Sweeping processes

and matrices

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Map

$$W : [Z(t), x_0] \to x(t)$$

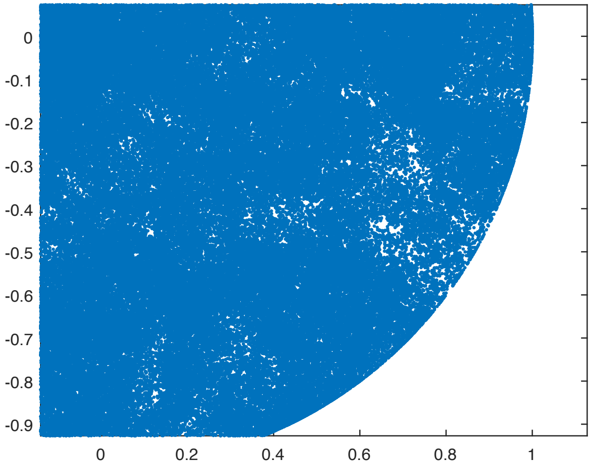
Definitions

- Differential inclusions
- Catching-up
- Minimal variation (lazy point) (?)

Not just convex sets!

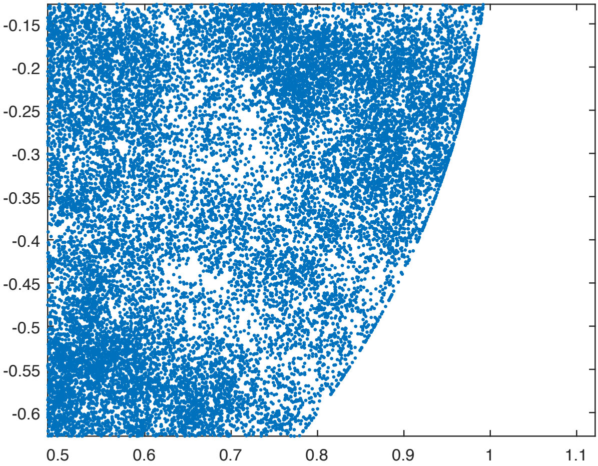
Brownian sweeping by a circle

Low resolution



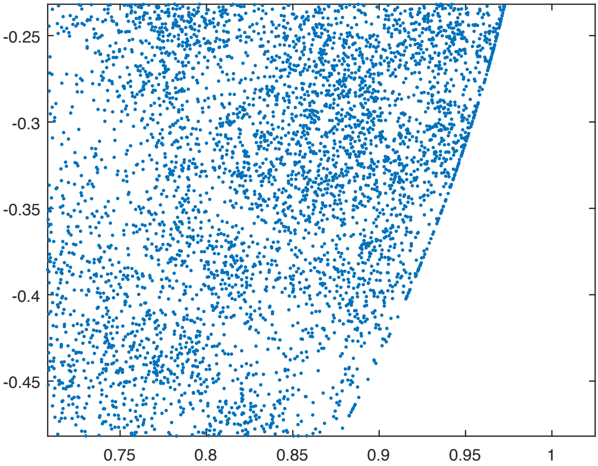
Brownian sweeping by a circle

Higher resolution



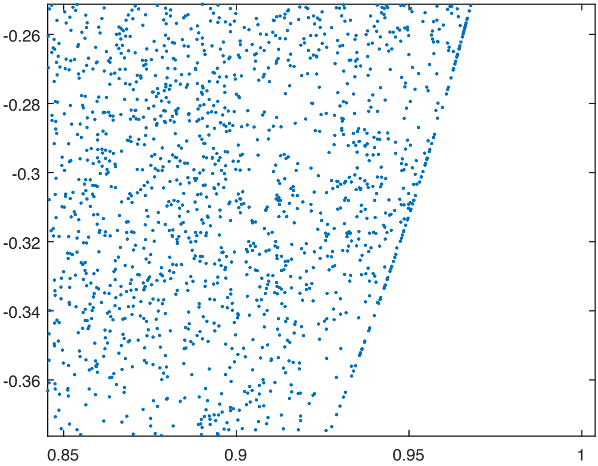
Brownian sweeping by a circle

High resolution



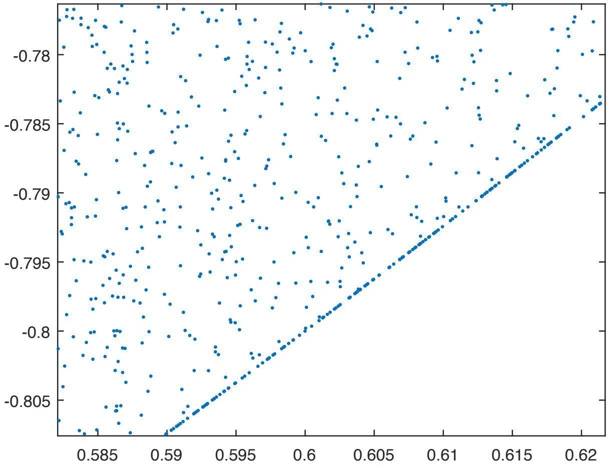
Brownian sweeping by a circle

Very high resolution



Brownian sweeping by a circle

20 000 000 points



Minimal variation

Theorem

For a scalar play, the solution has minimal variation among admissible paths starting at the same point.

This does not help much since there are many optimal paths. This is only natural: the variational principle does not depend on time direction. But it is at least rate-independent!

In dimensions 2 and more this is no longer true!

Weighted variation

Intuitively, it would be worth trying to use heavier weights for earlier times. Here we make a conjecture. It is true, clearly, for catching-up discrete-time processes. But this is not enough!

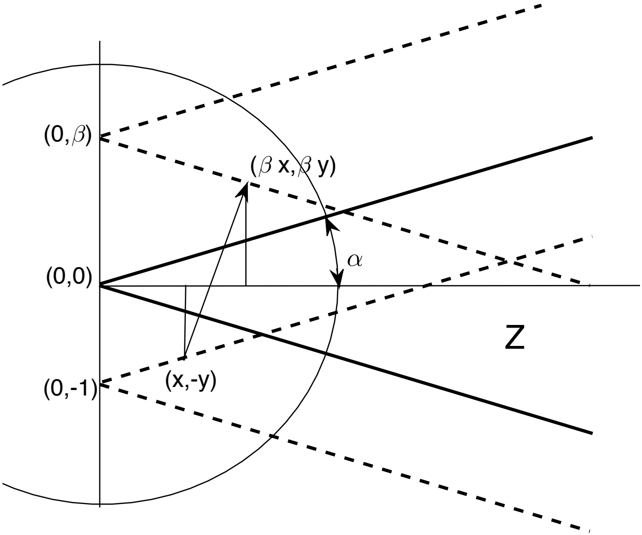
Conjecture

Let $z(t)$ be a solution of the sweeping process $Z(t)$. Denote by $y_a(t)$ one of the minimizers of

$$\int_0^T e^{-at} dV(y(t))$$

under constraints $y(0) = z(0)$, $y(t) \in Z(t)$ for all $t \in [0, T]$. Here $V(y(t))$ is the full variation of $y(\cdot)$ on $[0, t]$. Then $y_a(\cdot) \to z(\cdot)$ as $a \to \infty$.

Counterexample for $\mathbb{R}^2$



Counterexample

Let $Z(t) = Z + (0, h(t))$ (vertical oscillations only). We construct a self-similar linearly growing pair $h(t), z(t)$, where $z(t) = (x(t), y(t))$ is the unique solution. Due to rate-independence, we assume them piecewise-linear, say, $h(\beta^n) = (-\beta)^n$, $x(\beta^n) = x\beta^n$, and $y(\beta^n) = y \cdot (-\beta)^n$ for some $\beta > 1$.

We have $\Delta x = x(\beta) - x(1) = x(\beta - 1)$ and $\Delta y = y(\beta) - y(1) = y \cdot (\beta + 1)$. By the angles, we also have $\Delta x = \tan\alpha \Delta y$, that is, $x(\beta - 1) = \tan\alpha y(\beta + 1)$.

From the geometry, we also have $1 = y + x\tan\alpha$, and hence,

$$x = \frac{\tan\alpha(\beta + 1)}{\beta - 1 + (\beta + 1)\tan^2\alpha}.$$

Counterexample

The variation of solution on $[0,1]$ equals, of course, $x/\sin\alpha$ and is multiplied by β at each time step from $t=\beta^n$ to $t=\beta^{n+1}$. We construct another admissible path that would lie on the x-axis and would have variation $U(t)=\varepsilon V(t)$ for each $t\geq 0$.

So, what are the conditions of admissibility of such a path? Due to self-similarity, the only relation to hold is

$$U(1)\geq\frac{1}{\tan\alpha},\quad \text{that is,}\quad V(1)\geq\frac{1}{\varepsilon\tan\alpha}$$

which resolves to

$$\varepsilon\geq\sin\alpha\frac{\beta-1}{\beta+1}+\sin\alpha\tan^2\alpha.$$

Directional derivative

Map

Fix $Z(\cdot)$ and look at the map $F : x(0) \to x(T)$. For convex $Z(t)$, F is 1-Lipschitz, though non-smooth. What is the effect of small changes of $x(0)$ on $x(T)$? We may also introduce disturbance at $t = t_1, \ldots t_k \in [0, T]$.

Simple cases: catching-up

Projection on half-spaces

$$Z = \{z : \langle z, h\rangle \le a\}, \quad \langle h, h\rangle = 1$$

. Then $\Delta y = A\Delta x$, $A \in \Sigma = \{P, I\}$, where

$$P = I - hh' \qquad \text{(projection on a hyperplane, symmetric).}$$

$\langle z, h\rangle$ is the scalar product: $\langle z, h\rangle = z'h = h'z$.

Oblique projections

Here we project along $d \ne h$. Still we assume $\langle d, h\rangle > 0$. Then $y = x + bd$ and $\langle y, h\rangle = 0$. Hence $\langle x, h\rangle + b\langle d, h\rangle = 0$ and $b = -\frac{\langle x,h\rangle}{\langle d,h\rangle}$, that is,

$$P = I - dd'hh' \qquad \text{(no longer symmetric).}$$

Series of projections and more...

General matrices

$$\mathcal{A} = \{A_1 \dots A_k : k < \infty,\ A_i \in \Sigma\},$$

where Σ is a finite family of $n \times n$-matrices.

Say, we have a Σ-valued function of time $A(t)$, $0 \leq t \leq T$ (no continuity, of course, since Σ is discrete). Partitions

$$0 \leq t_1 < t_2 < \cdots < t_m \leq T$$

give products $A(t_1)A(t_2)\dots A(t_m)$.

Question

Is there a limit as partitions refine?

The set of finite partitions is a *directed set* with respect to inclusion.

Convergence of infinite products

By $\mathcal{L}(\Sigma)$ we denote the set of all finite products $\Pi_{i=1}^{m} A(i)$, $A(i) \in \Sigma$.

Definition

Σ is *product bounded* if there exists a $C > 0$ such that $\|A\| < C$ for all $A \in \mathcal{L}(\Sigma)$.

Definition

Σ is *LCP (left convergent products)* if, for any sequence $A(i) \in \Sigma,\ i = 1, 2, \ldots,$ there exists a limit matrix L_S such that

$$\lim_{m \to \infty} \|L_S^m - L_S\| = 0,$$

where $L_S^m = A(m) \ldots A(1), \quad m = 1, \ldots.$
…and RCP is LCP', that is, LCP for $\Sigma' = \{A_i'\}$.

They are different!

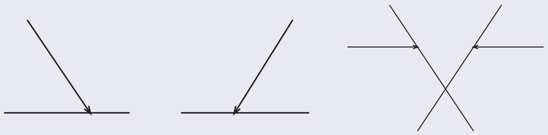
Example

LCP and RCP are not the same. Let

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $\Sigma = \{A_1, A_2\}$ is LCP and not RCP since $A_1 A_2 = A_2$ and $A_2 A_1 = A_1$.

Oblique projections



Equivalent definitions

Theorem

Σ is LCP if and only if all its paths $x_{i+1} = A(i)x_i$, $i = 1, \ldots,$ have bounded variation. Then the variation is also uniformly bounded over all $x_1 \in B(0,1)$.

Theorem

Σ is RCP if and only if each finite family of affine maps $x \to A(x+h) - h$, $A \in \Sigma$, $h \in \mathbb{R}^n$, generates a bounded semigroup.

Insertions and continuous products

Definition

Σ is called CP if substitutions of the form $B \to BA$ and $B \to AB$, $A \in \Sigma$, produce a converging sequence of finite products whenever we start with any finite product from $\mathcal{L}(\Sigma)$.

Theorem

Σ is CP iff it is LCP and RCP.

Continuous products of matrices

Let Σ be RCP and LCP. Then $\prod_{0 \le t \le T} A(t)$ is well defined for any map $A : [0, T] \to \Sigma$.

Good news and bad news

Add identity matrix

If Σ is CP then $\Sigma^* = \Sigma \cup I$ is also CP (follows from definitions).

Projections on half-spaces

Let us come back to sweeping processes. Any finite set of orthogonal projections is CP. Instead of linear maps (matrices), we consider projections on half spaces. But they are no longer CP (they are not LCP). Why? Because an insertion at the right affects the whole sequence of matrices at the left!

Question

For a general substitution rule (sort of $AB \to CBDA$), what are the conditions of CP property?

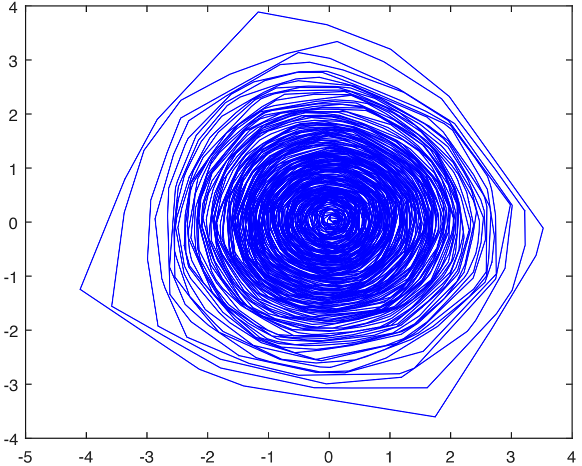
Hardness of recognition of CP property

Algorithmic insolvability

We have two options for Σ. If $\rho(\Sigma) < 1$, it is a CP family. The hardness of recognition of this case is still an open problem as far as we know. It is conjectured that the problem is algorithmically unsolvable for matrices with rational entries, the same way as it happens for a similar problem $\rho(\Sigma) \leq 1$.
The second case is $\rho(\Sigma) = 1$ and then the hardness of recognition is the same as in the first case

Strings

Embedded chain of sweeping sets.



Strings

Embedded chain of sweeping sets.