Optimal Control Problems with Mixed Constraints: Special Applications

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Control of state constrained dynamical systems
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We focus on the following

Problem

\[
\begin{aligned}
\text{(P)} & \quad \text{Minimize } \ell(x(0), x(1)) \\
\text{subject to} & \quad \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [0, 1] \\
& \quad (x(t), u(t)) \in S(t) \quad \text{a.e. } t \in [0, 1] \\
& \quad (x(0), x(1)) \in E
\end{aligned}
\]
Here we discuss some challenging problems related to $(P)$.

- **Main Focus on**
  
  How useful are known Necessary Conditions for problems related to OCP with MC?
Outline of this talk

- Regular Conditions for MC for Necessary Conditions.
- Different Problems (re)formulated as OCP with Mixed constraints:
  - Implicit Control Systems;
  - Applied Problems
    - Biomath Problems;
    - Robotics Problems;
    - Sweeping Process.

Known NCO for \((P)\) under regularity conditions on mixed constraints.

Is this enough for applications? We will see it is not!
Importance of Regularity Conditions

Under regularity conditions multipliers associated with mixed constraints are not measures!

Literature is sparse as far as NCO for irregular mixed constraints!!!
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### General Form of Mixed Constraints

\[(x, u) \in S(t)\]

### Radius Multifunction

Let \( R \) be multifunction:

\[R(t) \text{ open, convex and } \forall t \exists r_0 > 0: r_0 B \subset R(t).\]

Set

\[S_\varepsilon^*(t) := S(t) \cap (x^*(t) + \varepsilon B) \times R(t)\]
Regularity Conditions for General Form of MC

Let $R$ be multifunction:

- $R(t)$ open, convex and $\forall t \ \exists r_0 > 0: r_0 B \subseteq R(t)$. 

Regularity Condition (CdP2010) (BS):

There exists a measurable function $k$ such that, for almost every $t$ and $(x, u) \in S^\star$, $(\xi, \eta) \in N_P S(t)(x, u)$:

$$|\eta| \leq k(t) |\xi|$$

and $k(t) > 0$ a.e.
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Regularity Condition (CdP2010)

- **(BS):** There exists a measurable function $k$ such that, for almost every $t \in [0, 1]$,

$$(x, u) \in S^*_\epsilon(t), \ (\alpha, \beta) \in N^P_{S(t)}(x, u) \implies |\alpha| \leq k(t)|\beta|$$

and $k(t) \geq k_0 > 0$ a.e.
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Observation

Regularity Condition (CdP2010)

- **(BS):** There exists a measurable function $k$ such that, for almost every $t \in [0, 1]$,

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and $k(t) \geq k_0 > 0$ a.e.

This condition naturally excludes the case where

$$ S(T) = X(t) \times U(t) $$
Is \((BS)\) in some sense a \textit{minimal} regularity condition?
Is \( (BS) \) in some sense a \textit{minimal} regularity condition?

Can we have \textit{“treatable”} measures as multipliers for some irregular mixed constraints?
For $t \in [0, 1] = \bigcup_{i=1}^{7} l_i$ we have

\begin{align*}
(i) & \quad (\bar{x}(t), \bar{u}(t)) \in \partial S, \quad \text{if} \quad t \in l_4. \quad \text{(BS) does not holds.} \\
(ii) & \quad (\bar{x}(t), \bar{u}(t)) \in \partial S, \quad \text{if} \quad t \in l_6. \quad \text{(BS) does not holds.}
\end{align*}
In this situation known necessary conditions under regularity conditions do not apply!!!

**Multiplier???

Would the multiplier $\mu$ associated with Mixed Constraint $(x, u) \in S(t)$

$t \in l_4$ and $t \in l_6$ behave like a multiplier for state constraints of the form

$x(t) \leq x_4, \ t \in l_4, \quad x(t) \leq x_6, \ t \in l_6$?
Going Back to Regularity Conditions...

**General Form:** \((x(t), u(t)) \in S(t), \ S(t)\) locally closed.

**Special Forms of Mixed Constraints:**
- \(S(t) = \{(x, u) : h(t, x, u) = 0, \ u \in U(t)\}\): equality constraints,
- \(S(t) = \{(x, u) : g(t, x, u) \leq 0, \ u \in U(t)\}\): inequality constraints,
- \(S(t) = \{(x, u) : h(t, x, u) = 0, \ g(t, x, u) \leq 0, \ u \in U(t)\}\): both equality and inequality,
- \(S(t) = \{(x, u) : \phi(t, x, u) \in \Psi(t)\}\),

where \(t \rightarrow U(t), \ \Psi(t), \ S(t)\) are set-valued functions.
Regularity Conditions for MC

Taking advantage of specific forms of Mixed Constraints

define **Regularity conditions (RC)** such that

\[(RC) \implies (BS)\]

or

\[(RC) \iff (BS).\]
Regularity Conditions for Geometric Form of MC

Take $S(t) = \{(x, u) : \phi(t, x, u) \in \Psi(t)\}$ where

$$\phi(t, x, u) = (\phi_1(x, u), u), \quad \Psi(t) = \Psi_1 \times U.$$
Regularity Conditions for Geometric Form of MC

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Assume $\phi_1$ locally Lipschitz and
Regularity Conditions for Geometric Form of MC

Take \( S(t) = \{(x, u) : \phi(t, x, u) \in \Psi(t)\} \) where
\[
\phi(t, x, u) = (\phi_1(x, u), u), \quad \Psi(t) = \Psi_1 \times U.
\]

Assume \( \phi_1 \) locally Lipschitz and CCQ

there exists constant \( M \) such that, for almost every \( t \in [0, 1] \), all \((x, u) \in S_{\epsilon}^*(t)\) and all \((\lambda, \mu) \in N_{\Psi_1}^L(\phi_1(x, u)) \times N_{U}^L(u)\), we have
\[
(\alpha, \beta - \mu) \in \partial_{x,u}^L\langle \lambda, \phi_1(x, u) \rangle \implies |\alpha| \leq M|\beta|.
\]
Regularity Conditions for Geometric Form of MC

Take $S(t) = \{(x, u) : \phi(t, x, u) \in \Psi(t)\}$ where

$$\phi(t, x, u) = (\phi_1(x, u), u), \quad \Psi(t) = \Psi_1 \times U.$$ 

Assume $\phi_1$ locally Lipschitz and

CCQ

there exists constant $M$ such that, for almost every $t \in [0, 1]$, all $(x, u) \in S^*_\epsilon(t)$ and all $(\lambda, \mu) \in N^L_{\Psi_1}(\phi_1(x, u)) \times N^L_U(u)$, we have

$$(\alpha, \beta - \mu) \in \partial^L_{x, u} \langle \lambda, \phi_1(x, u) \rangle \implies |\alpha| \leq M|\beta|.$$ 

Then

$$\text{CCQ} \implies \text{BS} \text{ with } k(t) = ML^\phi_1(t)$$
Example: Implicit Systems

Minimize \( l(x(0), x(1)) \)

subject to

\[
\left\{ \begin{array}{l}
 f(t, x(t), \dot{x}(t), u(t)) \in \Psi \text{ a.e.,} \\
 u(t) \in U \text{ a.e.,} \\
 (x(0), x(1)) \in E.
\end{array} \right.
\]
Example: Implicit Systems

Minimize \( l(x(0), x(1)) \)
subject to
\[
\begin{align*}
  f(t, x(t), \dot{x}(t), u(t)) &\in \Psi \text{ a.e.}, \\
  u(t) &\in U \text{ a.e.}, \\
  (x(0), x(1)) &\in E.
\end{align*}
\]

Reformulation as a OCP with MC

\[
(f(t, x, \dot{x}, u), u) \in \Psi \times U
\]

reformulated as
\[
\begin{align*}
  \dot{x}(t) &= v(t), \\
  (f(t, x(t), v(t), u(t)), u(t)) &\in \Psi \times U.
\end{align*}
\]
### Strict Differentiable Version of MP

Let \((x^*, u^*)\) be a strong local minimizer for \((I)\) where \(f\) strictly differentiable and \(\Psi, U\) regular sets.

There exist \(p \in W^{1,1}([a, b]; \mathbb{R}^n)\), measurable functions \(\lambda(t) \in N_{\Psi}^C(x^*(t), \dot{x}^*(t), u^*(t))\) and \(\mu(t) \in N_U^C(u^*(t))\), and a scalar \(\lambda_0 \geq 0\) such that:

\[
||p||_\infty + \lambda_0 > 0,
\]

\[
(p(0), -p(1)) \in N_E^L(x^*(0), x^*(1)) + \lambda_0 \partial^L I(x^*(0), x^*(1)),
\]

\[
(-\dot{p}(t), 0, -\mu(t)) \in \partial^C_{x, \nu, u} \langle p(t), \dot{x}^*(t) \rangle - \partial^C_{x, \nu, u} \langle \lambda(t), f(t, x^*(t), \dot{x}^*(t), u^*(t)) \rangle
\]

and, for all \((\nu, u)\) such that \(u \in U\) and \(f(t, x^*(t), \nu, u) \in \Psi\), we have

\[
\langle p(t), \nu \rangle \leq \langle p(t), \dot{x}^*(t) \rangle.
\]
Assume further that \((x, v, u) \to f(x, v, u)\) is smooth, \(\Psi = \{0\}\). Then

**Smooth Version of MP: Adjoint Inclusion**

\[
\lambda(t) = f^+_v(t, x^*(t), \dot{x}^*(t), u^*(t))p(t)
\]

and

\[
\dot{p}(t) = (\nabla_x f(t, x^*(t), \dot{x}^*(t), u^*(t)))^T f^+_v(t, x^*(t), \dot{x}^*(t), u^*(t))p(t)
\]

\[
-\mu(t) = (\nabla_u f(t, x^*(t), \dot{x}^*(t), u^*(t)))^T f^+_v(t, x^*(t), \dot{x}^*(t), u^*(t))p(t),
\]

where, as before, \(\mu \in N^C(u^*(t))\) and

\[
f^+_v(t) = \left(\nabla_v f(t)(\nabla_v f(t))^T\right)^{-1} \nabla_v f(t),
\]
APPLIED PROBLEMS where mixed constraints (regular or irregular) play a role
SEIR Models with Vaccination

The SEIR model with vaccination is represented by the following diagram:

- **S** (Susceptible) transitions to **E** (Exposed) with rate $bN$.
- **E** (Exposed) transitions to **I** (Infected) with rate $\frac{cSI}{N}$.
- **I** (Infected) transitions to **R** (Removed) with rate $gI$.
- **R** (Removed) transitions back to **S** (Susceptible) with rate $uS$.

The diagram also shows:
- $dS$, $dE$, $dl+al$, and $dR$ as the rates for loss processes.
Control of Infectious Diseases

SEIR Models: Equations

\[
\begin{align*}
\dot{S}(t) &= bN(t) - dS(t) - cS(t)I(t) - u(t)S(t), \quad S(0) = S_0, \\
\dot{E}(t) &= cS(t)I(t) - (e + d)E(t), \quad E(0) = E_0, \\
\dot{I}(t) &= eE(t) - (g + a + d)I(t), \quad I(0) = I_0, \\
\dot{N}(t) &= (b - d)N(t) - aI(t), \quad N(0) = N_0, \\
u(t)S(t) &\leq \varphi(t), \quad \text{for a.e. } t \in [0, T], \\
u(t) &\in [0, 1] \text{ for a.e. } t \in [0, T],
\end{align*}
\]

Mixed Constraint: Bound vaccines per unit of time

\[u(t)S(t) \leq \varphi(t): \text{ number of vaccinated individuals bounded by } \varphi(t) > 0.\]
Path Planning of Autonomous Vehicles

Motion of a AUV driven by

\[ \dot{x}(t) = f(x(t)) + g(x(t))u(t) \]

should go from one point \( x(0) \) to a target \( T \) in \( T \) units of time such that the vehicle must "live"

- in \( \Omega \) for more than \( \theta_1 \) units of time (e.g., for communications): \( \Omega \) is a closed set.
- in \( \mathcal{O} \) for less than \( \theta_2 \) units of time (e.g., dangerous area): \( \mathcal{O} \) is an open set.
Formulation (by A. Désille, H. Zidani, dP)

\[
\begin{pmatrix}
\dot{z}_1(t) \\
\dot{z}_2(t)
\end{pmatrix}
= \begin{pmatrix}
\beta_1(t) \\
\beta_2(t)
\end{pmatrix}
\]

where

\[\beta_i(t) \in [0, 1], \ i = 1, 2, \text{ Set Control Constraints}\]
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where

- \( \beta_i(t) \in [0, 1], \ i = 1, 2, \) Set Control Constraints
- \( \beta_1(t)d(x(t), \Omega) \leq 0, \) Mixed Constraints
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- \( \beta_i(t) \in [0, 1], \ i = 1, 2, \) Set Control Constraints
- \( \beta_1(t)d(x(t), \Omega) \leq 0, \) Mixed Constraints
- \( (1 - \beta_2(t))d(x(t), \mathcal{O}^c) \leq 0, \) Mixed Constraints
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where

- \( \beta_i(t) \in [0, 1], \ i = 1, 2, \) Set Control Constraints
- \( \beta_1(t)d(x(t), \Omega) \leq 0, \) Mixed Constraints
- \( (1 - \beta_2(t))d(x(t), O^c) \leq 0, \) Mixed Constraints
- \( z_1(T) \geq \theta_1 \) and \( z_2(T) \leq \theta_2, \) End Point Constraints
Formulation (by A. Désille, H. Zidani, dP)

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where

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- \( \beta_1(t)d(x(t), \Omega) \leq 0, \) Mixed Constraints
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- \( z_1(T) \geq \theta_1 \) and \( z_2(T) \leq \theta_2, \) End Point Constraints

So we guarantee that

- the length of the time interval where \( x \) lives in \( \Omega \) is \( \geq \theta_1 \) and
- the length of the time interval where \( x \) lives in \( \mathcal{O} \) is \( \leq \theta_2. \)
Minimize $\phi(x(1))$

over processes $(x, u)$ such that

$$\dot{x}(t) \in f(x(t), u(t)) - N_C(x(t)), \quad \text{a.e. } t \in [0, 1],$$

$$u(t) \in U, \quad \text{a.e. } t \in [0, 1],$$

$$(x(0), x(1)) \in C_0 \times C_1.$$
Minimize \( \phi(x(1)) \)

over processes \((x, u)\) such that

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)) - \lambda(t)\nabla\psi(x(t)), \quad \text{a.e. } t \in [0, 1], \\
\psi(x(t)) &\leq 0 \quad \text{for all } t \in [0, 1] \\
-\lambda(t)\psi(x(t)) &\leq 0 \quad \text{a.e. } t \in [0, 1], \\
(u(t), \lambda(t)) &\in U \times \{\lambda \in \mathbb{R} : \lambda \geq 0\} \quad \text{a.e. } t \in [0, 1], \\
(x(0), x(1)) &\in C_0 \times C_1.
\end{align*}
\]
Challenges

(SP) and (AUV) Mixed Constraints are irregular!!!

- Can we have “treatable” measures as multipliers for these and others irregular mixed constraints?

- Can we use the special structure of problems to get “treatable” measures as multipliers for some irregular mixed constraints?
Literature on this sort of problems


- An Li and Jane J. Ye, Necessary optimality conditions for implicit control systems with applications to control of differential algebraic equations, Preprint, 2017.
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Thank You for your attention!
Questions / Remarks?