INEQUALITY PROBLEMS, SWEEPING PROCESS AND OPTIMAL CONTROL IN CONTACT MECHANICS

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Aims

- To draw the attention of the *Applied Mathematics* community to interesting issues on Variational and Hemivariational Inequalities with emphasis to the study of *Contact Mechanics*.

- To illustrate the cross fertilization between *Modeling and Applications* on the one hand, and *Nonlinear Functional Analysis* on the other hand.

CONTENTS

I. A viscoelastic contact problem

(the model, variational formulations, abstract tools, existence, uniqueness and equivalence results).

II. Variational-hemivariational inequalities for elastic contact problems

(existence, uniqueness and convergence results, optimal control, one-dimensional example).

I. A VISCOELASTIC CONTACT PROBLEM

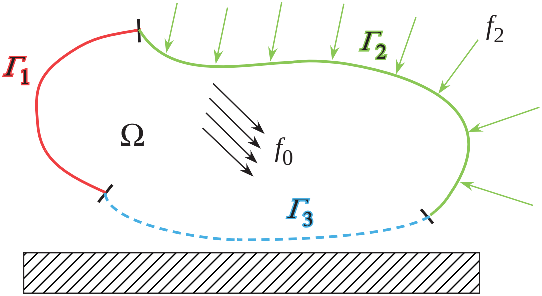

Figure: 1. Physical setting.

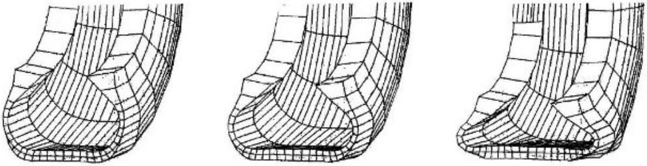

Fig. 5.20 – Écrasement statique - $F = 6$ tonnes

Fig. 5.21 – Écrasement statique - $F = 10$ tonnes

Fig. 5.23 – Écrasement statique - $F = 18$ tonnes

Figure: 2. Tire of the plane at landing.

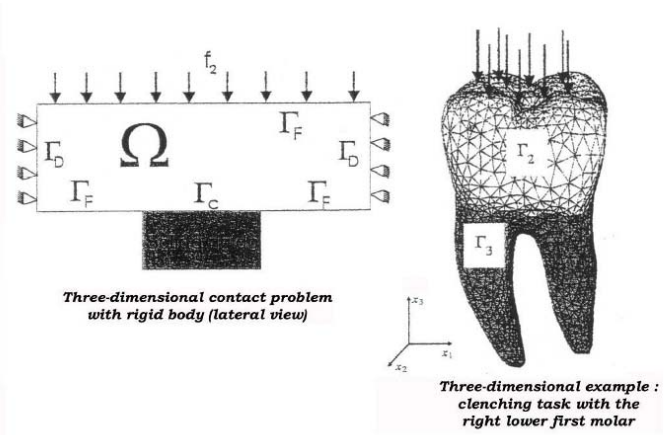

Figure: 3. Some examples of contact processes.

Notation

$I = [0, T]$ or $I = \mathbb{R}_+$ - interval of time ;

Ω - bounded domain of $\mathbb{R}^d$ $(d = 1, 2, 3)$;

Γ - boundary of Ω ;

Γ_1, Γ_2, Γ_3 - partition of Γ such that *meas* $\Gamma_1 > 0$;

$\boldsymbol{\nu}$ - unit outward normal on Γ ;

$\mathbb{S}^d$ - space of second order symmetric tensors on $\mathbb{R}^d$;

$\boldsymbol{\varepsilon}$ - the deformation operator;

v_ν, $\boldsymbol{v}_\tau$ - *normal* and *tangential* components of $\boldsymbol{v}$ on Γ;

σ_ν, $\boldsymbol{\sigma}_\tau$ - *normal* and *tangential* components of $\boldsymbol{\sigma}$ on Γ.

Problem $\mathcal{P}$. *Find a displacement field $\boldsymbol{u} : \Omega \times I \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times I \to \mathbb{S}^d$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{B}\varepsilon(\boldsymbol{u}(t)) + \int_0^t \mathcal{C}(t-s)\varepsilon(\dot{\boldsymbol{u}}(s))\,ds \qquad \text{in } \Omega,$$

$$\mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \qquad \text{in } \Omega,$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \qquad \text{on } \Gamma_1,$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \qquad \text{on } \Gamma_2,$$

$$F \le -\sigma_\nu(t) \le 0, \quad -\sigma_\nu(t) = \begin{cases} 0 & \text{if} \quad \dot{u}_\nu(t) < 0, \\ F & \text{if} \quad \dot{u}_\nu(t) > 0 \end{cases} \qquad \text{on } \Gamma_3,$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \qquad \text{on } \Gamma_3.$$

for all $t \in I$.

$$V = \{ \boldsymbol{v} = (v_i) \in H^1(\Omega)^d : v_i = 0 \text{ on } \Gamma_1 \},$$

$$Q = \{ \boldsymbol{\tau} = (\tau_{ij}) \ : \ \tau_{ij} = \tau_{ji} \in L^2(\Omega) \}.$$

Inner products:

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx.$$

Associated norms: $\| \cdot \|_V$ and $\| \cdot \|_Q$.

$$\mathbf{Q}_\infty = \{ \mathcal{E} = (e_{ijkl}) \ : \ e_{ijkl} = e_{jikl} = e_{klij} \in L^\infty(\Omega), \ \}.$$

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \max_{0 \le i,j,k,l \le d} \|e_{ijkl}\|_{L^\infty(\Omega)}$$

$(X, \| \cdot \|_X)$ - real normed space, $C(I; X)$ - standard notation.

Assumptions

(1) $\begin{cases} \text{(a) } \mathcal{A} \in \mathbf{Q}_\infty. \\ \text{(b) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \qquad \mathcal{A}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \geq m_{\mathcal{A}} \|\boldsymbol{\tau}\|^2 \ \forall \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. in } \Omega. \end{cases}$

(2) $\mathcal{B} \in \mathbf{Q}_\infty$.

(3) $\mathcal{C} \in C(I; \mathbf{Q}_\infty)$.

(4) $\boldsymbol{f}_0 \in C(I; L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in C(I; L^2(\Gamma_2)^d)$.

(5) $F \in L^2(\Gamma_3), \quad F \geq 0$ a.e. on Γ_3.

(6) $\boldsymbol{u}_0 \in V$.

Denote by $\boldsymbol{w} = \dot{\boldsymbol{u}}$ the **velocity field**. Then

$$(7) \quad \boldsymbol{u}(t) = \int_0^t \boldsymbol{w}(s)\, ds + \boldsymbol{u}_0 \quad \forall\, t \in I.$$

Moreover, a standard result implies that

$$(8) \quad \varepsilon(\boldsymbol{w}(t)) = \mathcal{A}^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}\boldsymbol{\sigma}(t) \quad \forall\, t \in I.$$

where $\mathcal{R} : C(I; Q) \to C(I; Q)$.

Define $A : V \to V$, $B : V \to V$, $C : C(I;V) \to C(I;V)$, $\mathcal{S} : C(I;V) \to C(I;V)$ by equalities

$$(A\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{A}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_Q, \quad (B\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{B}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_Q,$$

$$(C\boldsymbol{w}(t), \boldsymbol{v})_V = \Big(\int_0^t \mathcal{C}(t-s)\varepsilon(\boldsymbol{w}(s))\, ds, \varepsilon(\boldsymbol{v}) \Big)_Q,$$

$$(\mathcal{S}\boldsymbol{w}(t), \boldsymbol{v})_V = (B\Big(\int_0^t \boldsymbol{w}(s)\, ds + \boldsymbol{u}_0 \Big), \boldsymbol{v})_V + (C\boldsymbol{w}(t), \boldsymbol{v})_V,$$

for all $\boldsymbol{u}$, $\boldsymbol{v}, \in V$, $\boldsymbol{w} \in C(I;V)$, $t \in I$.

Remarks

1. A and B are **linear bounded symetric operators**.
2. C, $\mathcal{S}$ and $\mathcal{R}$ are **history-dependent operators.**

Definition. *An operator* $\mathcal{S}\colon C(I;X) \to C(I;Y)$ *is called a* **history-dependent** *operator if for any compact set* $K \subset I$ *there exists* $L_K > 0$ *such that*

$$\|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_Y \le L_K \int_0^t \|u_1(s) - u_2(s)\|_X \, ds$$
$$\text{for all} \;\; u_1,\, u_2 \in C(I;X), \quad t \in K.$$

Such operators arise in **Solid Mechanics** and **Contact Mechanics**.

Define $j : V \to \mathbb{R}$, $\boldsymbol{f} : I \to V$, $K : I \to 2^V$, $\Sigma : I \to 2^Q$ by

$$j(\boldsymbol{v}) = \int_{\Gamma_3} F v_\nu^+ \, da,$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v} \, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v} \, da,$$

$$K(t) = \boldsymbol{f}(t) - \partial j(0_V),$$

$$\Sigma(t) = \{ \boldsymbol{\sigma} \in Q \ : \ (\boldsymbol{\sigma}, \varepsilon(\boldsymbol{v}))_Q + j(v) \geq (\boldsymbol{f}(t), \boldsymbol{v})_V \ \ \forall \, \boldsymbol{v} \in V \},$$

for all $\boldsymbol{v} \in V$, $t \in I$.

Remarks

1. j is a **continuous seminorm** on V. Therefore, it is convex, positively homogenuous, Lipshitz continuous and $j(0_V) = 0$.

2. $\boldsymbol{f}$ has the regularity $\boldsymbol{f} \in C(I; V)$.

3. For all $t \in I$, $K(t)$ is a **nonempty closed convex** subset of V. Denote by $N_{K(t)}(u)$ the outward normal cone to $K(t)$ at u.

4. For all $t \in I$, $\Sigma(t)$ is a **nonempty closed convex** subset of Q. Moreover,

$$\Sigma(t) = \Sigma_0 + \varepsilon(\boldsymbol{f}(t)) \quad \forall\, t \in I$$

where

$$\Sigma_0 = \{\, \boldsymbol{\sigma} \in Q \;:\; (\boldsymbol{\sigma}, \varepsilon(\boldsymbol{v}))_Q + j(v) \geq 0 \;\; \forall\, \boldsymbol{v} \in V \,\}.$$

Variational formulations

(P_w) **Variational formulation in velocity:** *Find* $\boldsymbol{w} : I \to V$ *s.t.*

$$(A\boldsymbol{w}(t), \boldsymbol{v} - \boldsymbol{w}(t))_V + (\mathcal{S}\boldsymbol{w}(t), \boldsymbol{v}) - \boldsymbol{w}(t))_Q + j(\boldsymbol{v}) - j(\boldsymbol{w}(t))$$
$$\geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{w}(t))_V \qquad \forall\, \boldsymbol{v} \in V,\ t \in I.$$

(P_u) **Variational formulation in displacement:** *Find* $\boldsymbol{u} : I \to V$ *s.t.*

$$-\dot{\boldsymbol{u}}(t) \in N_{K(t)}(A\dot{\boldsymbol{u}}(t) + B\boldsymbol{u}(t) + C(\dot{\boldsymbol{u}}(t)) \qquad \forall\, t \in I.$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0.$$

(P_s) **Variational formulation in stress:** *Find* $\boldsymbol{\sigma} : I \to V$ *s.t.*

$$\boldsymbol{\sigma}(t) \in \Sigma(t), \quad (\mathcal{A}^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}\boldsymbol{\sigma}(t), \boldsymbol{\tau} - \boldsymbol{\sigma}(t))_Q \geq 0 \quad \forall\, \boldsymbol{\tau} \in \Sigma(t), t \in I.$$

Remarks

1. The variational formulations above are obtained by using **Green's formula** combined with the boundary conditions and the properties of the operators and functions defined above.

2. **Problems (P_w), (P_u) and (P_s)** represent variational formulations of the **the same** contact problem. Therefore, besides their unique solvability, there is a need to provide the **link** between their solutions.

3. **Problems (P_w), (P_u) and (P_s)** represent **different** mathematical objects: a history-dependent variational inequality of the second kind, a sweeping proces, and a time-dependent variational inequality of the first kind. Therefore, their solvability is made by using **different** functional arguments.

Equivalence result

Theorem 1. *Assume that* (1)–(6) *hold and let* $\boldsymbol{w} \in C(I;V)$, $\boldsymbol{u} \in C^1(I;V)$, $\boldsymbol{\sigma} \in C(I,Q)$ *be three functions which satisfy the equalities below, for all* $t \in I$*:*

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{w}(s)\,ds + \boldsymbol{u}_0,$$

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{B}\varepsilon(\boldsymbol{u}(t)) + \int_0^t \mathcal{C}(t-s)\varepsilon(\dot{\boldsymbol{u}}(s))\,ds.$$

Consider the following statements:

i) $\boldsymbol{w}$ *is a solution of* **Problem** (P_w).
ii) $\boldsymbol{u}$ *is a solution of* **Problem** (P_u).
iii) $\boldsymbol{\sigma}$ *is a solution of* **Problem** (P_s).

Then, if one of these statements holds, the reminders hold, too.

Proof

The prof of Theorem 1 is based on the defintions of the sets $K(t)$, $\Sigma(t)$ and operators A, B, C, $\mathcal{S}$, $\mathcal{R}$, combine with the properties of the seminorm j. $\square$

Mathematical tools

Theorem 2. *Assume that K is a nonempty closed convex subset of the Hilbert space X, $A : X \to X$ is a strongly monotone Lipschitz continous operator, $\mathcal{S} \colon C(I;X) \to C(I;X)$ is a history-depdendent operator, $f \in C(I;X)$ and $\varphi : X \times X \times K \to \mathbb{R}$ is such that*

$$\left\{\begin{array}{l}\text{(a) For all } z,\ u \in X,\ \varphi(z,u,\cdot) \text{ is convex and l.s.c. on } K.\\ \text{(b) There exist } \alpha > 0 \text{ and } 0 < \beta < m_A \text{ such that}\\ \qquad \varphi(z_1,u_1,v_2) - \varphi(z_1,u_1,v_1) + \varphi(z_2,u_2,v_1) - \varphi(z_2,u_2,v_2)\\ \qquad\quad \leq \alpha\|z_1 - z_2\|_X\,\|v_1 - v_2\|_X + \beta\,\|u_1 - u_2\|_X\,\|v_1 - v_2\|_X\\ \qquad\quad\ \textit{for all } z_1,\ z_2 \in X,\ u_1,\ u_2 \in X,\ v_1,\ v_2 \in K.\end{array}\right.$$

Then there exists a unique function $u \in C(I;X)$ such that

$$u(t) \in K, \quad (Au(t), v - u(t))_X + \varphi(\mathcal{S}u(t), u(t), v)$$
$$-\varphi(\mathcal{S}u(t), u(t), u(t)) \geq (f(t), v - u(t))_X \ \ \forall\, v \in K,\ t \in I.$$

The proof of Theorem 2 was obtained in

M. Sofonea & Y. Xiao, Fully history-dependent quasivariational inequalities in Contact Mechanics, *Applicable Analysis* **95** (2016), 2464–2484.

It is based on a fixed point result for history-dependent operators obtained in

M. Sofonea, C. Avramescu & A. Matei, A fixed point result with applications in the study of viscoplastic frictionless contact problems, *Communications on Pure and Applied Analysis* **7** (2008), 645–658.

Theorem 3. *Assume that X is a real Hilbert space and, moreover:*

- $A, B : X \to X$ *are linear bounded symetric positive operators, A being positively defined,*
- $j : X \to \mathbb{R}$ *is a convex positively homogenous Lipschtz continuous function such that* $j(0_X) = 0$,
- $f \in W^{1,1}(0, T; X)$, $K(t) = f(t) - \partial j(0_X)$ *for all* $t \in [0, T]$,
- $u_0 \in X$ *with* $Bu_0 \in K(0)$.

Then there exists a unique function $u \in W^{1,\infty}(0, T; X)$ *such that*

$$-\dot{u}(t) \in N_{K(t)}(A\dot{u}(t) + Bu(t)) \quad \text{a.e. } t \in (0, T),$$

$$u(0) = u_0.$$

The proof of Theorem 3 was obtained in the recent paper

S. Adly & T. Haddad, An implicit sweeping process approach to quasistatic evolution variational inequalities, Mechanics, *SIAM Journal of Mathematical Analysis*, in press.

It is based on time-discretization method and arguments of maximal monotone operators.

Existence and uniqueness results

Theorem 2 can be used to obtain the existence of a unique solution of **Problem** (P_w), with regularity $\boldsymbol{w} \in C(I; V)$, under assumptions (1)-(6). To this end, an appropriate choice of operators is needed.

It also can be used to obtain the existence of a unique solution of **Problem** (P_s), with regularity $\boldsymbol{\sigma} \in C(I; V)$, under assumptions (1)-(6). To this end, the **special structure**

$$\Sigma(t) = \Sigma_0 + \varepsilon(\boldsymbol{f}(t)) \quad \forall\, t \in I$$

is crucial.

Moreover, the hypothesis of Theorem 2 allow us to use it in the study of more complicate quasistatic contact problems with:

- nonlinear constitutive operators $\mathcal{A}$ and $\mathcal{B}$ (the operators A and B can be nonlinear);
- unilateral constraints in displacements or velocities (the abstract inequality allows unilateral constraints for the unknown).
- various nonlinear boundary conditions (j is not necessary positively homogenous and could depend on the solution).

Theorem 3 can be used to obtain the existence of a unique solution of **Problem** (P_u), with regularity $\boldsymbol{u} \in W^{1,\infty}(0, T; V)$, under stronger assumptions:

- $I = [0, T]$ with $T > 0$ (**instead of** $I = [0, T]$ or $I = \mathbb{R}_+$),
- $\mathcal{C} \equiv 0$ (**instead of** $\mathcal{C} \in C(I; \mathbf{Q}_\infty)$),
- $\boldsymbol{f}_0 \in W^{1,1}(0, T; L^2(\Omega)^d)$, $\quad \boldsymbol{f}_2 \in W^{1,1}(0, T; L^2(\Gamma_2)^d)$ (**instead of** $\boldsymbol{f}_0 \in C(I; L^2(\Omega)^d)$, $\quad \boldsymbol{f}_2 \in C(I; L^2(\Gamma_2)^d)$).
- $\boldsymbol{u}_0 \in V$ with $B\boldsymbol{u}_0 \in K(0)$ (**instead of** $\boldsymbol{u}_0 \in V$.)

Moreover, the regularity of the solution is weaker:
$\boldsymbol{u} \in W^{1,\infty}(0, T; V)$ **instead of** $\boldsymbol{u} \in C^1([0, T]; V)$.

Conclusions

- The weak formulations in terms of **VI** are more appropiate than the weak formulations in terms of **sweeping process**. The reason : the moving convex describes the boundary conditions (in contrast to plasticity problems, where the moving convex is related to the constitutive law).

- **Open problem :** extention of the results in the paper by *Adly and Haddad* to more general problems of the form

  $$-\dot{u}(t) \in N_{K(t)}(A\dot{u}(t)+Bu(t)+\mathcal{S}\dot{u}(t)) \text{ a.e. } t \in (0,T), u(0) = u_0,$$

  where $\mathcal{S}$ is history-dependent (linear) operator.

II. VARIATIONAL-HEMIVARIATIONAL INEQUALITIES FOR ELASTIC CONTACT PROBLEMS

Problem $\mathcal{P}$. *Given $f \in Y$ and $g > 0$, find $u \in K_g$ such that*

$$\langle Au, v-u\rangle + \varphi(u, v) - \varphi(u, u) + j^0(u; v-u) \geq (f, \pi v - \pi u)_Y \quad \forall\, v \in K_g.$$

Here:

X - reflexive Banach space of dual X^*,

$K \subset X$, $K_g = gK$,

$A : X \to X^*$, $\varphi : X \times X \to \mathbb{R}$, $j : X \to \mathbb{R}$,

Y- Hilbert space with the inner product $(\cdot, \cdot)_Y$,

$\pi : X \to Y$.

Main results

Under appropriate assumptions of the data, the following results have been obtained:

- existence of a unique solution $u = u(f, g)$ of **Problem $\mathcal{P}$**.
- a weak-strong convergence result:

$$f_n \rightharpoonup f \text{ in } Y, \quad g_n \to g \implies u(f_n, g_n) \to u(f, g) \text{ in } X.$$

- existence of the solution of two optimal control problems in which the controls are f and g, respectively.
- convergence results for the corresponding optimal control problems.

One-dimensional elastic contact problem

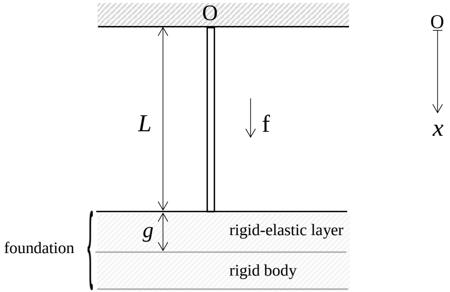

Figure: 4. Physical setting.

We consider the particular case $L = 1$, $f \in \mathbb{R}$ and we restrict to linear elasticity and monotone contact conditions.

Problem $\mathcal{P}^{1d}$. *Find a displacement field* $u\colon [0,1] \to \mathbb{R}$ *and a stress field* $\sigma\colon [0,1] \to \mathbb{R}$ *such that*

$$\sigma(x) = E\,u'(x) \quad \text{for } x \in (0,1),$$

$$\sigma'(x) + f = 0 \quad \text{for } x \in (0,1),$$

$$u(0) = 0,$$

$$u(1) \le g, \qquad \left.\begin{array}{ll} \sigma(1) = 0 & \text{if } \ u(1) < 0 \\ -F < \sigma(1) < 0 & \text{if } \ u(1) = 0 \\ \sigma(1) = -F & \text{if } \ 0 < u(1) < g \\ \sigma(1) \le -F & \text{if } \ u(1) = g \end{array}\right\}.$$

We use the space

$$V = \{ \, v \in H^1(0,1) \ : \ v(0) = 0 \, \}$$

and the set of **admissible displacement fields** defined by

$$K_g = \{ \, u \in V \mid u(1) \leq g \, \}.$$

The variational formulation of **Problem** $\mathcal{P}^{1d}$ is the following.

Problem $\mathcal{P}_V^{1d}$. *Find a displacement field* $u \in K_g$ *such that*

$$\int_0^1 Eu'(v'-u')\,dx + Fv(1)^+ - Fu(1)^+ \geq \int_0^1 f\,(v-u)\,dx \quad \forall\, v \in K_g.$$

Our **abstract results** (existence, uniqueness, continuous dependence, control) can be applied in the sudy of **Problem** $\mathcal{P}_V^{1d}$.

Moreover, a simple calculation allows to solve **Problems** $\mathcal{P}^{1d}$ **and** $\mathcal{P}_V^{1d}$. Four cases are possible, as shown below.

a) $f < 0 \implies \begin{cases} \sigma(x) = -fx + f, \\ u(x) = -\frac{f}{2E}x^2 + \frac{f}{E}x \end{cases}$

b) $0 \leq f < 2F \implies \begin{cases} \sigma(x) = -fx + \frac{f}{2}, \\ u(x) = -\frac{f}{2E}x^2 + \frac{f}{2}x \end{cases}$

c) $2F \leq f < 2Eg + 2F \implies \begin{cases} \sigma(x) = -fx + f - F, \\ u(x) = -\frac{f}{2E}x^2 + \frac{f-F}{E}x \end{cases}$

d) $2Eg + 2F \leq f \implies \begin{cases} \sigma(x) = -fx + \frac{2Eg+f}{2}, \\ u(x) = -\frac{f}{2E}x^2 + \frac{2Eg+f}{2E}x \end{cases}$

for all $x \in [0, 1]$.

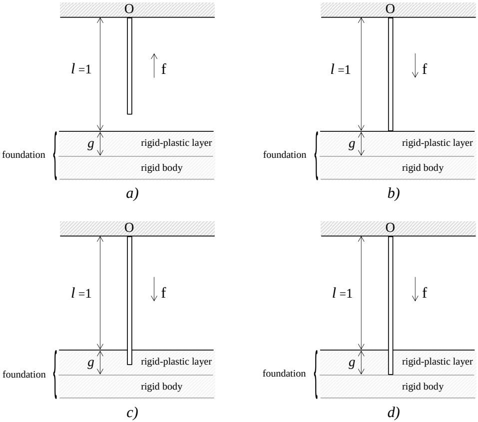

Figure: 5. The rod in contact with a foundation:

a) The case $f < 0$; b) The case $0 \leq f < 2F$;

c) The case $2F \leq f < 2Eg + 2F$; d) The case $2Eg + 2F \leq f$.

Consider the following **optimal control problem**.

Problem $\mathcal{Q}^{1d}$. *Find* $(u^*, g^*) \in \mathcal{V}_{ad}$ *such that*

$$\mathcal{L}(u^*, g^*) = \min_{(u,g)\in\mathcal{V}_{ad}} \mathcal{L}(u, g).$$

Here

$$\mathcal{L}(u, g) = \alpha\,|u(1) - \phi| + \beta\,|g|,$$

$$\mathcal{V}_{ad} = \{\,(u, g) \in K_g \times W \text{ such that } \mathcal{P}_V^{1d} \text{ holds}\,\}.$$

with $\phi \in \mathbb{R}$, $\alpha > 0$, $\beta > 0$, $W = [g_0, \infty)$ with $g_0 > 0$.

Mechanical interpretation : given f, we are looking for a thickness $g \geq g_0$ such that the displacement of the rod in $x = 1$ is as close as possible to the "desired displacement" ϕ. Furthermore, this choice has to fulfill a minimum expenditure condition.

$$\left(\mathcal{L}(u, g) = \alpha \, |u(1) - \phi| + \beta \, |g|.\right)$$

The existence of the solution of **Problem** $\mathcal{Q}^{1d}$ follows from our abstract existence result. An analytic expression can be provided. Some numerical examples show that the solution could be unique or not.

Numerical example

We take $E = 1$, $f = 10$, $F = 2$, $\phi = 4$ and $g_0 = 1$ which implies that $W = [1, +\infty)$. It is easy to see that

$$(s) \qquad u(x) = \begin{cases} -5x^2 + (g+5)\,x & \text{if } \; 1 \le g \le 3, \\ -5x^2 + 8x & \text{if } \; g > 3 \end{cases}$$

for all $x \in [0, 1]$ and, therefore,

$$\mathcal{L}(u, g) = \begin{cases} (\beta - \alpha)\,g + 4\alpha & \text{if } \; 1 \le g \le 3, \\ \beta\,g + \alpha & \text{if } \; g > 3. \end{cases}$$

We note that:

a) If $\beta > \alpha > 0$ then the optimal control problem $\mathcal{Q}^{1d}$ has a **unique solution** (u^*, g^*) where $g^* = 1$ and u^* is given by (s) with $g = g^*$.

b) If $\beta = \alpha$ then the optimal control problem $\mathcal{Q}^{1d}$ has an **infinity of solutions** of the form (u^*, g^*) where g^* is any value in the interval $[1, 3]$ and u^* is given by (s) with $g = g^*$.

c) If $0 < \beta < \alpha$ then the optimal control problem $\mathcal{Q}^{1d}$ has a **unique solution** (u^*, g^*) where $g^* = 3$ and u^* is given by (s) with $g = g^*$.

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Thank you for your attention!