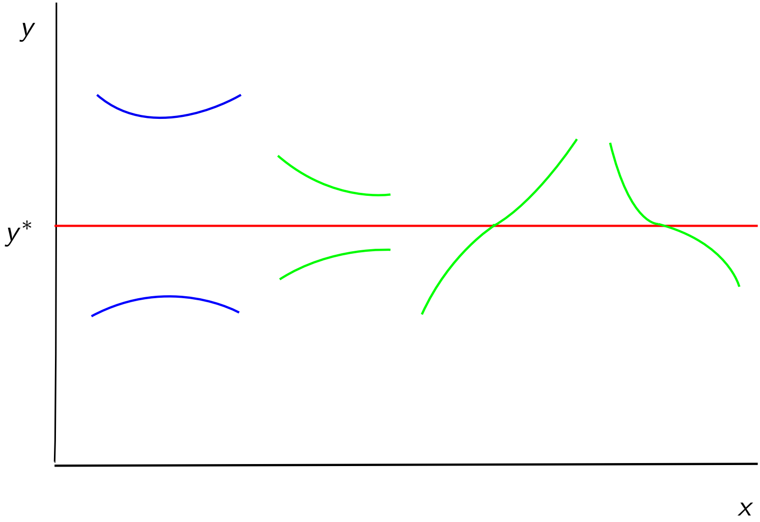
f, g *continuous*, $g > 0$, *then:*
y' *has at most one zero* x_0 *and either*

$$y(x) > y(x_0) > y^*, \; y(x_1) > y(x_0) < y(x_2)$$
$$\text{for } 0 \leq x_1 < x_0 < x_2 \text{ and } x > 0$$

or a symmetric w. r. to y^* *conclusion holds.*
Consequences:

- *A solution intersecting* $y = y^*$ *is monotone*
- *A solution convergent for* $t \to 0$ *or* $t \to \infty$ *converges monotonically*

$$y'' = f(x)y' + g(x,y)(y - y^*),\ g > 0$$



Outlines of proofs - solution properties 2

Assumptions satisfied for
• equation (1) with $y^* = c/b$ for $b \neq 0$, $y^* = -\infty$ for $b = 0 \implies y(x)$ either $\nearrow c/b$ or unbounded
• equation for derivative $y_1 = y'$

$$x^2 y_1'' = (a-2)xy_1' + (a+b)y_1 - 2c(y'-1)y_1'$$

with $y_1^* = 0 \implies y'$ decreases from 1 to 0
• equation for 2nd derivative $\implies$ second term for asymptotics at 0, positive profit
• equation for derivative of difference Δy_1 of solutions

$$x^2 \Delta y_1'' = (a-2)x\Delta y_1 + (a+b)\Delta y_1 - 2cy''\Delta y_1 - 2c(y'-1)\Delta y_1 - 2c\Delta y_1 \Delta y_1'$$

with $\Delta y^* = 0 \implies$ uniqueness.

Outlines of proofs - sufficiency

"Verification" theorem o (*Fleming-Soner)*; "transversality" condition

$$E_{F_0,x_0}e^{-\rho t}V(F_t,x_t)\to 0 \text{ for } t\to\infty$$

following from

$$V((F,x)\leq Fx.$$

In case discount outweighes profit from holding the asset the unique solution $0 \leq y(x) \leq x$ of the ODE

$$x^2 y'' = axy' + by - c(y'-1)^2, \qquad x > 0$$
$$y(0) = 0$$

is the only one generating by $V(F,x) = \frac{x^2}{\eta} y(\eta F/x)$ the solution of the HJB equation

$$\frac{1}{2}\sigma^2 z^2 D_{zz} V = -\lambda z D_z V - rFD_F + \rho V - \frac{(z - D_F V)^2}{4\eta} \text{ for } F > 0$$
$$V(0,z) = 0$$

consistent with the known requirements. This solution represents the value function. In addition, its V satisfies $D_F V > 0, D^2_{FF} < 0, D_x V > 0$ and yields positive profit. In the opposite case the ODE has no solution.

Open loop controls synthesize into optimal feedback $\Longrightarrow$
Pontrjagin maximum principle can be effectively employed
Dynamic programming equation is an unpleasant first order DE -
for infinite horizon problems implicit.

Example:Ramsey problem

$$\min_{c(.)} \int_0^\infty e^{-\rho t} U(c(t)) dt$$

subject to

$$dk/dt = f(k) - c(t),\ \ k(0) = k_0,\ \ k(t) \geq 0,\ \ c(t) \geq 0,$$

in particular

$$f(k) = k^\alpha - \mu k, U(c) = (1-\theta)^{-1} c^{1-\theta}, 0 < \theta < 1.$$

can be fully understood by variational (DP) methods (Jurča 2004)
DP equation

$$\max_c [(1-\theta)^{-1} c^{1-\theta} - V'(k)(f(k) - \theta] + \rho V = 0.$$

Minimum achieved for $c = V'(k)^{-\theta} \implies$

$$\frac{\theta}{\theta - 1} V'^{\frac{\theta-1}{\theta}} + f(k)V' - \rho V = 0 -$$

unpleasant implicit DE.

I. Ekeland 2010:

Thom's theory of Pfaff systems yield "fake" solutions $\implies$ need of "initial condition"

$$V(k_\infty) = \rho^{-1} f(k_\infty),$$

k_∞ "equlibrium" state provided by PMP theory

Stochastic problems

PMP not applicable because concept of open loop controls irrevelant DP much more pleasant because DP(HJB) equation well

behaved explicit equation

- uniformly parabolic for finite horizon and
- uniformly elliptic for autonomous infinite horizon problems

provided stochastic dynamics governed by Wiener process

$$dy_t = f(y_t) + \sigma dW_t$$

However: in economics and finance mostly stochastic dynamics governed by *geometric* Brownian motion

$$dy_t = y_t g(y_t) + \sigma y_t dW_t$$

leading to equation degenerating at $y_t = 0$.

Explicit solution approach

- linear-quadratic problem (*Rami et al. 2001*): more solutions
- Merton's consumption and investment problem (*Karatzas et al. 1986, Fleming et al. 1991*)

"Qualitative" approach

- Merton's problem (*Zariphapoulou 1994, Vila et al. 1997*)
- Speculator's problem (*PB et al. 2013*)

Intuitive explicit solutions

Common in macroeconomic models. Danger: possibility of multiple solutions, in particular in the absence of "initial condition".

For example, in our case

$$y(x) \equiv c/b$$

if $b > 0$.