Differential Sensitivity in Rate Independent Problems

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Parabolic control problem with hysteresis

Minimize

$$J(y,u) = \frac{1}{2}\int_\Omega (y(\cdot,T) - y_d)^2\,dx + \frac{\beta}{2}\iint_{\Omega_T} u^2\,dx\,dt$$

subject to $y = Su$, that is

$$\begin{aligned} y_t - \Delta y &= w + u && \text{in } \Omega_T = \Omega \times (0,T) \\ w &= \mathcal{W}[y] && \text{in } \Omega_T \end{aligned}$$

$$y(\cdot,0) = y_0 \text{ on } \Omega, \qquad y = 0 \text{ on } \Gamma_T$$

S = control-to-state mapping

$\mathcal{W}$ = scalar hysteresis operator

Parabolic control problem with hysteresis

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subject to $y = Su$, that is

$$\begin{aligned} y_t - \Delta y &= w + u && \text{in } \Omega_T = \Omega \times (0,T) \\ w &= \mathcal{W}[y] && \text{in } \Omega_T \end{aligned}$$

$$y(\cdot,0) = y_0 \text{ on } \Omega, \qquad y = 0 \text{ on } \Gamma_T$$

$\mathcal{W}$ is not smooth, nonlocal in time, not monotone

S is not smooth, problem not convex

Rate Independence

Input-output system $w = \mathcal{W}[v]$

Inputs $v = v(t),\ v : [0, T] \to X$

Outputs $w = w(t),\ w : [0, T] \to Y$

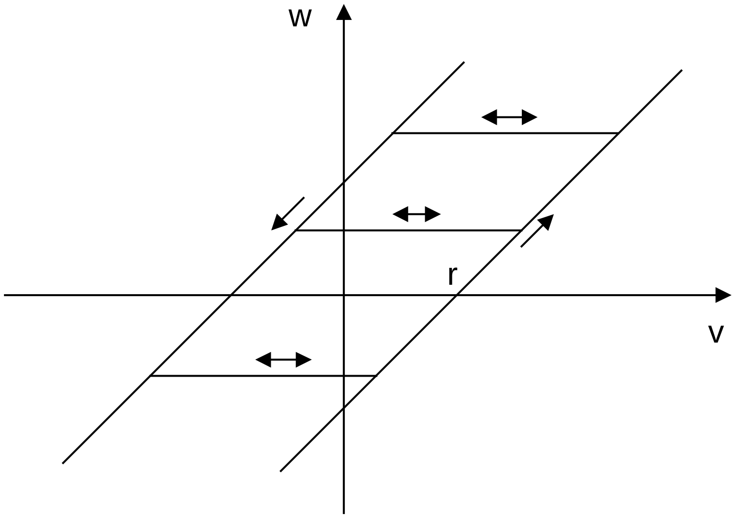
$\mathcal{W}$ is called rate independent, if

$$\mathcal{W}[v \circ \varphi] = \mathcal{W}[v] \circ \varphi$$

for all time transformations $\varphi : [0, T] \to [0, T]$.

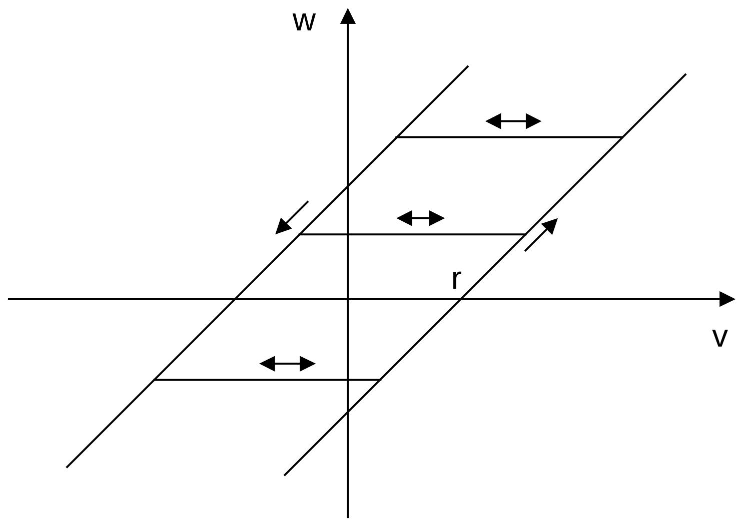
If the input v produces the output w, then the input $\tilde{v}(t) = v(2t)$ produces the output $\tilde{w}(t) = w(2t)$.

Play



$$w = \mathcal{P}_r[v] \qquad\qquad w(t) = (\mathcal{P}_r[v])(t)$$

Play

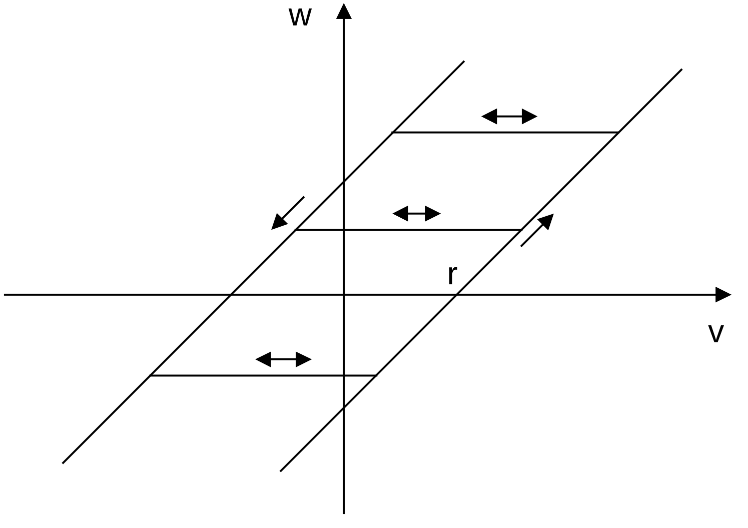


$\dot{w}(t) = 0$ if $|w(t) - v(t)| < r$ (interior)

$\dot{w}(t) = \dot{v}(t)$ if $v(t) - w(t) = r\,, \quad \dot{v}(t) > 0$ (boundary)

$v(t) - w(t) = -r\,, \; \dot{v}(t) < 0$

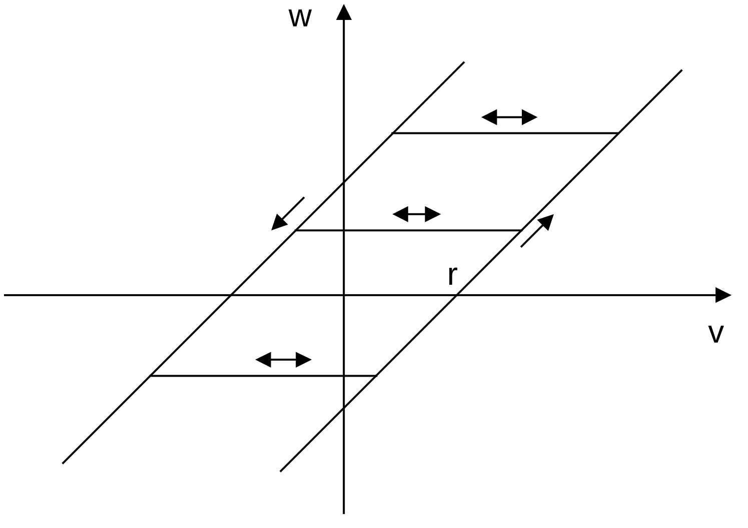
Play



$$\dot{w}(t) \cdot (v(t) - w(t) - \zeta) \geq 0\,, \quad \forall\, \zeta \in Z = [-r, r]$$

$$v(t) - w(t) \in Z\,, \quad w(0) = v(0) - z_0$$

Play

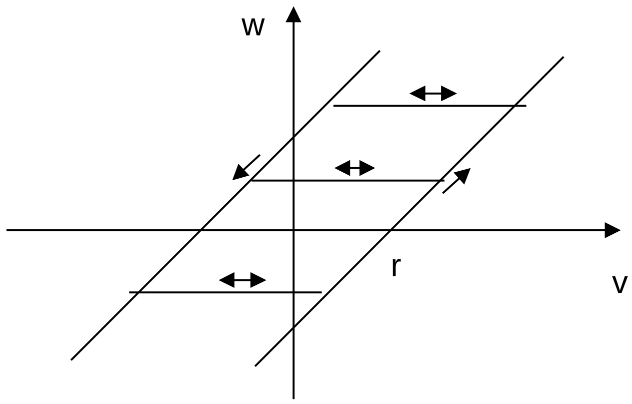


$$w = \mathcal{P}_r[v] \qquad\qquad w(t) = (\mathcal{P}_r[v])(t)$$

More precisely: $w = \mathcal{P}_r[v; z_0], \qquad z_0 \in [-r, r]$

$$w(0) = v(0) - z_0$$

Play: Derivatives ?



$$w = \mathcal{P}_r[v]$$

$$w(t) = (\mathcal{P}_r[v])(t)$$

variation $h = h(t)$

Fix a $t > 0$.

As $\lambda \downarrow 0$

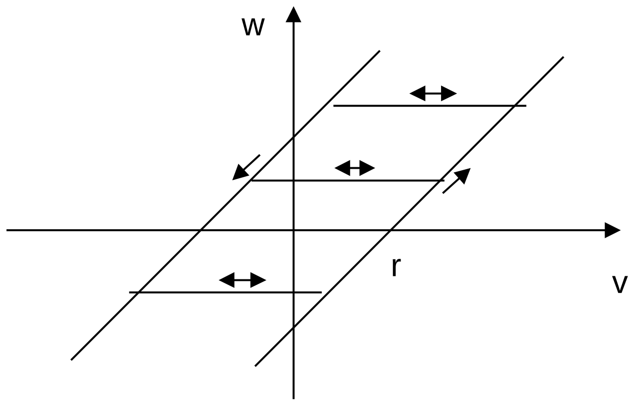
you expect:

$$\frac{\mathcal{P}_r[v + \lambda h](t) - \mathcal{P}_r[v](t)}{\lambda} \to \mathcal{P}'_r[v; h](t)$$

$$\frac{\mathcal{P}_r[v - \lambda h](t) - \mathcal{P}_r[v](t)}{\lambda} \to \mathcal{P}'_r[v; -h](t)$$

But: $\mathcal{P}'_r[v; -h] \neq -\mathcal{P}'_r[v; h]$

Play: Derivatives ?



$$w = \mathcal{P}_r[v]$$

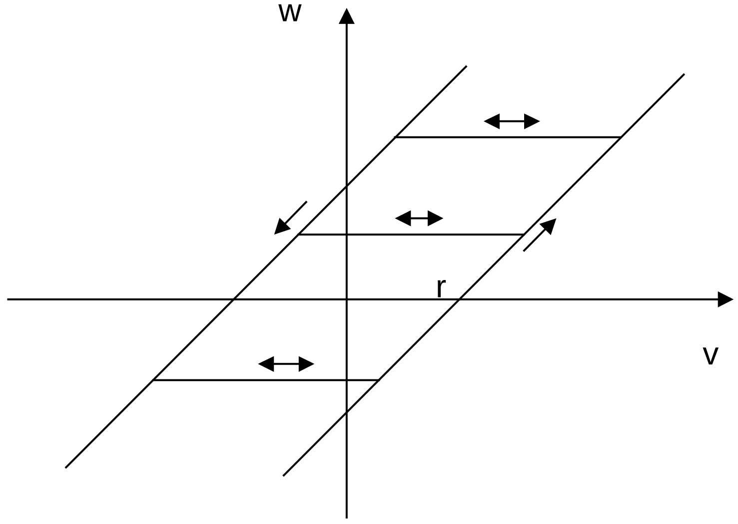
$$w(t) = (\mathcal{P}_r[v])(t)$$

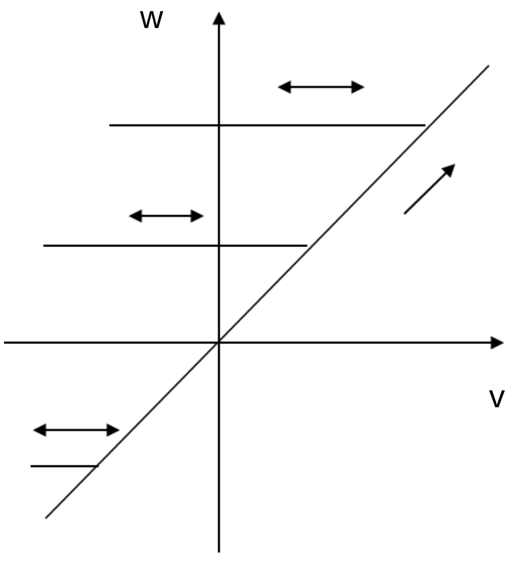
Moreover: Does

$$\lim_{\lambda \downarrow 0} \frac{\mathcal{P}_r[v + \lambda h] - \mathcal{P}_r[v]}{\lambda}\,, \qquad \mathcal{P}_r : X \to Y$$

exist ? In which function spaces X, Y ?

Play: Directional differentiability





$$w = \mathcal{P}_r[v]$$

$$w(t) = (\mathcal{P}_r[v])(t)$$

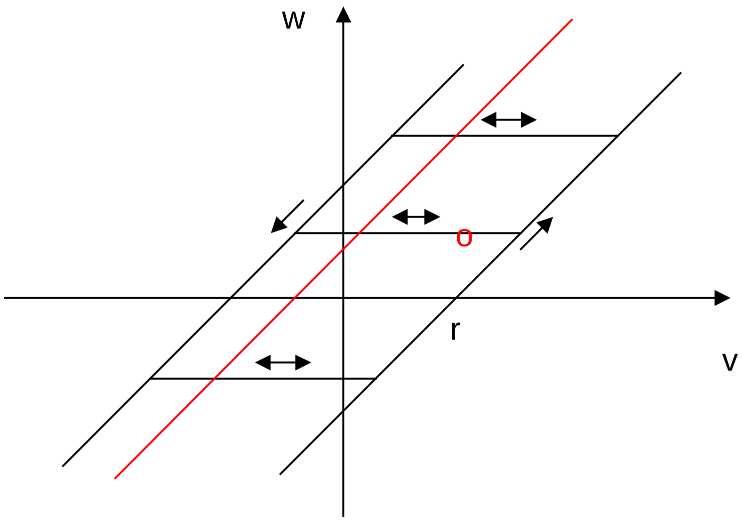
$$w(t) = \max_{0 \leq s \leq t} v(s)$$

(accumulated maximum)

M. Brokate, P. Krejci, Weak differentiability of scalar hysteresis operators,
Discrete Cont. Dyn. Syst. Ser. A 35 (2015), 2405-2421

Play: Local description

Locally, the play can be represented as a finite composition of variants of the accumulated maximum.



$$w = \mathcal{P}_r[v; z_0]$$

o $= (v(t_*), w(t_*))$

$$w(t) = \max\{w(t_*), \max_{t_* \le s \le t} (v(s) - r)\} \qquad \text{for } t \ge t_*$$

From maximum to play

Maximum: $$\varphi(v) = \max_{0 \le s \le T} v(s)$$ $$\varphi : C[0,T] \to \mathbb{R}$$

Accumulated maximum:
(gliding maximum)

$$(Fv)(t) = \max_{0 \le s \le t} v(s)$$

$$F : C[0,T] \to C[0,T]$$

Play:

$$w(t) = (\mathcal{P}_r[v; z_0])(t)$$

$$\mathcal{P}_r : C[0,T] \times [-r,r] \to C[0,T]$$

Maximum functional: Directional derivative

$$\varphi(v) = \max_{0 \le s \le T} v(s)$$

$$\varphi : C[0,T] \to \mathbb{R}$$

convex, Lipschitz

Directional derivative:

$$\varphi'(v;h) = \max_{s \in M(v)} h(s)$$

$$M(v) = \{\tau : v(\tau) = \max v\}$$

$$M : C[0,T] \rightrightarrows [0,T]$$

Directional derivative is a Hadamard derivative:

$$\varphi'(v;h) = \lim_{\lambda \downarrow 0} \frac{\varphi(v + \lambda h + o(\lambda)) - \varphi(v)}{\lambda}$$

Bouligand derivative

Assume: $F : U \subset X \to Y$ is directionally differentiable

The directional derivative is called a Bouligand derivative if

$$\lim_{\|h\| \to 0} \frac{\|F(v+h) - F(v) - F'(v;h)\|}{\|h\|} = 0$$

Still nonlinear w.r.t the direction h.
Better approximation property than the directional derivative:
Limit process is uniform w.r.t. the direction h

Maximum functional: Bouligand derivative

$$\varphi(v) = \max_{0 \le s \le T} v(s)$$

$$\varphi'(v;h) = \max_{s \in M(v)} h(s)$$

$$M(v) = \{\tau : v(\tau) = \max v\}$$

$$\varphi : C[0,T] \to \mathbb{R}$$

$$\varphi : W^{1,1}(0,T) \to \mathbb{R}$$

φ' is not a Bouligand derivative

$$\varphi : C^{0,\alpha}[0,T] \to \mathbb{R}$$

$$\varphi : W^{1,p}(0,T) \to \mathbb{R}$$

φ' is a Bouligand derivative

for all $0 < \alpha \le 1,\ 1 < p \le \infty$

Newton derivative

$$F : U \subset X \to Y$$
$$G : U \to \mathcal{L}(X;Y)$$

G is called a Newton derivative of F in U, if

$$\lim_{\|h\|\to 0} \frac{\|F(v+h) - F(v) - G(v+h)h\|}{\|h\|} = 0$$

for all $v \in U$

Semismooth Newton method: In order to solve $F(v) = 0$

$$G(v_k)(v_{k+1} - v_k) = -F(v_k)$$

Newton derivative (Set valued)

$$F : U \subset X \to Y$$

$$G : U \rightrightarrows \mathcal{L}(X;Y)$$

G is called a Newton derivative of F in U, if

$$\lim_{\|h\| \to 0} \sup_{L \in G(v+h)} \frac{\|F(v+h) - F(v) - Lh\|}{\|h\|} = 0$$

for all $v \in U$

Semismooth Newton method:

$$L_k(v_{k+1} - v_k) = -F(v_k) \qquad L_k \in G(v_k)$$

Maximum functional: Newton derivative

$$\varphi(v) = \max_{0 \le s \le T} v(s) \qquad \varphi : C[0,T] \to \mathbb{R}$$

Directional derivative:

$$\varphi'(v;h) = \max_{s \in M(v)} h(s) \qquad M(v) = \{\tau : v(\tau) = \max v\}$$

Candidate for the Newton derivative Φ:

$$\Phi(v) = \partial\varphi(v)$$

$$\Phi(v) = \{\mu : \mu \in C[0,T]^*, \\ \text{supp}(\mu) \subset M(v),\ \mu \ge 0,\ \|\mu\| = 1\ \}$$

If $M(v) = \{s\}$, then $\Phi(v) = \delta_s$.

Maximum functional: Newton derivative

$$\varphi(v) = \max_{0 \le s \le T} v(s) \qquad M(v) = \{\tau : v(\tau) = \max v\}$$

$$\Phi(v) = \{\mu : \mu \in C[0,T]^*, \\ \mathrm{supp}(\mu) \subset M(v),\ \mu \ge 0,\ \|\mu\| = 1\ \}$$

$\varphi : C[0,T] \to \mathbb{R}$

$\varphi : W^{1,1}(0,T) \to \mathbb{R}$

Φ is not a Newton derivative

$\varphi : C^{0,\alpha}[0,T] \to \mathbb{R}$

$\varphi : W^{1,p}(0,T) \to \mathbb{R}$

Φ is a Newton derivative

for all $0 < \alpha \le 1,\ 1 < p \le \infty$

Proof

$$M(v) = \{\tau : v(\tau) = \max v\}$$

$$\varphi(v+h) - \varphi(v) - \varphi'(v;h) \overset{!}{=} o(\|h\|)$$

$$\varphi(v+h) - \varphi(v) - \langle \mu, h \rangle \overset{!}{=} o(\|h\|) \qquad \text{if } \mu \in \Phi(v+h)$$

$$\varphi'(v;h) \le \varphi(v+h) - \varphi(v) \le \langle \mu, h \rangle$$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall \|h\|_\infty < \delta :$$

$$\langle \mu, h \rangle - \varphi'(v;h) \le \omega(h;\varepsilon) = \sup_{|t-s| \le \varepsilon} |h(t) - h(s)|$$

$$\le \|h\|_{C^{0,\alpha}} \cdot \varepsilon^\alpha$$

$$\langle \mu, h \rangle - \varphi'(v;h) \le \rho_v(\|h\|_\infty) \cdot \|h\|_{C^{0,\alpha}}$$

with $\rho_v(\delta) \to 0$ as $\delta \to 0$

Upper semicontinuity

$$\varphi(v) = \max_{0 \le s \le T} v(s) \qquad\qquad M(v) = \{\tau : v(\tau) = \max v\}$$

$$M : C[0,T] \rightrightarrows [0,T]$$

M is upper semicontinuous (usc):

$$A \subset [0,T] \text{ closed } \Rightarrow M^{-1}(A) \subset C[0,T] \text{ closed}$$

$$M^{-1}(A) := \{v : M(v) \cap A \neq \emptyset\}$$

Upper semicontinuity

$$\varphi(v) = \max_{0 \le s \le T} v(s) \qquad M(v) = \{\tau : v(\tau) = \max v\}$$

$$\Phi(v) = \{\mu : \mu \in C[0,T]^*,$$
$$\operatorname{supp}(\mu) \subset M(v),\ \mu \ge 0,\ \|\mu\| = 1\ \}$$

$$\Phi : C[0,T] \rightrightarrows C[0,T]^*$$

$\Phi(v)$ is convex and w^* seq compact

$\Phi : C[0,T] \rightrightarrows (C[0,T]^*, w^*)$ is usc

From maximum to play

Maximum: $$\varphi(v) = \max_{0 \le s \le T} v(s) \qquad \varphi : C[0,T] \to \mathbb{R}$$

Accumulated maximum:
(gliding maximum)

$$(Fv)(t) = \max_{0 \le s \le t} v(s)$$

$$F : C[0,T] \to C[0,T]$$

Play:

$$w(t) = (\mathcal{P}_r[v; z_0])(t)$$

$$\mathcal{P}_r : C[0,T] \times [-r,r] \to C[0,T]$$

Accumulated maximum

$$(Fv)(t) = \varphi_t(v) = \max_{0 \le s \le t} v(s) \qquad F : C[0,T] \to C[0,T]$$

$$M_t(v) = \{\tau : v(\tau) = \max_{[0,t]} v\,,\ \tau \in [0,t]\}$$

Directional derivative exists ``pointwise in time“:

$$\lim_{\lambda \downarrow 0} \frac{(F(v+\lambda h))(t) - (Fv)(t)}{\lambda} = \varphi_t'(v;h) = \max_{s \in M_t(v)} h(s)$$

Denote

$$F^{PD}(v;h)(t) = \max_{s \in M_t(v)} h(s)$$

Accumulated maximum

$$(Fv)(t) = \varphi_t(v) = \max_{0 \leq s \leq t} v(s) \qquad F : C[0,T] \to C[0,T]$$

The pointwise derivative

$$F^{PD}(v;h) : [0,T] \to \mathbb{R}$$

is a regulated function, but discontinuous in general.

Thus, $F : C[0,T] \to C[0,T]$ is not directionally differentiable

But

$$F : C[0,T] \to L^p[0,T], \; p < \infty,$$

is directionally differentiable.

Accumulated maximum: Newton derivative

$$(Fv)(t) = \varphi_t(v) = \max_{0 \le s \le t} v(s) \qquad F : C[0,T] \to C[0,T]$$

$$M_t(v) = \{\tau : v(\tau) = \max_{[0,t]} v \,,\ \tau \in [0,t]\}$$

$G(v) = \{\mu : [0,T] \to C[0,T]^*$ weakly measurable,

$$\text{supp}(\mu_t) \subset M_t(v),\ \mu_t \ge 0,\ \|\mu_t\| = 1\ \}$$

$$F : C^{0,\alpha}[0,T] \to L^s(0,T)$$

$$F : W^{1,p}(0,T) \to L^s(0,T)$$

G is a Newton derivative

for all $0 < \alpha,\ s < \infty,\ 1 < p$

Accumulated maximum: Usc

$$(Fv)(t) = \varphi_t(v) = \max_{0 \le s \le t} v(s) \qquad F : C[0,T] \to C[0,T]$$

$$M_t(v) = \{\tau : v(\tau) = \max_{[0,t]} v\,,\ \tau \in [0,t]\}$$

$(v,t) \mapsto M_t(v)$ defines a set-valued map

$$M : C[0,T] \times [0,T] \rightrightarrows [0,T]$$

which is upper semicontinuous.

Accumulated maximum: Usc

$$(Fv)(t) = \varphi_t(v) = \max_{0 \le s \le t} v(s) \qquad F : C[0,T] \to C[0,T]$$

$$M_t(v) = \{\tau : v(\tau) = \max_{[0,t]} v\,,\ \tau \in [0,t]\}$$

$$\Phi_t(v) = \{\mu : \mu \in C[0,T]^*,$$
$$\mathrm{supp}(\mu) \subset M_t(v),\ \mu \ge 0,\ \|\mu\| = 1\ \}$$

$(v,t) \mapsto \Phi_t(v)$ defines a set-valued map

$$\Phi : C[0,T] \times [0,T] \rightrightarrows (C[0,T]^*, w^*)$$

which is upper semicontinuous.

Measurable selectors of Φ

yield elements of the Newton derivative of F

Play: Newton derivative

$$w = \mathcal{P}_r[v; z_0]$$

Theorem. The play operator

$$\mathcal{P}_r : C^{0,\alpha}[0,T] \times [-r,r] \to L^s(0,T) \qquad \alpha > 0,\ s < \infty$$

$$\mathcal{P}_r : W^{1,p}[0,T] \times [-r,r] \to L^s(0,T) \qquad p > 1,\ s < \infty$$

is Newton differentiable.

The proof uses the chain rule for Newton derivatives, and yields a recursive formula based on the successive accumulated maxima.

M. Brokate, Newton and Bouligand derivatives of the scalar play and stop operator, arXiv:1607.07344, 2016

Play: Newton derivative

$$w = \mathcal{P}_r[v; z_0]$$

$$\mathcal{P}_r : (X \cap U_\delta) \times [-r, r] \to L^s(0, T), \qquad X = C^{0,\alpha} \text{ or } W^{1,p},$$

Newton derivative

$$G^r : (X \cap U_\delta) \times [-r, r] \rightrightarrows \mathcal{L}(X \times \mathbb{R}; L^s(0, T)),$$

The linear mappings $L^r \in G^r$ satisfy

$$\|L^r(h, q)\|_\infty \le \|h\|_\infty + |q|$$

$$\|L^r(h, q)\|_{L^s} \le T^{1/s}(c_X\|h\|_X + |q|)$$

Newton derivative is globally bounded

(in particular, the bound does not depend on the local description of the play in U_δ, nor on r)

Play: Newton derivative

$$w = \mathcal{P}_r[v]$$

$$\mathcal{P}_r : X \cap U_\delta \to L^s(0,T), \qquad X = C^{0,\alpha} \text{ or } W^{1,p},$$

Newton derivative $L^r \in G^r(v+h)$ satisfies

$$\|\mathcal{P}_r(v+h - \mathcal{P}_r(v) - L^r(h))\|_{L^s} \le \rho_v(\|h\|_X) \cdot \|h\|_X$$

where $\rho_v(\delta) \to 0$ as $\delta \to 0$

Can be improved to

$$\|\mathcal{P}_r(v+h - \mathcal{P}_r(v) - L^r(h))\|_{L^s} \le \rho_v(\|h\|_\infty) \cdot \|h\|_X$$

where $\rho_v(\delta) \to 0$ as $\delta \to 0$, and $|\rho_v| \le C$

Play: Bouligand derivative

$$w = \mathcal{P}_r[v; z_0]$$

Theorem. The play operator

$$\mathcal{P}_r : C^{0,\alpha}[0,T] \times \mathbb{R} \to L^s(0,T) \qquad \alpha > 0,\ s < \infty$$

$$\mathcal{P}_r : W^{1,p}[0,T] \times \mathbb{R} \to L^s(0,T) \qquad p > 1,\ s < \infty$$

is Bouligand differentiable.

A refined remainder estimate holds as in the Newton case.

Parabolic problem

$$y_t - \Delta y = w + u \qquad \text{in } \Omega_T$$
$$w = \mathcal{W}[y] \qquad \text{in } \Omega_T$$

$$y(\cdot, 0) = y_0 \text{ on } \Omega, \qquad y = 0 \text{ on } \Gamma_T$$

Solution operator $\quad y = Su$

M. Brokate, K. Fellner, M. Lang-Batsching, Weak differentiability of the control-to-state mapping in a parabolic control problem with hysteresis, Preprint IGDK 1754

Parabolic problem

$$y_t - \Delta y = w + u \qquad \text{in } \Omega_T$$
$$w = \mathcal{W}[y] \qquad \text{in } \Omega_T$$

$$y(\cdot, 0) = y_0 \text{ on } \Omega, \qquad y = 0 \text{ on } \Gamma_T$$

Solution operator $y = Su$

Operator $\mathcal{W}$ acts pointwise:

$$\mathcal{W}[y](x,t) = \mathcal{W}_p[y(x,\cdot)](t)$$

Assumptions: $$|\mathcal{W}_p[v](t)| \leq L \sup_{s \leq t} |v(s)| + c_0$$

$$|(\mathcal{W}_p[v] - \mathcal{W}_p[\tilde{v}])(t)| \leq L \sup_{s \leq t} (|v(s) - \tilde{v}(s)|$$

Then

$$\mathcal{W} : L^2(\Omega; C[0,T]) \to L^2(\Omega; C[0,T])$$

Parabolic problem

$$y_t - \Delta y = w + u \qquad \text{in } \Omega_T$$

$$w = \mathcal{W}[y] \qquad \text{in } \Omega_T$$

$$y(\cdot, 0) = y_0 \text{ on } \Omega, \qquad y = 0 \text{ on } \Gamma_T$$

Theorem: (Visintin, Hilpert)

Given $u \in L^2(\Omega_T)$, $y_0 \in V = H_0^1(\Omega)$, $\exists |$ sol'n

$$y \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; V)$$

$$w \in L^2(\Omega; C[0, T])$$

Solution operator $\qquad y = Su$

$$S : L^2(\Omega_T) \to H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; V)$$

First order problem

$$d_t - \Delta d = h + p \qquad \text{in } \Omega_T$$
$$p = \mathcal{W}'[y; d] \qquad \text{in } \Omega_T$$

$$d(\cdot, 0) = 0 \text{ on } \Omega, \qquad d = 0 \text{ on } \Gamma_T$$

Goal: $S(u + h) = y + d + o(\|h\|)$

Theorem: (variant of Visintin) $\exists|$ sol'n

$$d \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; V)$$
$$p \in L^2(\Omega; G[0, T])$$

Dependence on h ?

Auxiliary estimates

$$z_t - \Delta z = g \qquad \text{in } \Omega_T$$

$$z(\cdot, 0) = 0 \text{ on } \Omega, \qquad z = 0 \text{ on } \Gamma_T$$

Assume

$$|g(x,t)| \leq L \sup_{s \leq t} |z(x,s)| + |f(x,t)|$$

Then

$$\iint_{\Omega_T} z_t^2 \, dx \, dt + \sup_t \int_\Omega |\nabla z(\cdot, t)|^2 \, dx \leq C \iint_{\Omega_T} f^2 \, dx \, dt$$

$$\sup_{\Omega_T} |z| \leq C \int_0^T \sup_x |f(\cdot, s)| \, ds$$

First order problem

$$d_t - \Delta d = h + p \qquad \text{in } \Omega_T$$
$$p = \mathcal{W}'[y; d] \qquad \text{in } \Omega_T$$

$$d(\cdot, 0) = 0 \text{ on } \Omega, \qquad d = 0 \text{ on } \Gamma_T$$

Goal: $S(u+h) = y + d + o(\|h\|)$

Theorem:

$$\iint_{\Omega_T} d_t^2 \, dx \, dt + \sup_t \int_\Omega |\nabla d(\cdot, t)|^2 \, dx \le C \iint_{\Omega_T} h^2 \, dx \, dt$$

$$\sup_{\Omega_T} |d| \le C \int_0^T \sup_x |h(\cdot, s) \, ds$$

Sensitivity result

Goal: $S(u+h) = y + d + o(\|h\|)$

Theorem:

The control-to-state mapping has a Bouligand derivative when considered as a mapping

$$S : L^2(0,T;L^\infty(\Omega)) \to H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)$$

Proof: Estimates for the remainder problem

$$r := y_h - y - d, \; q := w_h - w - p$$

$$r_t - \Delta r = q \qquad \text{in } \Omega_T$$

$$q = \mathcal{W}[y_h] - \mathcal{W}[y] - \mathcal{W}'[y;d] \qquad \text{in } \Omega_T$$

$$r(\cdot,0) = 0 \text{ on } \Omega, \qquad r = 0 \text{ on } \Gamma_T$$

Sensitivity result

Theorem:

The control-to-state mapping has a Bouligand derivative when considered as a mapping

$$S : L^2(0,T;L^\infty(\Omega)) \to H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)$$

An analogous result holds for the Newton derivative, for the special case of the accumulated maximum.

Parabolic control problem with hysteresis

Minimize

$$J(y,u) = \frac{1}{2}\int_\Omega (y(\cdot,T) - y_d)^2\,dx + \frac{\beta}{2}\iint_{\Omega_T} u^2\,dx\,dt$$

subject to $y = Su$, that is

$$\begin{aligned} y_t - \Delta y &= w + u && \text{in } \Omega_T = \Omega \times (0,T) \\ w &= \mathcal{W}[y] && \text{in } \Omega_T \end{aligned}$$

$$y(\cdot,0) = y_0 \text{ on } \Omega, \qquad y = 0 \text{ on } \Gamma_T$$

Optimality condition

Reduced cost functional $j(u) = J(Su, u)$

At a minimizer u: for the directional derivative

$$j'(u;h) \geq 0 \qquad \text{for all } h$$

$$j'(u;h) = \int_\Omega (y(\cdot,T) - y_d)\, d(\cdot,T)\, dx + \beta \iint_{\Omega_T} uh\, dx\, dt$$

where d solves the first order problem

$$d_t - \Delta d = h + p \qquad \text{in } \Omega_T$$

$$p = \mathcal{W}'[y;d] \qquad \text{in } \Omega_T$$

$$d(\cdot,0) = 0 \text{ on } \Omega, \qquad d = 0 \text{ on } \Gamma_T$$