Control of State Constrained Dynamical Systems

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BV-norm continuity of sweeping processes driven by a set with constant shape

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The play operator

$$\mathcal{H} \text{ is a real Hilbert space}, \quad z_0 \in \mathcal{H}, \quad 0 < T < \infty,$$

$$\mathcal{Z} \in \mathscr{C}_{\mathcal{H}} := \Big\{ \mathcal{C} \subseteq \mathcal{H} \ : \ \mathcal{C} \text{ nonempty, closed, convex} \Big\}$$

$\forall u \in AC([0,T]\,;\mathcal{H})\ \exists! y \in AC([0,T]\,;\mathcal{H})$ such that

$$\begin{cases} u(t) - y(t) \in \mathcal{Z} & \forall t \in [0,T] \\ \langle z - u(t) + y(t), y'(t) \rangle \leq 0 & \text{for } \mathscr{L}^1\text{-a.e. } t \in [0,T], \forall z \in \mathcal{Z} \\ u(0) - y(0) = z_0 \in \mathcal{Z} \end{cases}$$

$$\begin{array}{rccc} \mathsf{P} \ : \ AC([0,T]\,;\mathcal{H}) & \longrightarrow & AC([0,T]\,;\mathcal{H}) \\ u & \longmapsto & y \end{array}$$ *play operator.*

The play operator

$$\mathcal{H} \text{ is a real Hilbert space,} \quad 0 < T < \infty,$$

$$\mathcal{Z} \in \mathscr{C}_{\mathcal{H}} := \Big\{ \mathcal{C} \subseteq \mathcal{H} \ : \ \mathcal{C} \text{ nonempty, closed, convex} \Big\}$$

$\forall u \in AC([0,T]\,;\mathcal{H})\ \exists! y \in AC([0,T]\,;\mathcal{H})$ such that

$$\begin{cases} u(t) \in y(t) + \mathcal{Z} & \forall t \in [0,T] \\ y'(t) \in N_{y(t)+\mathcal{Z}}(u(t)) & \text{for } \mathscr{L}^1\text{-a.e. } t \in [0,T] \\ y(0) = u(0) - z_0 \end{cases}$$

$$\begin{array}{rccc} \mathsf{P} \ : \ AC([0,T]\,;\mathcal{H}) & \longrightarrow & AC([0,T]\,;\mathcal{H}) \\ u & \longmapsto & y \end{array} \quad \textit{play operator.}$$

The play operator as a sweeping process

If $u \in AC([0,T]\,;\mathcal{H})$ then $y = \mathsf{P}(u)$ is the unique solution of

$$\begin{cases} y(t) \in u(t) - \mathcal{Z} & \forall t \in [0,T] \\ -y'(t) \in N_{u(t)-\mathcal{Z}}(y(t)) & \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,T] \\ y(0) = u(0) - z_0 \end{cases}$$

This is a particular case of sweeping process:

$\forall \mathcal{C} \in AC([0,T]\,;\mathscr{C}_{\mathcal{H}})\ \exists! y = \mathsf{M}(\mathcal{C}) \in AC([0,T]\,;\mathcal{H})$ such that

$$\begin{cases} y(t) \in \mathcal{C}(t) & \forall t \in [0,T] \\ -y'(t) \in N_{\mathcal{C}(t)}(y(t)) & \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,T] \\ y(0) = \mathrm{Proj}_{\mathcal{C}(0)}(y_0) \end{cases}$$

The sweeping processes

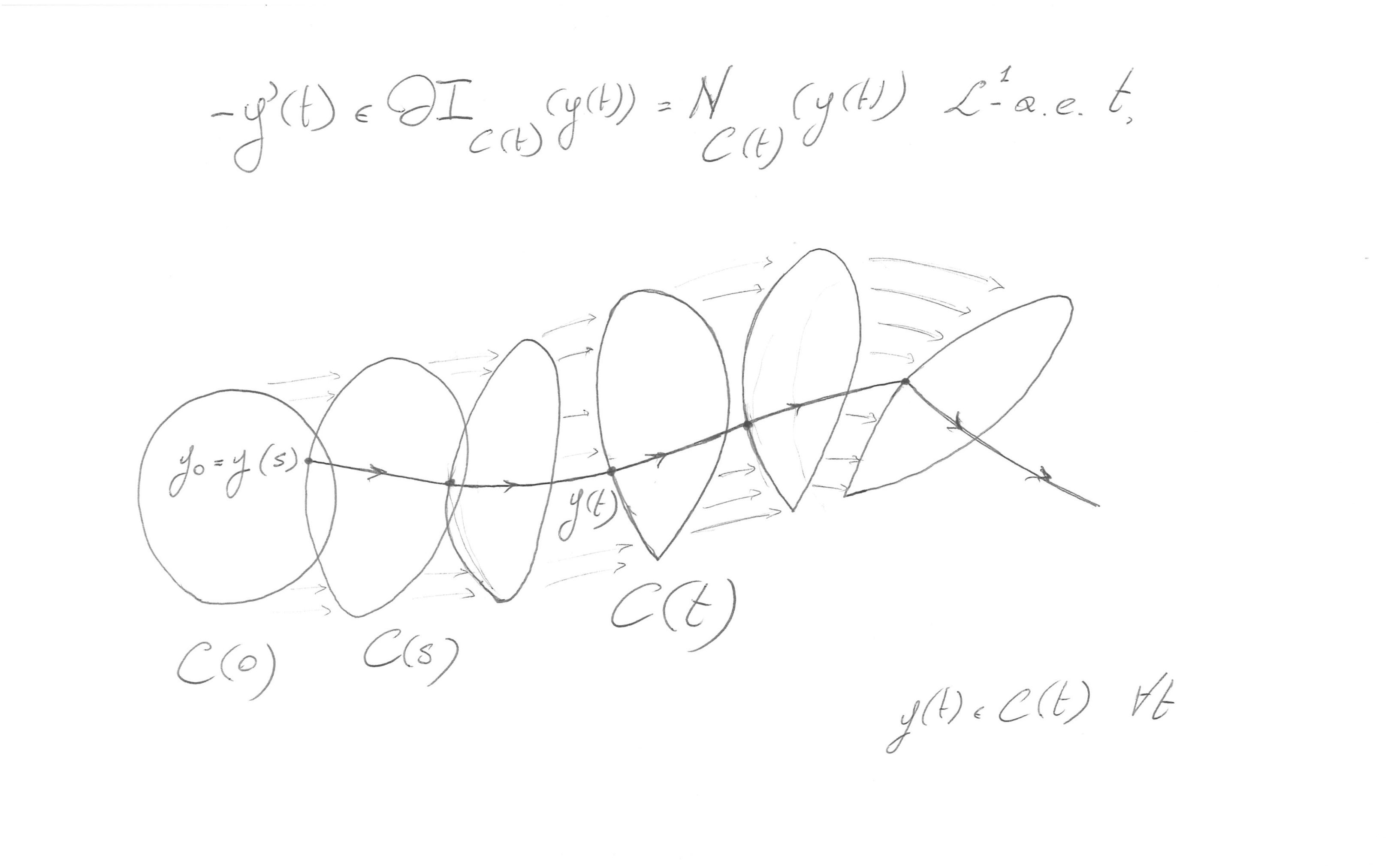
Theorem (J.J. Moreau, *Sém. d'Anal. Convexe* 1971)**.**
$\forall \mathcal{C} \in AC([0,T]\,;\mathscr{C}_{\mathcal{H}})\ \exists! y = \mathsf{S}(\mathcal{C}) \in AC([0,T]\,;\mathcal{H})$ *such that*

$$\begin{cases} y(t) \in \mathcal{C}(t) & \forall t \in [0,T] \\ -y'(t) \in N_{\mathcal{C}(t)}(y(t)) & \textit{for } \mathscr{L}^1\textit{-a.e. } t \in [0,T] \\ y(0) = \mathrm{Proj}_{\mathcal{C}(0)}(y_0) \end{cases}$$

$$\begin{array}{rccc} \mathsf{M} : & AC([0,T]\,;\mathscr{C}_{\mathcal{H}}) & \longrightarrow & AC([0,T]\,;\mathcal{H}) \\ & \mathcal{C} & \longmapsto & y \end{array}$$

$$\mathsf{P}(u) = \mathsf{M}(\mathcal{C}_u) \qquad \text{with } \mathcal{C}_u(t) := u(t) - \mathcal{Z}$$

A sweeping process



BV-solutions

Theorem (J.J. Moreau, *JDE*, 1977)**.**

$$\forall \mathcal{C} \in BV^{\mathrm{r}}([0,T]\,;\mathscr{C}_{\mathcal{H}}) \quad \exists! y \in BV^{\mathrm{r}}([0,T]\,;\mathcal{H}) \ : \ y(t) \in \mathcal{C}(t)\ \forall t \in [0,T]$$

$$\begin{cases} \mathrm{D}y = v\mu & \mu \textit{ Borel positive measure, } v \in \mathrm{L}^1(\mu;\mathcal{H}) \\ -v(t) \in N_{\mathcal{C}(t)}(y(t)) & \textit{for } \mu\textit{-a.e. } t \in [0,T] \\ y(0) = \mathrm{Proj}_{\mathcal{C}(0)}(y_0) \end{cases}$$

$$\begin{array}{rccc} \mathsf{M} \ : & BV^{\mathrm{r}}([0,T]\,;\mathscr{C}_{\mathcal{H}}) & \longrightarrow & BV^{\mathrm{r}}([0,T]\,;\mathcal{H}) \\ & \mathcal{C} & \longmapsto & y \end{array}$$

$$\begin{array}{rccc} \mathsf{P} \ : & BV^{\mathrm{r}}([0,T]\,;\mathcal{H}) & \longrightarrow & BV^{\mathrm{r}}([0,T]\,;\mathcal{H}) \\ & u & \longmapsto & y := \mathsf{M}(u - \mathcal{Z}) \end{array} \quad BV\textit{-play operator}$$

Variational formulation for P in BV.

$$\begin{array}{rccc} \mathsf{P} : & BV^r([0,T]\,;\mathcal{H}) & \longrightarrow & BV^r([0,T]\,;\mathcal{H}) \\ & u & \longmapsto & y := \mathsf{M}(\mathcal{C}_u) = \mathsf{M}(u - \mathcal{Z}) \end{array}$$

$y = \mathsf{P}(u) \in BV^r([0,T]\,;\mathcal{H})$ is the unique function s.t.

$$\begin{cases} u(t) - y(t) \in \mathcal{Z} & \forall t \in [0,T]\,, \\ \displaystyle\int_{[0,T]} \big\langle z(t) - u(t) + y(t), \mathrm{d}\mathrm{D}y(t) \big\rangle \leq 0 & \forall z \in BV([0,T]\,;\mathcal{Z}), \\ u(0) - y(0) = z_0 \in \mathcal{Z} \end{cases}$$

Rate independence

The solution operator of the sweeping processes

$$
\begin{array}{cccc}
\mathsf{M} \;:\; & BV^{\mathrm{r}}([0,T]\,;\mathscr{C}_{\mathcal{H}}) & \longrightarrow & BV^{\mathrm{r}}([0,T]\,;\mathcal{H}) \\
& \mathcal{C} & \longmapsto & y
\end{array}
$$

is rate independent:

$$
\mathsf{M}(\mathcal{C} \circ \phi) = \mathsf{M}(\mathcal{C}) \circ \phi
$$

if $\phi \in C([0,T]\,;\mathbb{R}) \cap BV([0,T]\,;\mathbb{R})$ increasing, $\phi([0,T]) = [0,T]$.

$\boldsymbol{BV}$-norm well posedness of P?

Is

$$\begin{array}{rcccl} \mathsf{P} & : & BV^r([0,T]\,;\mathcal{H}) & \longrightarrow & BV^r([0,T]\,;\mathcal{H}) \\ & & u & \longmapsto & y \end{array}$$

continuous w.r.t.

$$\begin{aligned} \|v\|_{BV} &:= \|v\|_\infty + \mathrm{V}(v,[0,T]) \\ &= \|v\|_\infty + \|\,\mathrm{D}v\| \\ &:= \|v\|_\infty + |\mathrm{D}v|([0,T]) \ ? \end{aligned}$$

Known results

BV-norm continuity of P known for:

1. $\mathsf{P} : AC([0,T]\,;\mathcal{H}) \longrightarrow AC([0,T]\,;\mathcal{H})$

   (P. Krejčí, *Eur. J. Appl. Math.*, 1991)

   Use:

   $$\mathsf{Q}(u) := 2\mathsf{P}(u) - u, \qquad \|\mathsf{Q}(u)'(t)\| = \|u'(t)\| \text{ for } \mathcal{L}^1\text{-a.e. } t.$$

2. $\mathsf{P} : BV^r([0,T]\,;\mathcal{H}) \longrightarrow BV^r([0,T]\,;\mathcal{H})$ if $\mathcal{Z}$ smooth, $\mathring{\mathcal{Z}} \neq \varnothing$

   (P. Krejčí, T. Roche, *DCDS*, 2011)

   Use a formula for the unit normal vector to $\mathcal{Z}$.

Idea: reparamatrize and reduce to $\boldsymbol{Lip}([0,T]\,;\mathcal{H})$

Reparametrize $u \in BV^r([0,T]\,;\mathcal{H})$ by the arclength

$$\ell_u(t) := \frac{T}{\mathrm{V}(u,[0,T])}\,\mathrm{V}(u,[0,t])$$

$\Longrightarrow \exists!\widetilde{u} : \ell_u([0,T]) \longrightarrow \mathcal{H}$ Lipschitz continuous such that

$$u = \widetilde{u} \circ \ell_u$$

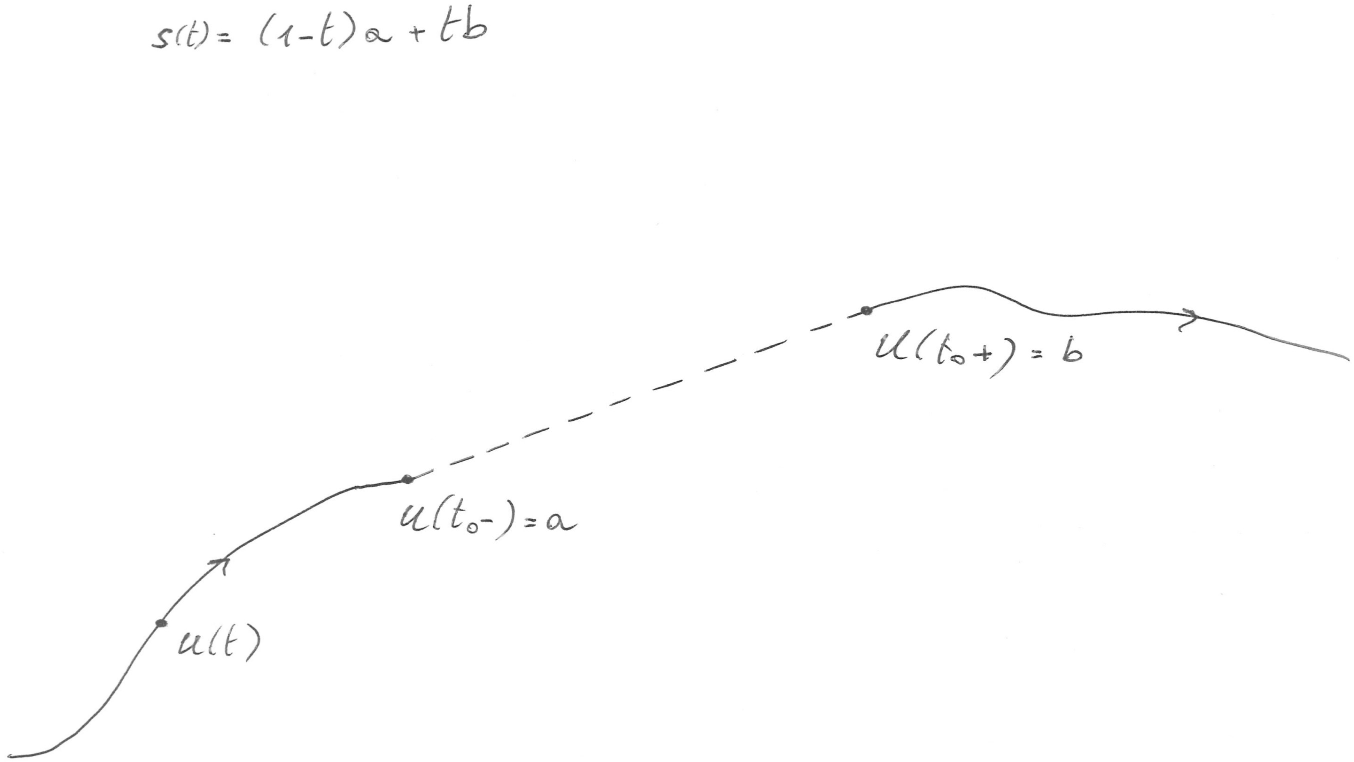
and hope that

$$\mathsf{P}(u) = \mathsf{P}(\widetilde{u} \circ \ell_u) = \mathsf{P}(\widetilde{u}) \circ \ell_u$$

?????????

We need to extend $\widetilde{u}$ to $[0,T]$ (i.e. to fill in the jumps).

Fill in the jump by a segment in $\mathcal{H}$



Fill in the jump by a segment in $\mathcal{H}$

A natural choice:

fill in the jumps with segments.

If $u \in BV^r([0,T]\,;\mathcal{H})$ and $\ell_u(t) := T\,\mathrm{V}(u,[0,t])/\,\mathrm{V}(u,[0,T])$ then $\exists! \widetilde{u} \in Lip([0,T]\,;\mathcal{H})$ such that

$$u = \widetilde{u} \circ \ell_u$$

$$\widetilde{u} \quad \text{is a segment } \textit{on}\, [\ell_u(t-), \ell_u(t+)]\,.$$

Is $\mathsf{P}(\widetilde{u}) \circ \ell_u = \mathsf{P}(u)$? the BV-solution?

No!

Theorem (V. Recupero, *Ann. SNS Pisa*, 2011)**.**

$$\mathsf{P}(\widetilde{u}) \circ \ell_u \neq \mathsf{P}(u).$$

Even if

$$\overline{\mathsf{P}}(u) := \mathsf{P}(\widetilde{u}) \circ \ell_u, \qquad u \in BV^{\mathrm{r}}([0,T]\,;\mathcal{H}),$$

has good properties: it is the unique continuous extension of $\mathsf{P} : Lip([0,T]\,;\mathcal{H}) \longrightarrow Lip([0,T]\,;\mathcal{H})$ *when*

the domain has the BV*-strict topology,*

the codomain has the L^1*-topology.*

Look at sweeping processes

If

$$\mathcal{C} \in BV^r([0,T]\,;\mathscr{C}_{\mathcal{H}})$$

$$\ell_{\mathcal{C}} : [0,T] \longrightarrow [0,T]\,, \quad \ell_{\mathcal{C}}(t) := T\,\mathrm{V}(\mathcal{C},[0,t])/\,\mathrm{V}(\mathcal{C},[0,T]),$$

then $\exists!\widetilde{\mathcal{C}} \in Lip(\ell_{\mathcal{C}}([0,T]);\mathscr{C}_{\mathcal{H}})$ such that $\mathcal{C} = \widetilde{\mathcal{C}} \circ \ell_{\mathcal{C}}$.

We need to extend $\widetilde{\mathcal{C}}$ to $[0,T]$,

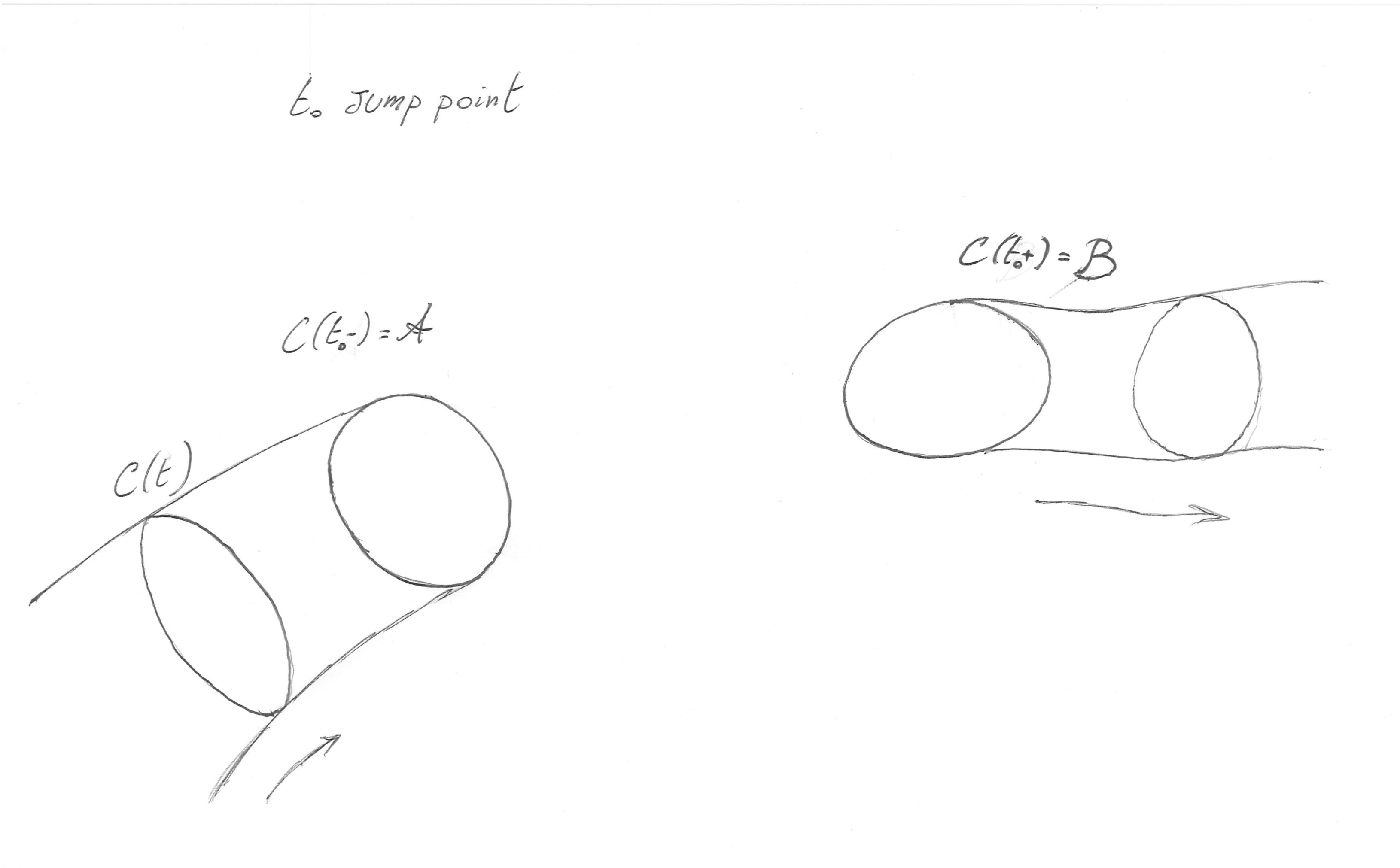
i.e.

we need to define $\widetilde{\mathcal{C}}$ on $[\ell_{\mathcal{C}}(t-), \ell_{\mathcal{C}}(t+)]$ for $t \in \mathrm{Discont}(\mathcal{C})$

i.e.

we need to fill in the jumps of $\mathcal{C}$ at every $t \in \mathrm{Discont}(\mathcal{C})$.

A jump for $\mathcal{C}(t)$



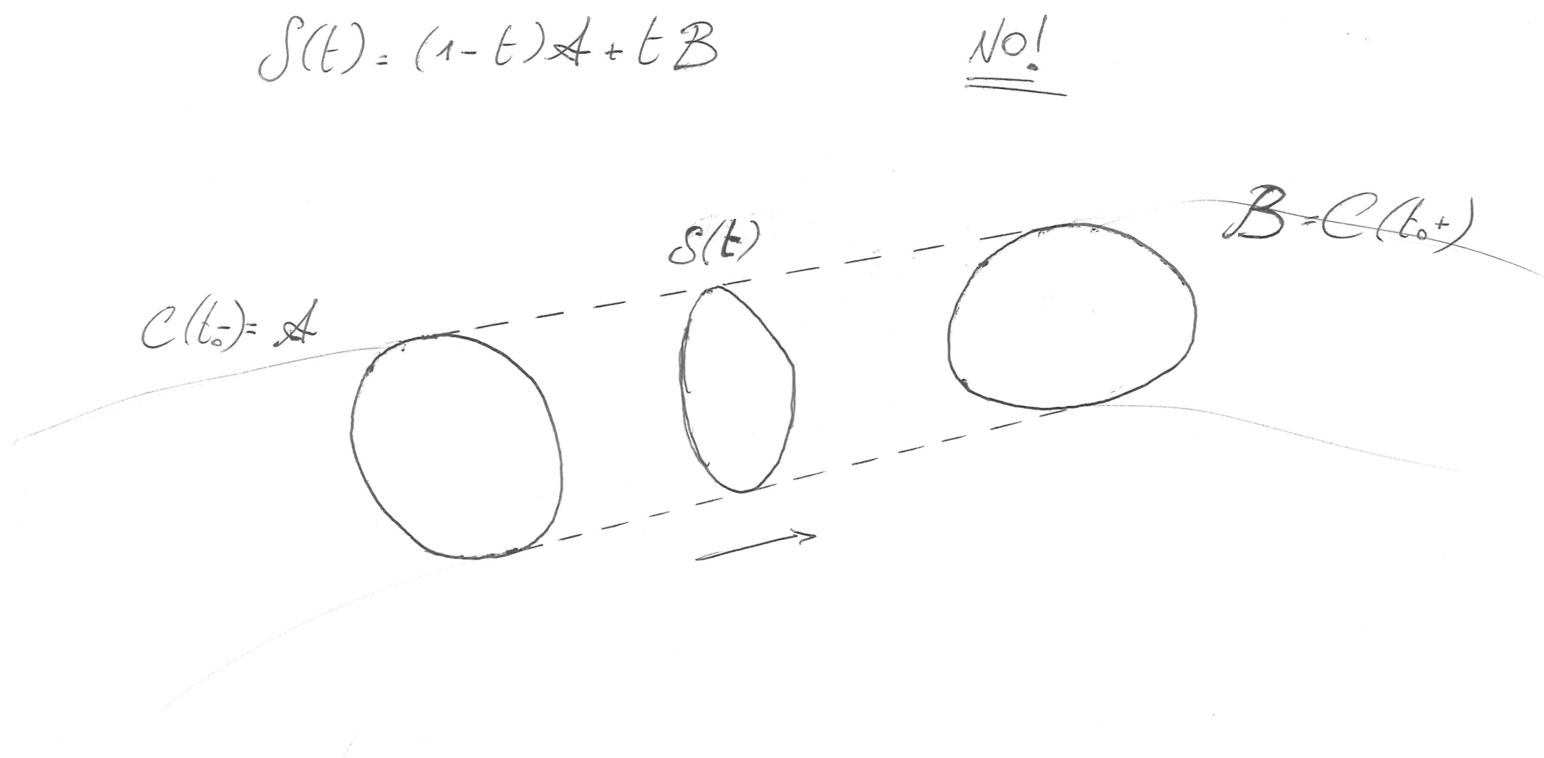
Convex-valued "segments"

Segments correspond to convex-valued geodesics of the form

$$\mathcal{S}(t) := (1-t)\mathcal{A} + t\mathcal{B}, \qquad t \in [0,1]\,.$$

connecting $\mathcal{A}$ to $\mathcal{B}$. Therefore $\mathcal{S}$ is not a good choice.

Minkowski sum-type geodesic



Another convex-valued curve is needed

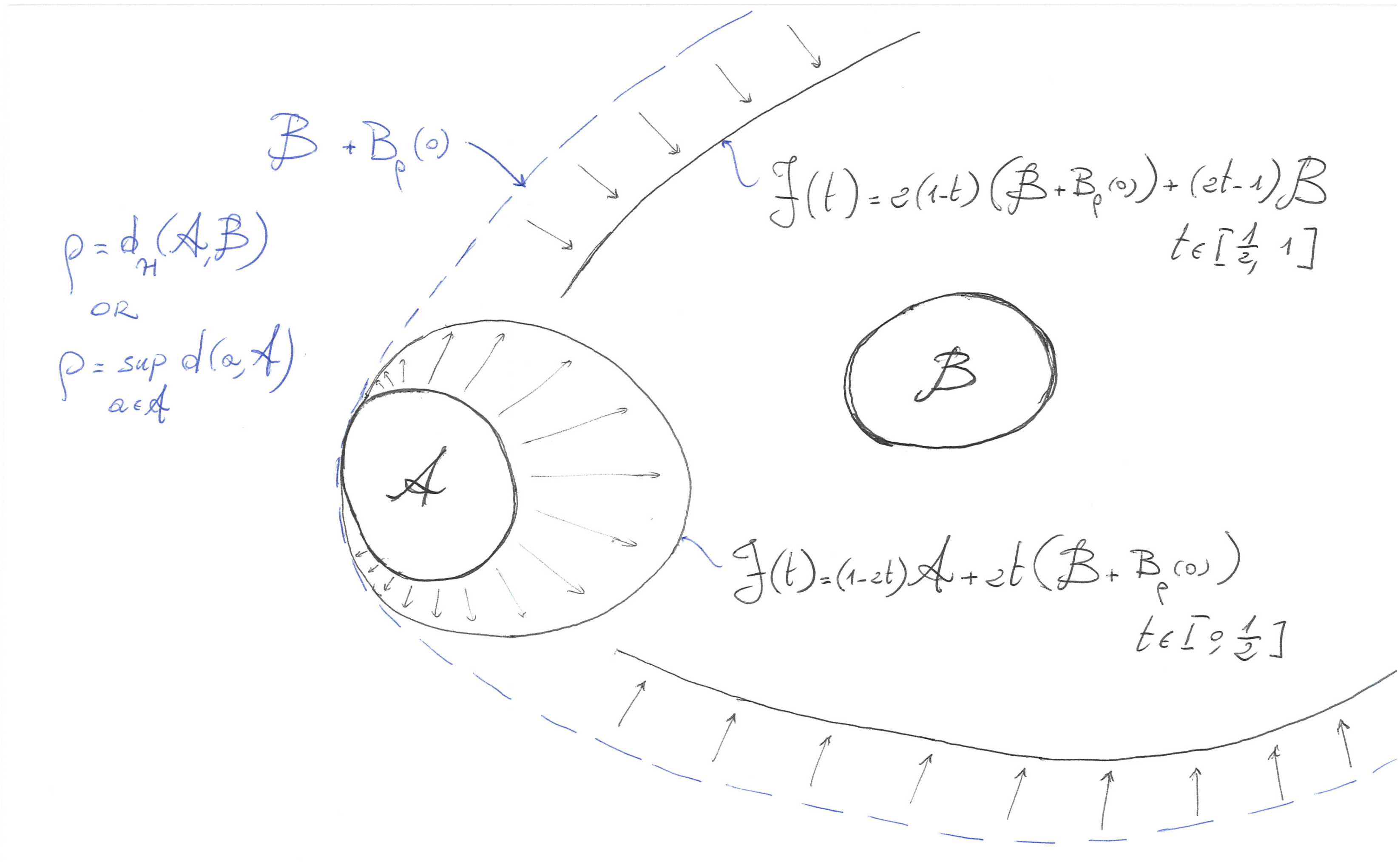
We look for a curve $\mathcal{F}(t)$ connecting $\mathcal{A}$ and $\mathcal{B}$ such that

any point $a \in \mathcal{A}$ is swept by $\mathcal{F}(t)$ to its projection on $\mathcal{B}$,

i.e. for *any* initial datum $a \in \mathcal{A}$, we want $\mathrm{Proj}_{\mathcal{B}}(a)$ to be the final point of trajectory of the solution of the sweeping process driven by $\mathcal{F}(t)$.

This is consistent with the catching-up algorithm.

A first attempt



A convex-valued geodesic which works

If $\rho := d_{\mathcal{H}}(\mathcal{A}, \mathcal{B})$

$$\mathcal{G}_{\mathcal{A},\mathcal{B}}(t) := (\mathcal{A} + B_{t\rho}(0)) \cap (\mathcal{B} + B_{(1-t)\rho}(0)), \quad t \in [0,1]\,,$$

is a geodesic connecting $\mathcal{A}$ to $\mathcal{B}$.

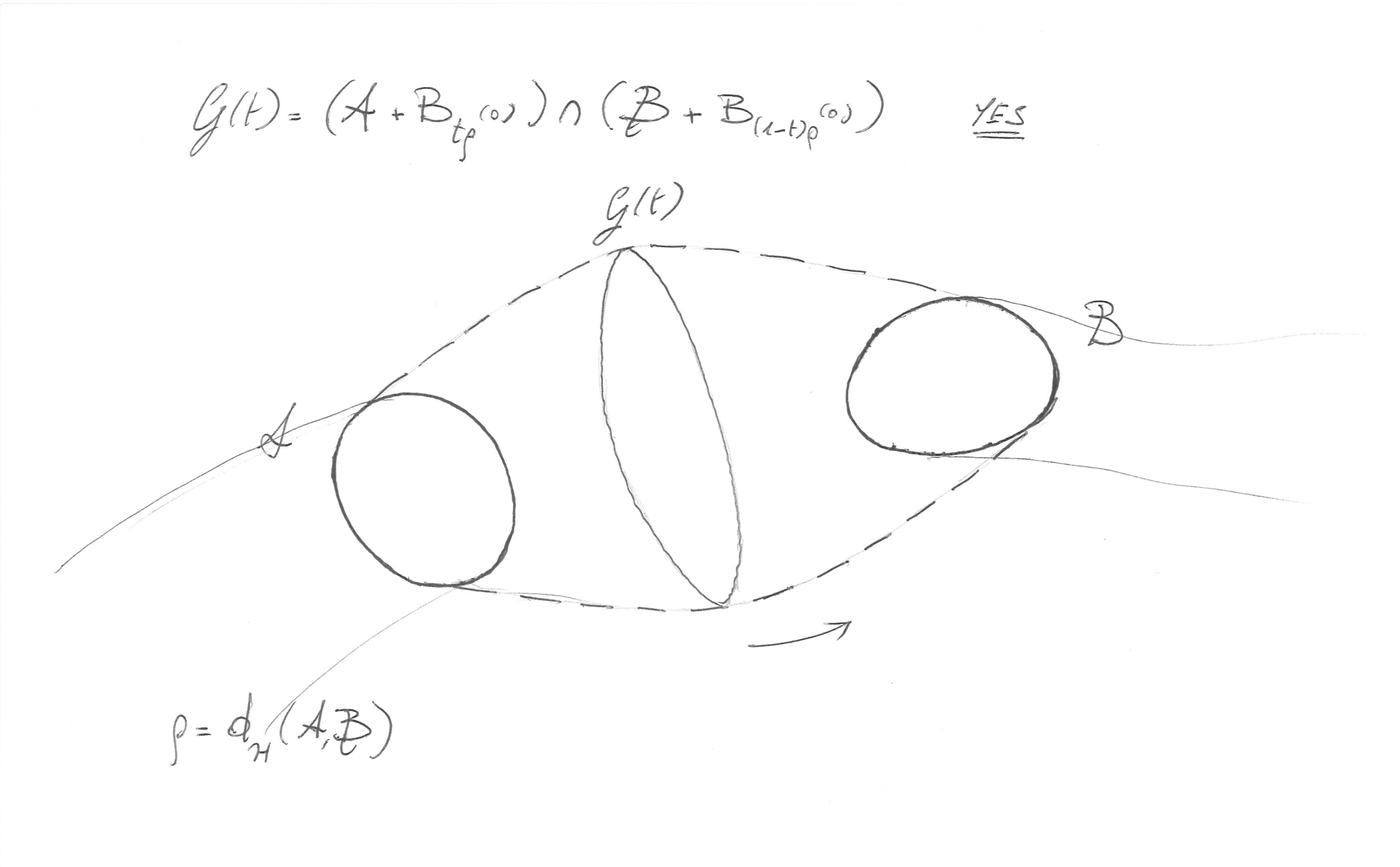
For any $a \in \mathcal{A}$, the solution $y \in Lip([0,1]\,;\mathcal{H})$ of

$$\begin{cases} y(t) \in \mathcal{G}(t) & \forall t \in [0,1]\,, \\ -y'(t) \in N_{\mathcal{G}(t)}(y(t)) & \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,1], \\ y(0) = a \end{cases}$$

satisfies

$$y(1) = \mathrm{Proj}_{\mathcal{B}}(a).$$

The geodesic $\mathcal{G}_{\mathcal{A},\mathcal{B}}$



Reduction from *BV* to *Lip*

Theorem (V. Recupero, *JCA*, 2016)**.**
If

$$\mathcal{C} \in BV([0,T]\,;\mathscr{C}_{\mathcal{H}}), \quad \ell_{\mathcal{C}}(t) := T\,\mathrm{V}(\mathcal{C},[0,t])/\,\mathrm{V}(\mathcal{C},[0,T]),$$

then $\exists!\widetilde{\mathcal{C}} \in Lip([0,T]\,;\mathscr{C}_{\mathcal{H}})$ *such that*

$$\mathcal{C} = \widetilde{\mathcal{C}} \circ \ell_{\mathcal{C}}$$

$$\widetilde{\mathcal{C}} \quad \textit{is the geodesic } \mathcal{G}_{\mathcal{C}(t-),\mathcal{C}(t+)} \quad \textit{on } [\ell_{\mathcal{C}}(t-),\ell_{\mathcal{C}}(t+)]\,.$$

and $\mathsf{M}(\widetilde{\mathcal{C}}) \circ \ell_{\mathcal{C}}$ *"easily" solves the* BV*-sweeping process, i.e.*

$$\mathsf{M}(\widetilde{\mathcal{C}}) \circ \ell_{\mathcal{C}} = \mathsf{M}(\mathcal{C})$$

existence, continuous dependence, convergence
of the catching-up algorithm are deduced from the Lip-case.

"Easily" means: basic measure theory tools

If

$$f \in Lip([0,T]\,;\mathcal{H}), \quad h : [0,T] \longrightarrow [0,T]\,\text{increasing}\ ,$$

then

$$\begin{cases} \mathrm{D}h(h^{-1}(B)) = \mathscr{L}^1(B) \quad \forall B \in \mathscr{B}(h(\mathrm{Cont}(h))). \\ \mathrm{D}(f \circ h) = g\,\mathrm{D}h \end{cases}$$

where

$$g(t) := \begin{cases} f'(h(t)) & \text{if } t \in \mathrm{Cont}(h) \\[2ex] \dfrac{f(h(t+)) - f(h(t-))}{h(t+) - h(t-)} & \text{if } t \in \mathrm{Discont}(h) \end{cases}$$

“Easily” means: simple measure theory proof

$$\mathrm{D}\mathsf{M}(\mathcal{C}) = \mathrm{D}\big(\mathsf{M}(\widetilde{\mathcal{C}}) \circ \ell_{\mathcal{C}}\big) = g\,\mathrm{D}\ell_{\mathcal{C}}$$

where

$$g(t) = \begin{cases} (\mathsf{M}(\widetilde{\mathcal{C}}))'(\ell_{\mathcal{C}}(t)) & \text{if } t \in \mathrm{Cont}(h) \\[2em] \dfrac{\mathsf{M}(\widetilde{\mathcal{C}})(\ell_{\mathcal{C}}(t+)) - \mathsf{M}(\widetilde{\mathcal{C}})(\ell_{\mathcal{C}}(t-))}{\ell_{\mathcal{C}}(t+) - \ell_{\mathcal{C}}(t-)} & \text{if } t \in \mathrm{Discont}(h) \end{cases}$$

If $\widehat{y} := \mathsf{M}(\widetilde{\mathcal{C}}) \in Lip([0,T]\,;\mathcal{H})$ then

$$Z := \{t \ : \ -\widehat{y}\,'(t) \notin N_{\widetilde{\mathcal{C}}(t)}(\widehat{y}(t))\} \Longrightarrow \ \mathcal{L}^1(Z) = 0.$$

On continuity points:

$$\begin{aligned} &\mathrm{D}\ell_{\mathcal{C}}(\{t \in \mathrm{Cont}(\mathcal{C}) \ : \ -\widehat{y}\,'(\ell_{\mathcal{C}}(t)) \notin N_{\widetilde{\mathcal{C}}(\ell_{\mathcal{C}}(t))}(\widehat{y}(\ell_{\mathcal{C}}(t)))\}) \\ = &\mathrm{D}\ell_{\mathcal{C}}(\{t \in \mathrm{Cont}(\mathcal{C}) \ : \ \ell_{\mathcal{C}}(t) \in Z\}) = \mathrm{D}\ell_{\mathcal{C}}(\ell_{\mathcal{C}}^{-1}(Z)) = \mathcal{L}^1(Z) = 0. \end{aligned}$$

On jumps, i.e. on $\mathrm{Discont}(\mathcal{C})$: direct computation.

We finally get

$$-\widehat{y}\,'(\ell_{\mathcal{C}}(t)) \in N_{\widetilde{\mathcal{C}}(\ell_{\mathcal{C}}(t))}(\widehat{y}(\ell_{\mathcal{C}}(t))) \quad \text{for } \mathrm{D}\ell_{\mathcal{C}}\text{-a.e. } t \in [0,T]. \quad \blacksquare$$

The play operator: $\mathcal{C}(t) = \mathcal{C}_u(t) = u(t) - \mathcal{Z}$

$$u \in BV^r([0,T]\,;\mathcal{H}),$$

$$y := \mathsf{P}(u) = \mathsf{M}(\mathcal{C}_u) = \mathsf{M}(\widetilde{\mathcal{C}}_u) \circ \ell_{\mathcal{C}_u} = \mathsf{M}(\widetilde{\mathcal{C}}_u) \circ \ell_u,$$

$$\mathsf{Q}(u) := 2\mathsf{P}(u) - u = \mathsf{Q}(\widetilde{\mathcal{C}}_u) \circ \ell_u$$

$$\mathrm{D}\mathsf{Q}(u) = g_u \, \mathrm{D}\ell_u$$

where

$$g_u(t) = \begin{cases} \dfrac{\mathsf{Q}(\widetilde{\mathcal{C}}_u)(\ell_u(t+)) - \mathsf{Q}(\widetilde{\mathcal{C}}_u)(\ell_u(t-))}{\ell_u(t+) - \ell_u(t-)} & \text{if } t \in \mathrm{Discont}(u) \\ \mathsf{Q}(\widetilde{\mathcal{C}}_u)'(\ell_u(t)) & \text{if } t \in \mathrm{Cont}(u) \end{cases}$$

$$\|g_u(t)\| = \|\mathsf{Q}(\widetilde{\mathcal{C}}_u)'(\ell_u(t))\| = \|\widetilde{u}'(\ell_u(t))\| = \mathrm{V}(u,[0,T])/T \;\; \forall t \in \mathrm{Cont}(u)$$

$$\|u_n - u\|_{BV} \to 0 \quad \overset{?}{\Longrightarrow} \quad \|\mathsf{Q}(u_n) - \mathsf{Q}(u)\|_{BV} \to 0$$

$$\|\mathsf{Q}(u_n) - \mathsf{Q}(u)\|_\infty \to 0, \qquad \|g_{u_n}\| \leq C,$$

hence

$$\begin{aligned}
&\mathrm{V}(\mathsf{Q}(u_n) - \mathsf{Q}(u), [0,T]) \\
&= \|\,\mathrm{D}(\mathsf{Q}(u_n)) - \mathrm{D}(\mathsf{Q}(u))\| \\
&= \|g_{u_n}\,\mathrm{D}\ell_{u_n} - g_u\,\mathrm{D}\ell_u\| \\
&\leq \|g_{u_n}\,\mathrm{D}(\ell_{u_n} - \ell_u)\| + \|(g_{u_n} - g_u)\,\mathrm{D}\ell_u\| \\
&\leq C\|\,\mathrm{D}(\ell_{u_n} - \ell_u)\| + \int_{[0,T]} \|g_{u_n} - g_u\|\,\mathrm{d}\mathrm{D}\ell_u
\end{aligned}$$

Notation: change the index u_n into n, delete the index u

$$\|u_n - u\|_{BV} \to 0 \quad \Longrightarrow \quad \| \mathrm{D}(\ell_{u_n} - \ell_u)\| \to 0,$$

$$\|g_n\| \leq C \quad \Longrightarrow \quad g_n \rightharpoonup z \qquad \text{in } L^p(\mathrm{D}\ell; \mathcal{H}), \ \ p \in \,]1, \infty[$$

thus for every bounded $\phi : [0, T] \longrightarrow \mathcal{H}$ Borel:

$$\lim_{n\to\infty} \int_{[0,T]} \langle \phi(t), g_n(t) \rangle \, \mathrm{d}\,\mathrm{D}\ell_n(t) = \int_{[0,T]} \langle \phi(t), z(t) \rangle \, \mathrm{d}\,\mathrm{D}\ell(t)$$

On the other hand by Dunford-Pettis theorem for measures

$$\mathrm{D}\mathsf{Q}(u_n) = g_n \, \mathrm{D}\ell_n \rightharpoonup \mathrm{D}\mathsf{Q}(u) = g \, \mathrm{D}\ell \ \ \text{(Dunford-Pettis)}$$

therefore

$$\lim_{n\to\infty} \int_{[0,T]} \langle \phi(t), g_n(t) \rangle \, \mathrm{d}\,\mathrm{D}\ell_n(t) = \int_{[0,T]} \langle \phi(t), g(t) \rangle \, \mathrm{d}\,\mathrm{D}\ell(t)$$

So we have found:

$$\lim_{n\to\infty}\int_{[0,T]}\langle\phi(t),g_n(t)\rangle\,\mathrm{d}\mathrm{D}\ell_n(t)=\int_{[0,T]}\langle\phi(t),z(t)\rangle\,\mathrm{d}\mathrm{D}\ell(t);$$

$$\lim_{n\to\infty}\int_{[0,T]}\langle\phi(t),g_n(t)\rangle\,\mathrm{d}\mathrm{D}\ell_n(t)=\int_{[0,T]}\langle\phi(t),g(t)\rangle\,\mathrm{d}\mathrm{D}\ell(t)$$

hence

$$\int_{[0,T]}\langle\phi(t),z(t)\rangle\,\mathrm{d}\mathrm{D}\ell_u(t)=\int_{[0,T]}\langle\phi(t),g(t)\rangle\,\mathrm{d}\mathrm{D}\ell_u(t)$$

$$\begin{aligned}
\Longrightarrow\quad & z\,\mathrm{D}\ell = g\,\mathrm{D}\ell \\
\Longrightarrow\quad & z = g \qquad\qquad \mathrm{D}\ell\text{-a.e.} \\
\Longrightarrow\quad & g_n \rightharpoonup g \qquad\qquad \text{in } L^p(\mathrm{D}\ell;\mathcal{H}),\ \ p\in\,]1,\infty[
\end{aligned}$$

For every $t \in \mathrm{Cont}(u)$:

$$\|g_n(t)\| = \frac{\mathrm{V}(u_n, [0,T])}{T} \quad \to \quad \frac{\mathrm{V}(u, [0,T])}{T} = \|g(t)\|$$

For every $t \in \mathrm{Discont}(u)$:

$$\|g_n(t)\| = \frac{\mathsf{Q}(\widetilde{\mathcal{C}}_n)(\ell_n(t+)) - \mathsf{Q}(\widetilde{\mathcal{C}}_n)(\ell_n(t-))}{\ell_n(t+) - \ell_n(t-)} \to$$

$$\to \frac{\mathsf{Q}(\widetilde{\mathcal{C}})(\ell(t+)) - \mathsf{Q}(\widetilde{\mathcal{C}})(\ell(t-))}{\ell(t+) - \ell(t-)} = \|g(t)\|$$

$\Longrightarrow \|g_n\|_{L^p(\mathrm{D}\ell;\mathcal{H})} \to \|g\|_{L^p(\mathrm{D}\ell;\mathcal{H})}, \ \ p \in \,]1, \infty[$

$\Longrightarrow g_n \to g$ in $L^p(\mathrm{D}\ell;\mathcal{H}), \ \ p \in \,]1, \infty[$

$\Longrightarrow g_n \to g$ in $L^1(\mathrm{D}\ell;\mathcal{H}) \ \ (\mathrm{D}\ell([0,T]) = T < \infty)$

$$
\begin{aligned}
&\mathrm{V}(\mathsf{Q}(u_n) - \mathsf{Q}(u), [0,T]) \\
&= \|\, \mathrm{D}(\mathsf{Q}(u_n)) - \mathrm{D}(\mathsf{Q}(u))\| \\
&= \|g_n \,\mathrm{D}\ell_n - g\,\mathrm{D}\ell\| \\
&\leq \|g_n \,\mathrm{D}(\ell_n - \ell)\| + \|(g_n - g)\,\mathrm{D}\ell\| \\
&\leq C\|\,\mathrm{D}(\ell_n - \ell)\| + \int_{[0,T]} \|g_n - g\| \,\mathrm{d}\,\mathrm{D}\ell \to 0
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{V}(\mathsf{Q}(u_n) - \mathsf{Q}(u), [0,T]) \\
&= \| \mathrm{D}(\mathsf{Q}(u_n)) - \mathrm{D}(\mathsf{Q}(u)) \| \\
&= \| g_n \, \mathrm{D}\ell_n - g \, \mathrm{D}\ell \| \\
&\leq \| g_n \, \mathrm{D}(\ell_n - \ell) \| + \| (g_n - g) \, \mathrm{D}\ell \| \\
&\leq C \| \mathrm{D}(\ell_n - \ell) \| + \int_{[0,T]} \| g_n - g \| \, \mathrm{d}\,\mathrm{D}\ell \to 0
\end{aligned}
$$

Theorem (J. Kopfová, V. Recupero, *JDE* 2016)**.**
P *is BV-norm continuous.*

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