Optimal Control Problems with Mixed Constraints: Special Applications

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Control of state constrained dynamical systems
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We focus on the following

Problem

$$
(\mathrm{P}) \quad \begin{cases} \text{Minimize } l(x(0), x(1)) \\ \text{subject to} \\ \qquad \dot{x}(t) = f(t, x(t), u(t)) & \text{a.e.} \quad t \in [0,1] \\ \qquad (x(t), u(t)) \in S(t) & \text{a.e.} \quad t \in [0,1] \\ \qquad (x(0), x(1)) \in E \end{cases}
$$

Goal

Here we discuss some challenging problems related to (P).

- **Main Focus on**

  How useful are known Necessary Conditions for problems related to OCP with MC?

Outline

Outline of this talk

- Regulary Conditions for MC for Necessary Conditions.
- Different Problems (re)formulated as OCP with Mixed constraints:
  - Implicit Control Systems;
  - Applied Problems
    - Biomath Problems;
    - Robotics Problems;
    - Sweeping Process.

Known NCO for (P) under regularity conditions on mixed constraints.

Is this enough for applications? We will see it is not!

Importance of Regularity Conditions

Under regularity conditions multipliers associated with mixed constraints are not meaures!

Literature is sparse as far as NCO for irregular mixed constraints!!!

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See V. Dmitruk, On the development of Pontryagins maximum principle in the works of A. Ya. Dubovitskii and A. A. Milyutin, Control Cybernet., 38 (2009) *on Dubovitskii–Milyutin scheme for irregular mixed constraints* : multipliers in dual of (L^∞) .

General Form of Mixed Constraints

$$(x, u) \in S(t)$$

Radius Multifunction

Let R be multifunction:

$R(t)$ open, convex and $\forall\ t\ \ \exists r_0 > 0$: $r_0\mathbf{B} \subset R(t)$.

Set

$$S^*_\epsilon(t) := S(t) \cap (x^*(t) + \epsilon\mathbf{B}) \times R(t)$$

Regularity Conditions for General Form of MC

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Regularity Condition (CdP2010)

- **(BS)**: There exists a measurable function k such that, for almost every $t \in [0, 1]$,

$$(x, u) \in S^*_\epsilon(t), \ \ (\alpha, \beta) \in N^P_{S(t)}(x, u) \Longrightarrow |\alpha| \leq k(t)|\beta|$$

and $k(t) \geq k_0 > 0$ a.e.

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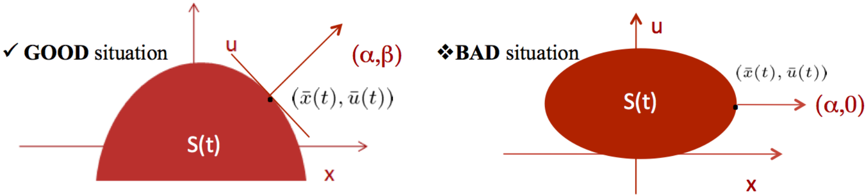

Observation

Regularity Condition (CdP2010)

- **(BS)**: There exists a measurable function k such that, for almost every $t \in [0,1]$,

$$(x,u) \in S_\epsilon^*(t), \quad (\alpha,\beta) \in N_{S(t)}^P(x,u) \Longrightarrow |\alpha| \le k(t)|\beta|$$

and $k(t) \ge k_0 > 0$ a.e.

This condition naturally excludes the case where

$$S(T) = X(t) \times U(t)$$

Question

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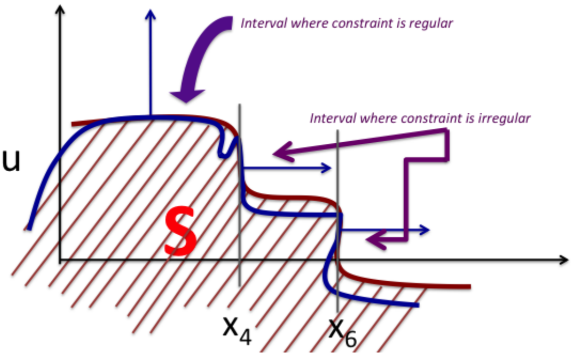
- Can we have **“treatable”** measures as multipliers for some irregular mixed constraints?



For $t \in [0,1] = \bigcup_{i=1}^{7} I_i$ we have

$$\begin{cases} (i) \quad (\bar{x}(t), \bar{u}(t)) \in \partial S, \quad \text{if} \quad t \in I_4. \quad \text{(BS) does not holds.} \\ (ii) \quad (\bar{x}(t), \bar{u}(t)) \in \partial S, \quad \text{if} \quad t \in I_6. \quad \text{(BS) does not holds.} \end{cases}$$

Question

In this situation known necessary conditions under regularity conditions do not apply!!!

Multiplier???

Would the multiplier μ associated with Mixed Constraint

$$(x, u) \in S(t)$$

$t \in I_4$ and $t \in I_6$ behave like

a multiplier for state constraints of the form

$$x(t) \leq x_4, \quad t \in I_4, \qquad x(t) \leq x_6, \quad t \in I_6?$$

Going Back to Regularity Conditions...

General Form: $(x(t), u(t)) \in S(t),\ S(t)$ locally closed.

Special Forms of Mixed Constraints:

- $S(t) = \{(x, u) :\ h(t, x, u) = 0,\ u \in U(t)\}$: equality constraints,
- $S(t) = \{(x, u) :\ g(t, x, u)) \leq 0,\ u \in U(t)\}$: inequality constraints,
- $S(t) = \{(x, u) :\ h(t, x, u) = 0,\ g(t, x, u) \leq 0,\ u \in U(t)\}$: both equality and inequality,
- $S(t) = \{(x, u) : \phi(t, x, u) \in \Psi(t)\}$,

where $t \to U(t),\ \Psi(t),\ S(t)$ are set-valued functions.

Regularity Conditions for MC

Taking advantage of specific forms of Mixed Constraints

define **Regularity conditions (RC)** such that

$$(RC) \Longrightarrow (BS)$$

or

$$(RC) \Longleftrightarrow (BS).$$

Regularity Conditions for Geometric Form of MC

Take $S(t) = \{(x, u) : \phi(t, x, u) \in \Psi(t)\}$ where

$$\phi(t, x, u) = (\phi_1(x, u), u), \quad \Psi(t) = \Psi_1 \times U.$$

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Assume ϕ_1 locally Lipschitz and

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Assume ϕ_1 locally Lipschitz and

CCQ

there exists constant M such that, for almost every $t \in [0,1]$, all $(x,u) \in S^*_\varepsilon(t)$ and all $(\lambda, \mu) \in N^L_{\Psi_1}(\phi_1(x,u)) \times N^L_U(u)$, we have

$$(\alpha, \beta - \mu) \in \partial^L_{x,u} \langle \lambda, \phi_1(x,u) \rangle \implies |\alpha| \leq M|\beta|.$$

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Then

$$\textbf{CCQ} \implies \textbf{BS} \text{ with } k(t) = ML_x^{\phi_1}(t)$$

Example: Implicit Systems

$$
(I) \qquad \begin{cases} \text{Minimize} \quad l(x(0), x(1)) \\ \text{subject to} \\ \qquad\qquad f(t, x(t), \dot{x}(t), u(t)) \in \Psi \text{ a.e.,} \\ \qquad\qquad u(t) \in U \text{ a.e.,} \\ \qquad\qquad (x(0), x(1)) \in E. \end{cases}
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$$

Reformulation as a OCP with MC

$$
\boxed{(f(t, x, \dot{x}, u), u) \in \Psi \times U}
$$

reformulated as

$$
\begin{array}{rcl} \dot{x}(t) & = & v(t), \\ (f(t, x(t), v(t), u(t)), u(t)) & \in & \Psi \times U. \end{array}
$$

MP for OCP with Implicit Systems

Strict Differentiable Version of MP

Let (x^*, u^*) be a strong local minimizer for (I) where f strictly differentiable and Ψ, U *regular* sets.
There exist $p \in W^{1,1}([a,b];\mathbf{R}^n)$, measurable functions $\lambda(t) \in N^C_{\Psi_1}(x^*(t), \dot{x}^*(t), u^*(t))$ and $\mu(t) \in N^C_U(u^*(t))$, and a scalar $\lambda_0 \geq 0$ such that:

$$||p||_\infty + \lambda_0 > 0,$$

$$(p(0), -p(1)) \in N^L_E(x^*(0), x^*(1)) + \lambda_0 \partial^L l(x^*(0), x^*(1)),$$

$$(-\dot{p}(t), 0, -\mu(t)) \in \partial^C_{x,v,u}\langle p(t), \dot{x}^*(t)\rangle - \partial^C_{x,v,u}\langle \lambda(t), f(t, x^*(t), \dot{x}^*(t), u^*(t))\rangle$$

and, for all (v, u) such that $u \in U$ and $f(t, x^*(t), v, u) \in \Psi$, we have

$$\langle p(t), v\rangle \leq \langle p(t), \dot{x}^*(t)\rangle.$$

MP for OCP with Implicit Systems

Assume further that $(x, v, u) \to f(x, v, u)$ is smooth, $\Psi = \{0\}$. Then

Smooth Version of MP: Adjoint Inclusion

$$\lambda(t) = f_v^+(t, x^*(t), \dot{x}^*(t), u^*(t))p(t)$$

and

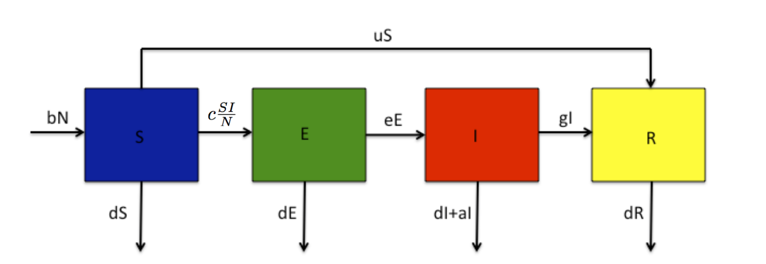
$$\begin{aligned} \dot{p}(t) &= (\nabla_x f(t, x^*(t), \dot{x}^*(t), u^*(t)))^T f_v^+(t, x^*(t), \dot{x}^*(t), u^*(t))p(t) \\ -\mu(t) &= (\nabla_u f(t, x^*(t), \dot{x}^*(t), u^*(t)))^T f_v^+(t, x^*(t), \dot{x}^*(t), u^*(t))p(t), \end{aligned}$$

where, as before, $\mu \in N_U^C(u^*(t))$ and

$$f_v^+(t) = \left(\nabla_v f(t)(\nabla_v f(t))^T\right)^{-1} \nabla_v f(t),$$

APPLIED PROBLEMS where mixed constraints (regular or irregular) play a role

SEIR Models with Vaccination



Control of Infectious Diseases

SEIR Models: Equations

$$\begin{cases} \dot{S}(t) & = bN(t) - dS(t) - cS(t)I(t) - u(t)S(t), \; S(0) = S_0, \\ \dot{E}(t) & = cS(t)I(t) - (e+d)E(t), \; E(0) = E_0, \\ \dot{I}(t) & = eE(t) - (g+a+d)I(t), \; I(0) = I_0, \\ \dot{N}(t) & = (b-d)N(t) - aI(t), \; N(0) = N_0, \\ u(t)S(t) & \leq \varphi(t), \text{ for a.e. } \quad t \in [0, T], \\ u(t) & \in [0,1] \text{ for a.e. } \quad t \in [0, T], \end{cases}$$

Mixed Constraint: Bound vaccines per unit of time

$u(t)S(t) \leq \varphi(t)$: number of vaccinated individuals bounded by $\varphi(t) > 0$.

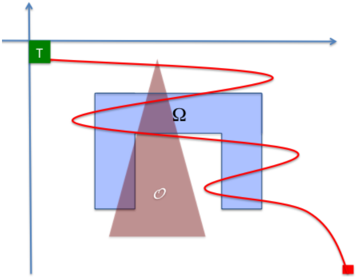
Path Planning of Autonomous Vehiches

Motion of a AUV driven by

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t)$$

should go from one point $x(0)$ to a target $\mathcal{T}$ in T units of time such that the vehicle must "live "

- in Ω for more than θ_1 units of time (e.g., for communications): Ω is a closed set.
- in $\mathcal{O}$ for less than θ_2 units of time (e.g., dangerous area): $\mathcal{O}$ is an open set.



Formulation (by A. Désille, H. Zidani, dP)

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix}$$ where

- $\beta_i(t) \in [0, 1]$, $i = 1, 2$, Set Control Constraints

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- $\beta_1(t) d(x(t), \Omega) \leq 0$, Mixed Constraints

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- $z_1(T) \geq \theta_1$ and $z_2(T) \leq \theta_2$, End Point Constraints

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So we guarantee that

- the *length* of the time interval where x lives in Ω is $\geq \theta_1$ and
- the *length* of the time interval where x lives in $\mathcal{O}$ is $\leq \theta_2$.

SP

$$
(SP) \begin{cases} \text{Minimize} \;\; \phi(x(1)) \\ \text{over processes } (x,u) \text{ such that} \\ \qquad \dot{x}(t) \in f(x(t),u(t)) - N_C(x(t)), \quad \text{a.e. } \; t \in [0,1], \\ \qquad u(t) \in U, \quad \text{a.e. } \; t \in [0,1], \\ \qquad (x(0),x(1)) \in C_0 \times C_1 \, . \end{cases}
$$

where $C := \{x \in \mathbf{R}^n : \; \psi(x) \leq 0\}$ for $\psi : \mathbf{R}^n \to \mathbf{R}$ *strictly convex* and $C^2(\mathbf{R}^n; \mathbf{R})$ function.

SP as OCP with MC

$$
(M)\begin{cases}
\text{Minimize } \ \phi(x(1)) \\
\text{over processes } (x,u) \text{ such that} \\
\quad \dot{x}(t) = f(x(t),u(t)) - \lambda(t)\nabla\psi(x(t)), \quad \text{a.e. } \ t\in[0,1], \\
\quad \psi(x(t)) \leq 0 \qquad\qquad\qquad \text{for all } \quad t\in[0,1] \\
\quad -\lambda(t)\psi(x(t)) \leq 0 \qquad\qquad\qquad \text{a.e. } \ t\in[0,1], \\
\quad (u(t),\lambda(t)) \in U\times\{\lambda\in\mathbf{R} : \lambda\geq 0\}\,, \quad \text{a.e. } \ t\in[0,1], \\
\quad (x(0),x(1)) \in C_0\times C_1 \ .
\end{cases}
$$

Challenges

(SP) and (AUV) Mixed Constraints **are irregular**!!!

- Can we have "treatable" measures as multipliers for these and others irregular mixed constraints?

- Can we use the special structure of problems to get "treatable" measures as multipliers for some irregular mixed constraints?

Literature on this sort of problems

- dP, *On necessary conditions for implicit control systems*, Pure and Applied Functional Analysis, Vol.1, N.2, pp. 185-196, 2016.
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Thank You for your attention!
Questions / Remarks?