Investigation of Optimality Conditions in Control Problems with State Constraints

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Problem Statement

Consider the following optimal control problem with inequality state constraints.

$$
\begin{array}{ll}
\text{Minimize} & \varphi(p), \\
\text{subject to} & \dot{x} = f(x, u, t), \\
 & p = (x_0, x_1) \in S, \\
 & u(t) \in U \ \text{ a.a. } t \in [t_0, t_1], \\
 & g(x(t), t) \leq 0 \ \ \forall\, t \in [t_0, t_1].
\end{array}
\tag{1}
$$

Here, $\dot{x} = \dfrac{dx}{dt}$, $t \in [0, 1]$ designates time, x is the state variable which takes values in $\mathbb{R}^n$; $x_0 = x(t_0)$, $x_1 = x(t_1)$; vector $u \in U \subset \mathbb{R}^m$ is the control parameter. The sets S, U are closed. Vector $p = (x_0, x_1)$ is termed endpoint. A measurable function $u : [t_0, t_1] \to U$ is termed control (or, control function).

Maximum Principle

Definition

The control process $(x^*(t), u^*(t))$ of (1) satisfies the *maximum principle* provided that there exist Lagrange multipliers: a number $\lambda \in [0,1]$, a function of bounded variation $\psi : [t_0, t_1] \to \mathbb{R}^n$, and a Borel measure $\eta \in C^*([t_0, t_1])$, $\eta \geq 0$, such that

$$\lambda + \sup_{t \in [t_0, t_1]} |\psi(t)| = 1,$$

$$d\psi(t) = -H'_x(x^*(t), u^*(t), \psi(t), t)dt + g'_x(x^*(t), t)d\eta, \quad t \in [t_0, t_1],$$

$$(\psi(t_0), -\psi(t_1)) \in \lambda\varphi'(p^*) + N_S(p^*),$$

$$\max_{u \in U} H(x^*(t), u, \psi(t), t) = H(x^*(t), u^*(t), \psi(t), t) \;\text{ a.a. } t \in [t_0, t_1],$$

$$\int_{[t_0, t_1]} \Big\langle g(x^*(t), t), d\eta \Big\rangle = 0.$$

Here, $H(x, u, \psi, t) := \langle \psi, f(x, u, t) \rangle$, $p^* = (x^*(t_0), x^*(t_1))$.

A.Ya Dubovitskii and A.A. Milyutin proposed the following theorem.

Theorem

Suppose that the control process $(x^*(t), u^*(t))$ *is optimal to problem (1). Then, the process* $(x^*(t), u^*(t))$ *satisfies the maximum principle.*

Suppose that one of the endpoints, either x_0, or x_1, is fixed. In this case, as is easy to verify, the Dubovitskii-Milyutin maximum principle can be satisfied by any feasible pair control/trajectory. Indeed, in order to ensure this we only need to consider the set of Lagrange multipliers (in case the left endpoint is fixed):

$$\lambda = 0, \quad \eta = \delta(t_0), \quad \psi(\cdot) : \ \psi(t) = 0 \ \forall\, t \in (t_0, t_1],$$

and the multipliers

$$\lambda = 0, \quad \eta = \delta(t_1), \quad \psi(\cdot) : \ \psi(t) = 0 \ \forall\, t \in [t_0, t_1),$$

if the right endpoint is fixed.

Degeneracy of the Maximum Principle

A.Ya. Dubovitskii and V.A. Dubovitskii found the following interesting example (1985).

Example

Consider $n = m = k = 1,\ t_0 = 0,\ t_1 = 1,$

$$\begin{cases} \int_0^1 u(t)dt \to \min, \\ \dot{x} = tu, \\ x_0 = 0, \\ u(t) \in [-1, 1], \\ x(t) \geq 0. \end{cases}$$

The optimal process is $x = u = 0$, but there are only degenerate multipliers satisfying the maximum principle for it.

Conditions for Non-degeneracy

Definition

The state constraints are said to be *regular* provided that for all $x, t : g(x,t) \leq 0$ there exists $z = z(x,t)$ such that

$$\left\langle \frac{\partial g^j}{\partial x}(x,t), z \right\rangle > 0 \;\; \forall j \in J(x,t).$$

Here, $J(x,t) := \{j : g^j(x,t) = 0\}$.

Conditions for Non-degeneracy

Definition

The state constraints are said to be *compatible* at p^* with the endpoint constraints provided that

$$\exists\, \varepsilon > 0 : \ \{p \in \mathbb{R}^{2n} : \ |p^* - p| \leq \varepsilon, \ p \in S\} \subseteq$$

$$\{p \in \mathbb{R}^{2n} : \ g(x_0, t_0) \leq 0, \ g(x_1, t_1) \leq 0\}.$$

The compatibility of constraints is not an extra requirement. It can always be achieved by replacing the set S with the set

$$S \cap \{p \in \mathbb{R}^{2n} : \ g(x_0, t_0) \leq 0, \ g(x_1, t_1) \leq 0\}.$$

Conditions for Non-degeneracy

Consider the function

$$\Gamma(x,u,t) = g'_x(x,t)f(x,u,t) + g'_t(x,t).$$

Definition

A feasible arc $x^*(t)$ is said to be *controllable* at the endpoints w.r.t. the state constraints, provided that there exist vectors $\gamma_r \in \operatorname{conv}\Gamma(x_r, U, r)$ such that:

$$(-1)^r \gamma_r^j < 0 \quad \forall j \in J(x_r, r), \quad r = 0, 1.$$

Non-degenerate Maximum Principle

Theorem

Suppose that the control process $(x^*(t), u^*(t))$ is optimal in problem (1). Suppose that the state constraints are regular and compatible with the endpoints constraints at p^*, and the controllability condition is satisfied.
Then, the process $(x^*(t), u^*(t))$ satisfies the maximum principle, and the following strengthened non-triviality condition is valid

$$\lambda + \ell(t \in [t_0, t_1] : |\psi(t)| \neq 0) > 0,$$

where ℓ stands for the Lebesgue measure.

Pointwise Controllability

Definition

A feasible arc $x^*(t)$ is said to be *controllable* w.r.t. the state constraints, provided that it is controllable at the endpoints and for every $t \in (t_0, t_1)$ there exist vectors $\gamma_{r,t} \in \mathbb{R}^k$, $r = 0, 1$, such that

$$\gamma_{r,t} \in \operatorname{conv} \Gamma(x^*(t), U, t),$$

$$(-1)^r \gamma^j_{r,t} < 0 \;\; \forall j \in J(x^*(t), t), \quad r = 0, 1.$$

Pointwise Controllability

Theorem

Suppose that the control process $(x^*(t), u^*(t))$ is optimal in problem (1). Suppose that the state constraints are regular and compatible with the endpoints constraints at p^*, and the pointwise controllability condition is satisfied.
Then, the process $(x^*(t), u^*(t))$ satisfies the maximum principle, and the following strengthened non-triviality condition is valid

$$\lambda + |\psi(t)| \neq 0 \;\; \forall\, t \in (t_0, t_1).$$

Maximum Principle in the Gamkrelidze form

Consider the extended Hamilton-Pontryagin function:

$$\bar{H}(x,u,\psi,\mu,t) = \langle \psi, f(x,u,t)\rangle - \langle \mu, \Gamma(x,u,t)\rangle .$$

Definition

The control process $(x^*(t), u^*(t))$ of (1) satisfies the *non-degenerate maximum principle* provided that there exist Lagrange multipliers $\lambda \in [0,1]$, $\psi \in W_{1,\infty}([t_0,t_1])$, and decreasing functions μ^j, $j = 1, ..., k$, such that

$$\dot{\psi}(t) = -\bar{H}'_x(x^*(t), u^*(t), \psi(t), \mu(t), t) \text{ a.a. } t \in [t_0, t_1],$$

$$(\psi(t_0), -\psi(t_1)) \in \lambda\varphi'(p^*) + (\mu(t_0)g'_x(x_0^*, t_0), -\mu(t_1)g'_x(x_1^*, t_1)) + N_S(p^*),$$

Definition

$$\max_{u\in U} \bar{H}(x^*(t), u, \psi(t), \mu(t), t) = \bar{H}(x^*(t), u^*(t), \psi(t), \mu(t), t) \text{ a.a. } t \in [t_0, t_1],$$

$$\int_{t_0}^{t_1} \Big\langle g(x^*(t), t), d\mu(t) \Big\rangle = 0,$$

$$\lambda + \ell\Big(t \in [t_0, t_1] : \psi(t) - \mu(t) g'_x(x^*(t), t) \neq 0\Big) > 0.$$

Theorem

Suppose that the control process $(x^*(t), u^*(t))$ *is optimal to problem (1). Suppose that the state constraints are regular and compatible with the endpoints constraints at the point* p^**, and the controllability condition is satisfied.*
Then, the process $(x^*(t), u^*(t))$ *satisfies non-degenerate maximum principle in the Gamkrelidze form.*

REMARK. Suppose that the set $N_S(p^*)$ is convex. Then, in the Maximum Principle, we can additionally require that μ is continuous at t_0, t_1, and $\mu(t_1) = 0$,

+ the conventional transversality condition is valid

$$(\psi(t_0), -\psi(t_1)) \in \lambda\varphi'(p^*) + N_S(p^*),$$

+ the conventional non-triviality condition is valid

$$\lambda + \max_{t\in[t_0,t_1]} |\psi(t)| + \mathrm{Var}\,|_{t_0}^{t_1}\mu \neq 0.$$

Relation between the two sets of NOC

Under the assumption that g is twice continuously differentiable, the two above forms of the Maximum Principle are equivalent. This is so due to the following Lagrange multipliers change:

$$\mu(t) := \int_{[t,t_1]} d\eta \ \ \forall\, t < t_1, \ \ \mu(t_1) = 0,$$

$$\phi(t) := \psi(t) + \mu(t) g'_x(x^*(t), t), \ \ t \in [t_0, t_1].$$

Some Further Refinements

These results were later refined regarding the continuity properties of the measure. Namely, it was proved that, under the regularity assumptions proposed by Gamkrelidze, the measure-multiplier is continuous. Moreover, under the additional assumption that the data are twice continuously differentiable w.r.t. u, it was proved that the measure enjoys even the Hölder property, that is,

$$|\mu(t) - \mu(s)| \leq \text{const} \sqrt{|t - s|} \quad \forall\, t, s \in [t_0, t_1].$$

The above results were also generalized onto the control problems with equality state constraints.

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