Average Cost Minimization Problems

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Conference, Control of state constrained dynamical systems
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Outline

Joint work with Piernicola Bettiol

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An Optimal Control Problem

Cauchy Problem

$$\dot{x}(t) = f(t, x(t)) \qquad x(0) = x_0$$

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◆ **Predicting but no altering the evolution**

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Control System

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad x(0) = x_0$$

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Control System with unknown parameters

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+ Performance Criterion ?

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+ Performance Criterion ? **Optimal Control**

An Optimal Control Problem

For a given μ (probability measure on Ω) and $g(x,\omega)$

$$
\begin{cases}
\text{minimize} \quad \int_\Omega g(x(T,\omega);\omega)\, d\mu(\omega) \quad \textbf{average cost} \\
\text{over } u : [0,T] \to \mathbb{R}^m \text{ and } W^{1,1} \text{ arcs } \{x(.,\omega)\} \\
\text{such that} \quad u(t) \in U(t) \quad \text{a.e. } t \in [0,T] \\
\text{and, for each } \omega \in \Omega, \\
\qquad \dot{x}(t,\omega) = f(t, x(t,\omega), u(t), \omega) \quad \text{a.e. } t \in [0,T], \\
\qquad x(0,\omega) = x_0 \quad \text{and} \quad x(T,\omega) \in C(\omega)\,.
\end{cases}
$$

Ω (set of unknown parameters) is a complete separable metric space

An Optimal Control Problem

For a given μ (probability measure on Ω) and $g(x,\omega)$

$$\begin{cases} \text{minimize} \quad \int_\Omega g(x(T,\omega);\omega)\, d\mu(\omega) \quad \textbf{average cost} \\ \text{over } u : [0,T] \to \mathbb{R}^m \text{ and } W^{1,1} \text{ arcs } \{x(.,\omega)\} \\ \text{such that} \quad u(t) \in U(t) \quad \text{a.e. } t \in [0,T] \\ \text{and, for each } \omega \in \Omega, \\ \qquad \dot{x}(t,\omega) = f(t, x(t,\omega), u(t), \omega) \quad \text{a.e. } t \in [0,T], \\ \qquad x(0,\omega) = x_0 \quad \text{and} \quad x(T,\omega) \in C(\omega)\,. \end{cases}$$

Ω (set of unknown parameters) is a complete separable metric space

Goal: characterize the optimal control independently of the unknown parameter action

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Example from aerospace engineering: Spacecraft[1]

Dynamics

$$\dot{q} = \frac{1}{2}Q(r)q$$
$$\dot{r} = I^{-1}(-r \times I \cdot r - r \times m_c(\delta) - A(\delta)u)$$
$$\dot{\delta} = u$$

$q \in \mathbb{R}^4$ (attitude), $r \in \mathbb{R}^3$ (body rate), δ N_c-vector of gimbals angles (associate with the onboard control moment gyros CMG), I inertia matrix, $Q(r)$ a given matrix, $m_c(\delta)$ angular momentum of CMG, $A(\delta)$ is a $3 \times N_c$ matrix associated with the control $u \in U$.

Goal: **minimize the time between two collects of images**

[1] Ross, I. M., Karpenko M., and Proulx J. R. "A Lebesgue-Stieltjes Framework For Optimal Control and Allocation." *American Control Conference (ACC)* 2015.

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Example from aerospace engineering: Spacecraft[2]

if $\delta(0) =$mean value of δ_0 **Uncontrollable system!**

[2] Ross, I. M., Karpenko M., and Proulx J. R. "A Lebesgue-Stieltjes Framework For Optimal Control and Allocation." *American Control Conference (ACC)* 2015.

Example from aerospace engineering: Spacecraft[2]

Average cost problem

Motivating problem

Link with previous works

Novelty and necessary conditions

Proof

Conclusion and Perspective

if $\delta(0) = \omega \in \Omega$ **('uncertainty' set) not necessarily compact**

[2] Ross, I. M., Karpenko M., and Proulx J. R. "A Lebesgue-Stieltjes Framework For Optimal Control and Allocation." *American Control Conference (ACC)* 2015.

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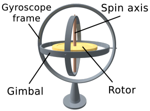
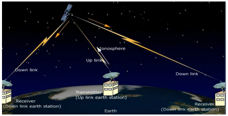
Example from aerospace engineering: Spacecraft[2]

if $\delta(0) = \omega \in \Omega$ **('uncertainty' set) not necessarily compact**

Optimal control problem with average cost

Satisfactory results ✔





[2] Ross, I. M., Karpenko M., and Proulx J. R. "A Lebesgue-Stieltjes Framework For Optimal Control and Allocation." *American Control Conference (ACC)* 2015.

Some literature on average control...

- Zuazua, Average control, *Automatica* 50 (12), 2014
- Agrachev, Baryshnikov, and Sarychev, Ensemble controllability by Lie algebraic methods, *ESAIM: Control, Optimisation and Calculus of Variations* 22 (4), 2016
- Caillau, Cerf, Sassi, Trélat, and Zidani, Solving chance-constrained optimal control problems in aerospace engineering via Kernel Density Estimation, *preprint*, 2016

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BUT...

NO Results For Necessary Optimality Conditions

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Minimax problem

$$\text{minimize } \max_{\omega \in \Omega} g(x(T,\omega);\omega) \quad \text{over } u \in U(t)$$

◆ **Standard criterion, unknown parameter ω and Ω is a compact metric space**

Goal: **characterize a solution considered for the worst performance case for all the values of the uncertain parameter $\omega \in \Omega$**

Works by: Vinter[3], Boltyanski[4], Karamzin, Oliveira, Pereira, Silva [5]

[3] Vinter, R. B. "Minimax optimal control." *SIAM journal on control and optimization* 44.3 (2005).

[4] Boltyanski, V. G. "Robust maximum principle." *Advanced Motion Control, 2006. 9th IEEE International Workshop on.* IEEE, 2012.

[5] Karamzin, D. et al. "Minimax optimal control problem with state constraints." *European Journal of Control* 32 (2016).

Novelty

$$\text{minimize} \int_{\Omega} g(x(T,\omega);\omega)\, d\mu(\omega) \quad \text{over } u \in U(t)$$

- the **probability measure** μ is given

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◆ the **probability measure** μ is given

◆ **integrate** over Ω (‘uncertainty’ set) instead of maximizing over $\omega \in \Omega$

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$$\text{minimize } \int_\Omega g(x(T,\omega);\omega)\, d\mu(\omega) \quad \text{over } u \in U(t)$$

◆ the **probability measure** μ is given

◆ **integrate** over Ω ('uncertainty' set) instead of maximizing over $\omega \in \Omega$

◆ Ω is merely a **complete separable** metric space, not necessarily compact

Assumptions on the data

- $\exists\, c > 0$ and $k_f(.) \in L^1$ s.t. $|f(t,x,u,\omega)| \leq c$

$$\left|f(t,x,u,\omega) - f(t,x',u,\omega)\right| \leq k_f(t)|x - x'|$$

for all x, x', $u \in U(t)$, $\omega \in \Omega$ a.e. $t \in [0,T]$

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  for all x, x', $u \in U(t)$, $\omega \in \Omega$ a.e. $t \in [0,T]$
- $f(t,x,U(t),\omega)$ closed for all $t,\ x,\ u$

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  for all x, x', $u \in U(t)$, $\omega \in \Omega$ a.e. $t \in [0,T]$
- $f(t,x,U(t),\omega)$ closed for all $t,\ x,\ u$
- $\exists\ k_g > 0$ and $M_g > 0$ s.t. for all $\omega \in \Omega$ $\ |g(x,\omega)| \leq M_g$ for all x, $|g(x,\omega) - g(x',\omega)| \leq k_g|x - x'|$ for all x, x'.

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- $\exists$ a modulus of continuity $\theta_g(.)$ s.t. for all $\omega \in \Omega$ and x

$$|g(x,\omega_1) - g(x,\omega_2)| \leq \theta_g(\rho_\Omega(\omega_1,\omega_2)) \quad \text{for all } \omega_1,\omega_2 \in \Omega\,.$$

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Assumptions on the data

- $\exists\, c > 0$ and $k_f(.) \in L^1$ s.t. $|f(t,x,u,\omega)| \leq c$

  $$\left|f(t,x,u,\omega) - f(t,x',u,\omega)\right| \leq k_f(t)|x - x'|$$

  for all x, x', $u \in U(t)$, $\omega \in \Omega$ a.e. $t \in [0,T]$
- $f(t,x,U(t),\omega)$ closed for all $t,\ x,\ u$
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- $\exists$ modulus of continuity $\theta_f(.)$ s.t. for all $\omega,\omega_1,\omega_2 \in \Omega$,

  $$\int_0^T \sup_{x,\,u} |f(t,x,u,\omega_1) - f(t,x,u,\omega_2)|\, dt \leq \theta_f(\rho_\Omega(\omega_1,\omega_2)).$$

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Necessary optimality conditions

Theorem (Bettiol-Khalil 2017)

Let $(\bar{u}, \{\bar{x}(.,\omega) : \omega \in \Omega\})$ be a $W^{1,1}-$local minimizer in which μ is given. Then, there exist $\lambda \geq 0$, a $\mathcal{L} \times \mathcal{B}_\Omega$ measurable function $p(.,.) : [0,T] \times \Omega \to \mathbb{R}^n$, and a countable dense subset $\widehat{\Omega}$ of supp(μ)

- $p(.,\omega) \in W^{1,1}([0,T],\mathbb{R}^n)$ *for all* $\omega \in \widehat{\Omega}$;
- $$\int_\Omega p(t,\omega) \cdot f(t,\bar{x}(t,\omega),\bar{u}(t),\omega)\, d\mu(\omega) = \max_{u \in U(t)} \int_\Omega p(t,\omega) \cdot f(t,\bar{x}(t,\omega),u,\omega)\, d\mu(\omega) \quad a.e.\ t \in [0,T]\ ;$$
- $p(.,\omega) \in co\, \mathcal{P}(\omega)$ *for all* $\omega \in \widehat{\Omega}$ *where*

$$\mathcal{P}(\omega) := \Big\{ q(.,\omega) \in W^{1,1}\ :\ \|q(.,.)\|_{L^\infty} \leq 1,\ \lambda + \sum_{\omega \in \widehat{\Omega}} \max_{t \in [0,T]} |q(t,\omega)| = 1,$$
$$-\dot{q}(t,\omega) \in co\, \partial_x [q(t,\omega) \cdot f(t,\bar{x}(t,\omega),\bar{u}(t),\omega)] \quad a.e.\ t \in [0,T],$$
$$\text{and} \quad -q(T,\omega) \in \lambda \partial_x g(\bar{x}(T,\omega);\omega) + N_{C(\omega)}(\bar{x}(T,\omega)) \Big\}.$$

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What if we add more regularity?

- $g(., \omega)$ is differentiable for each $\omega \in \Omega$, and $\nabla_x g(.,.)$ is continuous
- $f(t, ., u, \omega)$ is continuously differentiable on $\bar{x}(t, \omega) + \delta\mathbb{B}$ for all $u \in U(t)$ and $\omega \in \Omega$ a.e. $t \in [0, T]$, and $\omega \to \nabla_x f(t, x, u, \omega)$ is uniformly continuous with respect to $(t, x, u) \in \{(t', x', u') \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \mid u' \in U(t')\}$
- $C(\omega) := \mathbb{R}^n$

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What if we add more regularity?

Theorem (Bettiol-Khalil 2017)

There exists a $\mathcal{L} \times \mathcal{B}_\Omega$ measurable function $p(.,.)$ s.t.

- $p(.,\omega) \in W^{1,1}([0,T],\mathbb{R}^n)$ *for all* $\omega \in \Omega$
- $\int_\Omega p(t,\omega) \cdot f(t,\bar{x}(t,\omega),\bar{u}(t),\omega)\, d\mu(\omega)$
  $= \max_{u \in U(t)} \int_\Omega p(t,\omega) \cdot f(t,\bar{x}(t,\omega),u,\omega)\, d\mu(\omega)$ *a.e. t*
- $-\dot{p}(t,\omega) = [\nabla_x f(t,\bar{x}(t,\omega),\bar{u}(t),\omega)]^T p(t,\omega)$ *a.e. t, for all* $\omega \in \Omega$
- $-p(T,\omega) = \nabla_x g(\bar{x}(T,\omega);\omega)$, *for all* $\omega \in \Omega$.

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Steps of the proof (inspired by Vinter[6])

❶ **approximate** μ by convex combination of Dirac measures (**finite support**)

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❹ 'double' **limit-taking**: ➢ adjoint system/transversality condition

➢ Weierstrass condition (weak* −convergence of measures)

[6] Vinter, R. B. "Minimax optimal control." *SIAM journal on control and optimization* 44.3 (2005).

Conclusion and Perspectives

Conclusion: establish necessary optimality conditions for average cost minimization problems using approach of the minimax problem

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Perspectives:

➢ Add a state constraint condition (work in progress)

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Conclusion: establish necessary optimality conditions for average cost minimization problems using approach of the minimax problem

Perspectives:

➢ Add a state constraint condition (work in progress)

➢ Study stronger necessary optimality conditions (nondegeneracy, normality)

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Perspective: add a state constraint (in progress)

◆ **Theoretical reasons:** preliminary results

[7] Ross, I. M., Karpenko M., and Proulx J. R. "Path constraints in tychastic and unscented optimal control: Theory, application and experimental results." *American Control Conference (ACC)*. IEEE, 2016.

Perspective: add a state constraint (in progress)

- **Theoretical reasons:** preliminary results
- **Applications**[7]**:** aerospace engineering

Dynamics

$$\dot{q} = \frac{1}{2}Q(r)q$$
$$\dot{r} = I^{-1}(-r \times I \cdot r - r \times m_c(\delta) - A(\delta)u)$$
$$\dot{\delta} = u$$

State constraint $t \mapsto S(\delta) := \sqrt{\det[A(\delta)A^T(\delta)]} \geq \alpha \quad \forall t$

($\alpha > 0$ is an engineering decision)

[7] Ross, I. M., Karpenko M., and Proulx J. R. "Path constraints in tychastic and unscented optimal control: Theory, application and experimental results." *American Control Conference (ACC)*. IEEE, 2016.

Grazie!