The speculator: a case study of the dynamic programming equation of an infinite horizon stochastic problem

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A major speculator maximizes his profit from selling a stock $F_0$ of foreign currency under a floating exchange rate $z$ by choosing the selling rate $u$. More generally $F$ other assets like shares etc...
(A. Černý: Currency Crises: Introduction of Spot Speculators, Int. J. Financ. Econ. 4(1999)): For given \( F_0, z_0 \) maximize \( J(F_0, z_0, u) \) over \( u \), where

\[
J(F_0, z_0, u) = E \int_0^{T : F_T = 0} e^{-\rho t} u_t (z_t - \eta u_t) dt
\]

subject to

\[
\frac{dF}{dt} = rF - u \quad (2)
\]

\[
dz = \lambda z dt + \sigma z dW \quad (3)
\]

\( r, \rho, \eta, \sigma > 0 \), \( W \) the standard Wiener process.
Ito formula $\rightarrow$ DP (HJB) equation for the value function

\[ V(F, z) = \sup_u J(F, z, u) : \]

\[ \rho V = \sup_u \{u(z - \eta u) + D_F V(rF - u) + D_z V \lambda z + \frac{1}{2}\sigma z^2 D_{zz} V\}; \]

supremum achieved by $u = (z - D_F V)/\eta$:

\[ \frac{1}{2}\sigma^2 z^2 D_{zz} V = -\lambda z D_z V - rF D_F + \rho V - \frac{(z - D_F V)^2}{4\eta} \quad \text{for } F > 0 \]  \hspace{1cm} (4)

\[ V(0, z) = 0. \]  \hspace{1cm} (5)
If $\lambda + r < \rho$ (discount prevails) then

$$V(F, z) \leq Fz.$$ 

Indeed, for arbitrary $T > 0$

$$E \int_0^T e^{-\rho t} u_t(z_t - \eta u_t) dt \leq E \int_0^T e^{-\rho t} (rF_t - dF_t/dt)z_t dt$$

$$= F_0z_0 - EF_Tz_T + E \int_0^T e^{-\rho t} F_t z_t [(-\rho + r + \lambda) dt + \sigma dW_t] dt$$

$$\leq F_0z_0 + E \int_0^T e^{-\rho t} (-\rho + r + \lambda) F_t z_t dt + \sigma E \int_0^T e^{-\rho t} F_t z_t dW_t dt$$

$$\leq F_0z_0$$

because $F_t \geq 0$, $z_t > 0$ and

$$E \int_0^T \phi_t dW_t = 0$$

for $\phi$ continuous square integrable.
Reduction of HJB to ODE equation

Homogeneity  \( V(\kappa F, \kappa z) = \kappa^2 V(F, z) \implies \)

\[
V(F, z) = \left( \frac{z}{\eta} \right)^2 V\left( \frac{F}{z}, \eta \right) = \frac{z^2}{\eta} y(x), \quad x = \eta F/z, \quad y(x) = \frac{x^2}{\eta} V(x, \eta);
\]

\( y(x) \) satisfies

\[
x^2 y'' = axy' + by - c(y' - 1)^2, \quad x > 0 \tag{6}
\]

\[
y(0) = 0, \tag{7}
\]

with \( c > 0; \rho > \lambda + r > 0 \iff a + b > 0. \)

Questions: Existence, uniqueness (of solutions relevant for original problem), verification of optimality
PB, A. Černý, M. Winkler 2013:

• If $a + b > 0$, $\forall x_0 > 0$, $0 < y_0 < x_0$ $\exists$ solution on $[0, x_0]$ such that $y(x_0) = y_0$,

• If $a + b = 0$ then $y(x) = x$ is a solution on $[0, \infty)$,

• No solution for $a + b < 0$. 
• each solution satisfies

\[ y'(0) = 1, \ \lim_{x \to 0} \frac{y'(x) - 1}{\sqrt{x}} = -\sqrt{\frac{a+b}{c}} \]

• there is a unique solution such that \( 0 \leq y(x) \leq x \) on \([0, \infty)\), satisfies \( y' > 0, \ y'' < 0, \ y''' > 0 \),

\[ \lim_{x \to \infty} y'(x) = 0, \ \frac{y(x)}{x} < \frac{1 + y'(x)}{2} \]

• this solution represents the value function of the problem

Note \( y(x) \leq x \iff V(F, x) \leq Fx \)
$y = x$

$y = y(x)$
• Existence for $a + b > 0$ on $[\epsilon, x_0]$, $0 < \epsilon < x_0$ follows from Nagumo (1938) because $\overline{y}(x) = x$, $\underline{y}(x)$ are super- resp. subsolution. For $\epsilon = 0, x_0 = \infty$ it follows by principle of nested intervals.

• Nonexistence for $a + b < 0$: Transformation $x(t) = -t^{-1}$ of $0 < x < \infty$ to $(-\infty < t < 0)$ plus comparison of solutions
Proposition $\text{const} \neq y$ solving

\[ y'' = f(x)y' + g(x, y)(y - y^*) \text{ on } (0, \infty) \]

$f, g$ continuous, $g > 0$, then:

$y'$ has at most one zero $x_0$ and either

\[ y(x) > y(x_0) > y^*, \quad y(x_1) > y(x_0) < y(x_2) \]

for $0 \leq x_1 < x_0 < x_2$ and $x > 0$

or a symmetric w. r. to $y^*$ conclusion holds.

Consequences:

- A solution intersecting $y = y^*$ is monotone
- A solution convergent for $t \to 0$ or $t \to \infty$ converges monotonically
\[ y'' = f(x)y' + g(x, y)(y - y^*), \quad g > 0 \]
Assumptions satisfied for

- equation (1) with \( y^* = c/b \) for \( b \neq 0 \), \( y^* = -\infty \) for \( b = 0 \) \( \implies y(x) \) either \( \nearrow c/b \) or unbounded

- equation for derivative \( y_1 = y' \)

\[
x^2 y_1'' = (a - 2)xy_1' + (a + b)y_1 - 2c(y' - 1)y_1'
\]

with \( y_1^* = 0 \) \( \implies y' \) decreases from 1 to 0

- equation for 2nd derivative \( \implies \) second term for asymptotics at 0, positive profit

- equation for derivative of difference \( \Delta y_1 \) of solutions

\[
x^2 \Delta y_1'' = (a - 2)x \Delta y_1 + (a + b)\Delta y_1 - 2c y'' \Delta y_1 - 2c(y' - 1)\Delta y_1 - 2c \Delta y_1 \Delta y_1'
\]

with \( \Delta y^* = 0 \) \( \implies \) uniqueness.
"Verification" theorem (Fleming-Soner); "transversality" condition

\[ E_{F_0,x_0} e^{-\rho t} V(F_t, x_t) \to 0 \text{ for } t \to \infty \]

following from

\[ V((F, x) \leq Fx. \]
Conclusions

In case discount outweighs profit from holding the asset the unique solution $0 \leq y(x) \leq x$ of the ODE

$$x^2 y'' = a x y' + b y - c (y' - 1)^2, \quad x > 0$$

$y(0) = 0$

is the only one generating by $V(F, x) = \frac{x^2}{\eta} y(\eta F/x)$ the solution of the HJB equation

$$\frac{1}{2} \sigma^2 z^2 D_{zz} V = -\lambda z D_z V - r F D_F + \rho V - \frac{(z - D_F V)^2}{4 \eta} \quad \text{for } F > 0$$

$V(0, z) = 0$

consistent with the known requirements. This solution represents the value function. In addition, its $V$ satisfies $D_F V > 0$, $D_{FF}^2 < 0$, $D_x V > 0$ and yields positive profit. In the opposite case the ODE has no solution.
Deterministic optimal control problems

Open loop controls synthesize into optimal feedback $\implies$
Pontryagin maximum principle can be effectively employed
Dynamic programming equation is an unpleasant first order DE -
for infinite horizon problems implicit.
Example: Ramsey problem

\[ \min_{c(\cdot)} \int_0^\infty e^{-\rho t} U(c(t)) dt \]

subject to

\[ \frac{dk}{dt} = f(k) - c(t), \quad k(0) = k_0, \quad k(t) \geq 0, \quad c(t) \geq 0, \]

in particular

\[ f(k) = k^\alpha - \mu k, \quad U(c) = (1 - \theta)^{-1} c^{1-\theta}, \quad 0 < \theta < 1. \]

can be fully understood by variational (DP) methods (Jurča 2004)

DP equation

\[ \max_c [(1 - \theta)^{-1} c^{1-\theta} - V'(k)(f(k) - \theta)] + \rho V = 0. \]

Minimum achieved for \( c = V'(k)^{-\theta} \quad \implies \]

\[ \frac{\theta}{\theta - 1} V'^{\theta-1} + f(k)V' - \rho V = 0 \]

unpleasant implicit DE.
Thom’s theory of Pfaff systems yield ”fake” solutions $\implies$ need of ”initial condition”

$$V(k_\infty) = \rho^{-1}f(k_\infty),$$

$k_\infty$ ”equilibrium” state provided by PMP theory
Stochastic problems

PMP not applicable because concept of open loop controls irrelevant. DP much more pleasant because DP(HJB) equation well behaved explicit equation

- uniformly parabolic for finite horizon and
- uniformly elliptic for autonomous infinite horizon problems

provided stochastic dynamics governed by Wiener process

\[ dy_t = f(y_t) + \sigma dW_t \]

However: in economics and finance mostly stochastic dynamics governed by \textit{geometric} Brownian motion

\[ dy_t = y_t g(y_t) + \sigma y_t dW_t \]

leading to equation degenerating at \( y_t = 0 \).
Explicit solution approach


"Qualitative" approach

- Speculator’s problem (*PB et al.* 2013)
Intuitive explicit solutions

Common in macroeconomic models. Danger: possibility of multiple solutions, in particular in the absence of ”initial condition”.

For example, in our case

\[ y(x) \equiv \frac{c}{b} \]

if \( b > 0 \).