Differential Sensitivity in Rate Independent Problems

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Parabolic control problem with hysteresis

Minimize

$$J(y, u) = \frac{1}{2} \int_\Omega (y(\cdot, T) - y_d)^2 \, dx + \frac{\beta}{2} \iint_{\Omega_T} u^2 \, dx \, dt$$

subject to $y = Su$, that is

$$y_t - \Delta y = w + u \quad \text{in } \Omega_T = \Omega \times (0, T)$$
$$w = \mathcal{W}[y] \quad \text{in } \Omega_T$$

$$y(\cdot, 0) = y_0 \text{ on } \Omega, \quad y = 0 \text{ on } \Gamma_T$$

$S = \text{control-to-state mapping}$
$\mathcal{W} = \text{scalar hysteresis operator}$
Parabolic control problem with hysteresis

Minimize

\[ J(y, u) = \frac{1}{2} \int_{\Omega} (y(\cdot, T) - y_d)^2 \, dx + \frac{\beta}{2} \int_{\Omega_T} u^2 \, dx \, dt \]

subject to \( y = Su, \) that is

\[ y_t - \Delta y = w + u \quad \text{in} \quad \Omega_T = \Omega \times (0, T) \]
\[ w = \mathcal{W}[y] \quad \text{in} \quad \Omega_T \]
\[ y(\cdot, 0) = y_0 \text{ on } \Omega, \quad y = 0 \text{ on } \Gamma_T \]

\( \mathcal{W} \) is not smooth, nonlocal in time, not monotone

\( S \) is not smooth, problem not convex
Rate Independence

Input-output system \[ w = \mathcal{W}[v] \]

Inputs \[ v = v(t), \; v : [0, T] \rightarrow X \]

Outputs \[ w = w(t), \; w : [0, T] \rightarrow Y \]

\( \mathcal{W} \) is called rate independent, if

\[ \mathcal{W}[v \circ \varphi] = \mathcal{W}[v] \circ \varphi \]

for all time transformations \( \varphi : [0, T] \rightarrow [0, T] \).

If the input \( v \) produces the output \( w \), then the input \( \tilde{v}(t) = v(2t) \) produces the output \( \tilde{w}(t) = w(2t) \).
Play

\[ w = \mathcal{P}_r[v] \quad w(t) = (\mathcal{P}_r[v])(t) \]
\[ w(t) = 0 \quad \text{if} \quad |w(t) - v(t)| < r \quad \text{(interior)} \]

\[ w(t) = \dot{v}(t) \quad \text{if} \quad v(t) - w(t) = r, \quad \dot{v}(t) > 0 \quad \dot{v}(t) < 0 \quad \text{(boundary)} \]
\[ \dot{w}(t) \cdot (v(t) - w(t) - \zeta) \geq 0, \quad \forall \zeta \in Z = [-r, r] \]
\[ v(t) - w(t) \in Z, \quad w(0) = v(0) - z_0 \]
\[ w = \mathcal{P}_r[v] \quad \quad \quad w(t) = (\mathcal{P}_r[v])(t) \]

More precisely: \[ w = \mathcal{P}_r[v; z_0], \quad z_0 \in [-r, r] \]
\[ w(0) = v(0) - z_0 \]
Play: Derivatives?

Fix a $t > 0$.

As $\lambda \downarrow 0$

you expect:

$$
\frac{P_r[v + \lambda h](t) - P_r[v](t)}{\lambda} \rightarrow P_r'[v; h](t)
$$

$$
\frac{P_r[v - \lambda h](t) - P_r[v](t)}{\lambda} \rightarrow P_r'[v; -h](t)
$$

But: $P_r'[v; -h] \neq -P_r'[v; h]$
Play: Derivatives?

Moreover: Does

\[ w = \mathcal{P}_r[v] \]

\[ w(t) = (\mathcal{P}_r[v])(t) \]

Moreover: Does

\[ \lim_{\lambda \downarrow 0} \frac{\mathcal{P}_r[v + \lambda h] - \mathcal{P}_r[v]}{\lambda}, \quad \mathcal{P}_r : X \to Y \]

exist? In which function spaces \( X, Y \)?
Play: Directional differentiability

\[ w = \mathcal{P}_r[v] \]
\[ w(t) = (\mathcal{P}_r[v])(t) \]

\[ w(t) = \max_{0 \leq s \leq t} v(s) \]
(accumulated maximum)

Locally, the play can be represented as a finite composition of variants of the accumulated maximum.

\[ w = \mathcal{P}_r[v; z_0] \]

\[ \mathbf{0} = (v(t_*), w(t_*)) \]

\[ w(t) = \max\{w(t_*), \max_{t_* \leq s \leq t} (v(s) - r)\} \quad \text{for } t \geq t_* \]
From maximum to play

Maximum: \[ \varphi(v) = \max_{0 \leq s \leq T} v(s) \quad \varphi : C[0, T] \to \mathbb{R} \]

Accumulated maximum: (gliding maximum) \[ (Fv)(t) = \max_{0 \leq s \leq t} v(s) \]
\[ F : C[0, T] \to C[0, T] \]

Play:
\[ w(t) = (\mathcal{P}_r[v; z_0])(t) \]
\[ \mathcal{P}_r : C[0, T] \times [-r, r] \to C[0, T] \]
Maximum functional: Directional derivative

\[ \varphi(v) = \max_{0 \leq s \leq T} v(s) \]

\[ \varphi : C[0, T] \to \mathbb{R} \]

convex, Lipschitz

Directional derivative:

\[ \varphi'(v; h) = \max_{s \in M(v)} h(s) \]

\[ M(v) = \{ \tau : v(\tau) = \max v \} \]

\[ M : C[0, T] \Rightarrow [0, T] \]

Directional derivative is a Hadamard derivative:

\[ \varphi'(v; h) = \lim_{\lambda \downarrow 0} \frac{\varphi(v + \lambda h + o(\lambda)) - \varphi(v)}{\lambda} \]
Bouligand derivative

Assume: \( F : U \subset X \rightarrow Y \) is directionally differentiable

The directional derivative is called a Bouligand derivative if

\[
\lim_{\|h\| \rightarrow 0} \frac{\| F(v + h) - F(v) - F'(v; h) \|}{\|h\|} = 0
\]

Still nonlinear w.r.t the direction \( h \).
Better approximation property than the directional derivative:
Limit process is uniform w.r.t. the direction \( h \)
Maximum functional: Bouligand derivative

$$\varphi(v) = \max_{0 \leq s \leq T} v(s)$$

$$\varphi'(v; h) = \max_{s \in M(v)} h(s)$$

$$M(v) = \{\tau : v(\tau) = \max v\}$$

$$\varphi : C[0, T] \to \mathbb{R}$$

$$\varphi : W^{1,1}(0, T) \to \mathbb{R}$$

$$\varphi : C^{0, \alpha}[0, T] \to \mathbb{R}$$

$$\varphi : W^{1,p}(0, T) \to \mathbb{R}$$

for all $0 < \alpha \leq 1, 1 < p \leq \infty$
Newton derivative

\[ F : U \subset X \rightarrow Y \]
\[ G : U \rightarrow \mathcal{L}(X; Y) \]

\( G \) is called a Newton derivative of \( F \) in \( U \), if

\[ \lim_{\|h\| \to 0} \frac{\|F(v + h) - F(v) - G(v + h)h\|}{\|h\|} = 0 \]

for all \( v \in U \)

Semismooth Newton method: In order to solve \( F(v) = 0 \)

\[ G(v_k)(v_{k+1} - v_k) = -F(v_k) \]
Newton derivative (Set valued)

\[ F : U \subset X \rightarrow Y \]
\[ G : U \Rightarrow \mathcal{L}(X; Y) \]

\( G \) is called a Newton derivative of \( F \) in \( U \), if

\[
\lim_{\|h\| \rightarrow 0} \sup_{L \in G(v+h), L \in G(v)} \frac{\|F(v + h) - F(v) - Lh\|}{\|h\|} = 0
\]

for all \( v \in U \)

Semismooth Newton method:

\[
L_k(v_{k+1} - v_k) = -F(v_k) \quad L_k \in G(v_k)
\]
Maximum functional: Newton derivative

\[ \varphi(v) = \max_{0 \leq s \leq T} v(s) \quad \varphi : C[0, T] \to \mathbb{R} \]

Directional derivative:

\[ \varphi'(v; h) = \max_{s \in M(v)} h(s) \quad M(v) = \{ \tau : v(\tau) = \max v \} \]

Candidate for the Newton derivative \( \Phi \):

\[ \Phi(v) = \partial \varphi(v) \]

\[ \Phi(v) = \{ \mu : \mu \in C[0, T]^*, \quad \text{supp}(\mu) \subset M(v), \quad \mu \geq 0, \quad \|\mu\| = 1 \} \]

If \( M(v) = \{ s \} \), then \( \Phi(v) = \delta_s \).
Maximum functional: Newton derivative

$$\varphi(v) = \max_{0 \leq s \leq T} v(s) \quad \quad M(v) = \{ \tau : v(\tau) = \max v \}$$

$$\Phi(v) = \{ \mu : \mu \in C[0, T]^*, \quad \supp(\mu) \subset M(v), \mu \geq 0, \|\mu\| = 1 \}$$

$$\varphi : C[0, T] \rightarrow \mathbb{R}$$

$$\varphi : W^{1,1}(0, T) \rightarrow \mathbb{R}$$

$$\varphi : C^{0,\alpha}[0, T] \rightarrow \mathbb{R}$$

$$\varphi : W^{1,p}(0, T) \rightarrow \mathbb{R}$$

for all $$0 < \alpha \leq 1, 1 < p \leq \infty$$

$$\Phi$$ is not a Newton derivative

$$\Phi$$ is a Newton derivative
Proof

\[ M(v) = \{ \tau : v(\tau) = \max v \} \]

\[ \varphi(v + h) - \varphi(v) - \varphi'(v; h) = o(\|h\|) \]

\[ \varphi(v + h) - \varphi(v) - \langle \mu, h \rangle = o(\|h\|) \quad \text{if } \mu \in \Phi(v + h) \]

\[ \varphi'(v; h) \leq \varphi(v + h) - \varphi(v) \leq \langle \mu, h \rangle \]

\[ \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall \|h\|_\infty < \delta : \]

\[ \langle \mu, h \rangle - \varphi'(v; h) \leq \omega(h; \varepsilon) = \sup_{|t-s| \leq \varepsilon} |h(t) - h(s)| \]

\[ \leq \|h\|_{C^0,\alpha} \cdot \varepsilon^\alpha \]

\[ \langle \mu, h \rangle - \varphi'(v; h) \leq \rho_v(\|h\|_\infty) \cdot \|h\|_{C^0,\alpha} \]

with \( \rho_v(\delta) \to 0 \) as \( \delta \to 0 \)
Upper semicontinuity

\[ \varphi(v) = \max_{0 \leq s \leq T} v(s) \quad M(v) = \{ \tau : v(\tau) = \max v \} \]

\[ M : C[0, T] \Rightarrow [0, T] \]

\( M \) is upper semicontinuous (usc):

\[ A \subset [0, T] \text{ closed} \Rightarrow M^{-1}(A) \subset C[0, T] \text{ closed} \]

\[ M^{-1}(A) := \{ v : M(v) \cap A \neq \emptyset \} \]
Upper semicontinuity

\[ \varphi(v) = \max_{0 \leq s \leq T} v(s) \quad \text{and} \quad M(v) = \{ \tau : v(\tau) = \max v \} \]

\[ \Phi(v) = \{ \mu : \mu \in C[0, T]^* , \text{ supp}(\mu) \subset M(v), \mu \geq 0, \|\mu\| = 1 \} \]

\[ \Phi : C[0, T] \Rightarrow C[0, T]^* \]

\( \Phi(v) \) is convex and \( w^* \) seq compact

\[ \Phi : C[0, T] \Rightarrow (C[0, T]^*, w^*) \text{ is usc} \]
From maximum to play

Maximum:
\[ \varphi(v) = \max_{0 \leq s \leq T} v(s) \quad \varphi : C[0, T] \to \mathbb{R} \]

Accumulated maximum:
(gliding maximum)
\[ (Fv)(t) = \max_{0 \leq s \leq t} v(s) \]
\[ F : C[0, T] \to C[0, T] \]

Play:
\[ w(t) = (\mathcal{P}_r[v; z_0])(t) \]
\[ \mathcal{P}_r : C[0, T] \times [-r, r] \to C[0, T] \]
Accumulated maximum

\[(Fv)(t) = \varphi_t(v) = \max_{0 \leq s \leq t} v(s) \quad F : C[0, T] \to C[0, T]\]

\[M_t(v) = \{ \tau : v(\tau) = \max_{[0,t]} v, \tau \in [0, t] \}\]

Directional derivative exists ``pointwise in time``:

\[\lim_{\lambda \downarrow 0} \frac{(F(v + \lambda h))(t) - (Fv)(t)}{\lambda} = \varphi'_t(v; h) = \max_{s \in M_t(v)} h(s)\]

Denote

\[F^{PD}(v; h)(t) = \max_{s \in M_t(v)} h(s)\]
Accumulated maximum

\[(F v)(t) = \varphi_t(v) = \max_{0 \leq s \leq t} v(s)\]  \hspace{1cm} F : C[0, T] \rightarrow C[0, T]

The pointwise derivative

\[F^{PD}(v; h) : [0, T] \rightarrow \mathbb{R}\]

is a regulated function, but discontinuous in general.

Thus, \[F : C[0, T] \rightarrow C[0, T]\] is not directionally differentiable.

But \[F : C[0, T] \rightarrow L^p[0, T], \ p < \infty,\]

is directionally differentiable.
Accumulated maximum: Newton derivative

\[(Fv)(t) = \varphi_t(v) = \max_{0 \leq s \leq t} v(s) \]

\[F : C[0, T] \to C[0, T] \]

\[M_t(v) = \{ \tau : v(\tau) = \max_{[0,t]} v, \tau \in [0, t] \} \]

\[G(v) = \{ \mu : [0, T] \to C[0, T]^* \text{ weakly measurable,} \]
\[\text{supp}(\mu_t) \subset M_t(v), \mu_t \geq 0, \|\mu_t\| = 1 \} \]

\[F : C^{0,\alpha}[0, T] \to L^s(0, T) \]
\[F : W^{1,p}(0, T) \to L^s(0, T) \]

\[G \text{ is a Newton derivative} \]

for all \(0 < \alpha, s < \infty, 1 < p\)
Accumulated maximum: $\text{Usc}$

$$(Fv)(t) = \varphi_t(v) = \max_{0 \leq s \leq t} v(s) \quad F : C[0, T] \to C[0, T]$$

$$M_t(v) = \{ \tau : v(\tau) = \max_{[0,t]} v, \tau \in [0, t] \}$$

$$(v, t) \mapsto M_t(v)$$ defines a set-valued map

$$M : C[0, T] \times [0, T] \Rightarrow [0, T]$$

which is upper semicontinuous.
Accumulated maximum: Usc

\[(Fv)(t) = \varphi_t(v) = \max_{0 \leq s \leq t} v(s) \quad F : C[0,T] \rightarrow C[0,T]\]

\[M_t(v) = \{ \tau : v(\tau) = \max_{[0,t]} v, \tau \in [0,t] \}\]

\[\Phi_t(v) = \{ \mu : \mu \in C[0,T]^*, \supp(\mu) \subset M_t(v), \mu \geq 0, \|\mu\| = 1 \}\]

\((v, t) \mapsto \Phi_t(v)\) defines a set-valued map

\[\Phi : C[0,T] \times [0,T] \rightrightarrows (C[0,T]^*, w^*)\]

which is upper semicontinuous.

Measurable selectors of \(\Phi\)

yield elements of the Newton derivative of \(F\)
Play: Newton derivative

\[ w = \mathcal{P}_r[v; z_0] \]

**Theorem.** The play operator

\[
\mathcal{P}_r : C^{0,\alpha}[0, T] \times [-r, r] \to L^s(0, T) \quad \alpha > 0, \ s < \infty
\]

\[
\mathcal{P}_r : W^{1,p}[0, T] \times [-r, r] \to L^s(0, T) \quad p > 1, \ s < \infty
\]

is Newton differentiable.

The proof uses the chain rule for Newton derivatives, and yields a recursive formula based on the successive accumulated maxima.

Play: Newton derivative

\[ w = \mathcal{P}_r[v; z_0] \]

\[ \mathcal{P}_r : (X \cap U_\delta) \times [-r, r] \to L^s(0, T), \quad X = C^{0,\alpha} \text{ or } W^{1,p}, \]

Newton derivative

\[ G^r : (X \cap U_\delta) \times [-r, r] \Rightarrow \mathcal{L}(X \times \mathbb{R}; L^s(0, T)), \]

The linear mappings \( L^r \in G^r \) satisfy

\[ \|L^r(h, q)\|_\infty \leq \|h\|_\infty + |q| \]

\[ \|L^r(h, q)\|_{L^s} \leq T^{1/s}(c_X \|h\|_X + |q|) \]

Newton derivative is **globally bounded**

(in particular, the bound does not depend on the local description of the play in \( U_\delta \), nor on \( r \))
Play: Newton derivative

\[ w = \mathcal{P}_r[v] \]

\[ \mathcal{P}_r : X \cap U_\delta \to L^s(0, T), \quad X = C^{0, \alpha} \text{ or } W^{1, p}, \]

Newton derivative \( L^r \in G^r(v + h) \) satisfies

\[ \| \mathcal{P}_r(v + h - \mathcal{P}_r(v) - L^r(h)) \|_{L^s} \leq \rho_v(\|h\|_X) \cdot \|h\|_X \]

where \( \rho_v(\delta) \to 0 \) as \( \delta \to 0 \)

Can be improved to

\[ \| \mathcal{P}_r(v + h - \mathcal{P}_r(v) - L^r(h)) \|_{L^s} \leq \rho_v(\|h\|_{\infty}) \cdot \|h\|_X \]

where \( \rho_v(\delta) \to 0 \) as \( \delta \to 0 \), and \( |\rho_v| \leq C \)
Play: Bouligand derivative

\[ w = \mathcal{P}_r[v; z_0] \]

**Theorem.** The play operator

\[
\mathcal{P}_r : C^{0,\alpha}[0, T] \times \mathbb{R} \to L^s(0, T) \quad \alpha > 0, \ s < \infty \\
\mathcal{P}_r : W^{1,p}[0, T] \times \mathbb{R} \to L^s(0, T) \quad p > 1, \ s < \infty
\]

is Bouligand differentiable.

A refined remainder estimate holds as in the Newton case.
Parabolic problem

\[ \dot{y} - \Delta y = w + u \quad \text{in } \Omega_T \]
\[ w = \mathcal{W}[y] \quad \text{in } \Omega_T \]
\[ y(\cdot, 0) = y_0 \text{ on } \Omega, \quad y = 0 \text{ on } \Gamma_T \]

Solution operator \quad y = Su

M. Brokate, K. Fellner, M. Lang-Batsching, Weak differentiability of the control-to-state mapping in a parabolic control problem with hysteresis, Preprint IGDK 1754
Parabolic problem

\[ y_t - \Delta y = w + u \quad \text{in } \Omega_T \]
\[ w = \mathcal{W}[y] \quad \text{in } \Omega_T \]
\[ y(\cdot, 0) = y_0 \text{ on } \Omega, \quad y = 0 \text{ on } \Gamma_T \]

Solution operator \[ y = S u \]

Operator \( \mathcal{W} \) acts pointwise:
\[ \mathcal{W}[y](x, t) = \mathcal{W}_p[y(x, \cdot)](t) \]

Assumptions:
\[ |\mathcal{W}_p[v](t)| \leq L \sup_{s \leq t} |v(s)| + c_0 \]
\[ |(\mathcal{W}_p[v] - \mathcal{W}_p[\tilde{v}])(t)| \leq L \sup_{s \leq t} (|v(s) - \tilde{v}(s)|) \]

Then
\[ \mathcal{W} : L^2(\Omega; C[0, T]) \rightarrow L^2(\Omega; C[0, T]) \]
Parabolic problem

\[ y_t - \Delta y = w + u \quad \text{in } \Omega_T \]
\[ w = \mathcal{W}[y] \quad \text{in } \Omega_T \]
\[ y(\cdot, 0) = y_0 \text{ on } \Omega, \quad y = 0 \text{ on } \Gamma_T \]

**Theorem:** (Visintin, Hilpert)

Given \( u \in L^2(\Omega_T), \ y_0 \in V = H^1_0(\Omega), \ \exists! \ \text{sol’n} \)
\[ y \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; V) \]
\[ w \in L^2(\Omega; C[0, T]) \]

Solution operator \( y = Su \)
\[ S : L^2(\Omega_T) \rightarrow H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; V) \]
First order problem

\[ d_t - \Delta d = h + p \quad \text{in} \quad \Omega_T \]
\[ p = \mathcal{W}'[y; d] \quad \text{in} \quad \Omega_T \]
\[ d(\cdot, 0) = 0 \text{ on } \Omega, \quad d = 0 \text{ on } \Gamma_T \]

Goal: \( S(u + h) = y + d + o(\|h\|) \)

**Theorem:** (variant of Visintin) \( \exists \text{ sol'n} \)

\[ d \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; V) \]
\[ p \in L^2(\Omega; G[0, T]) \]

Dependence on \( h \) ?
Auxiliary estimates

\[ z_t - \Delta z = g \quad \text{in } \Omega_T \]

\[ z(\cdot, 0) = 0 \text{ on } \Omega, \quad z = 0 \text{ on } \Gamma_T \]

Assume

\[ |g(x, t)| \leq L \sup_{s \leq t} |z(x, s)| + |f(x, t)| \]

Then

\[
\iint_{\Omega_T} z_t^2 \, dx \, dt + \sup_t \int_{\Omega} |\nabla z(\cdot, t)|^2 \, dx \leq C \iint_{\Omega_T} f^2 \, dx \, dt
\]

\[
\sup_{\Omega_T} |z| \leq C \int_0^T \sup_x |f(\cdot, s)| \, ds
\]
First order problem

\[ d_t - \Delta d = h + p \quad \text{in } \Omega_T \]

\[ p = \mathcal{W}'[y; d] \quad \text{in } \Omega_T \]

\[ d(\cdot, 0) = 0 \text{ on } \Omega, \quad d = 0 \text{ on } \Gamma_T \]

Goal: \( S(u + h) = y + d + o(\|h\|) \)

Theorem:

\[
\int_\Omega \int_{\Omega_T} \frac{d_t^2}{dx^2} dt + \sup_t \int_\Omega |\nabla d(\cdot, t)|^2 dx \leq C \int_\Omega \int_{\Omega_T} h^2 dx dt
\]

\[
\sup_{\Omega_T} |d| \leq C \int_0^T \sup_x |h(\cdot, s)| ds
\]
Sensitivity result

Goal:  \( S(u + h) = y + d + o(\|h\|) \)

Theorem:

The control-to-state mapping has a Bouligand derivative when considered as a mapping

\[ S : L^2(0, T; L^\infty(\Omega)) \to H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; V) \]

Proof: Estimates for the remainder problem

\[ r := y_h - y - d, \quad q := w_h - w - p \]

\[ r_t - \Delta r = q \quad \text{in} \quad \Omega_T \]

\[ q = \mathcal{W}[y_h] - \mathcal{W}[y] - \mathcal{W}'[y; d] \quad \text{in} \quad \Omega_T \]

\[ r(\cdot, 0) = 0 \text{ on } \Omega, \quad r = 0 \text{ on } \Gamma_T \]
Sensitivity result

Theorem:

The control-to-state mapping has a Bouligand derivative when considered as a mapping

$$S : L^2(0, T; L^\infty(\Omega)) \to H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; V)$$

An analogous result holds for the Newton derivative, for the special case of the accumulated maximum.
Parabolic control problem with hysteresis

Minimize

$$J(y, u) = \frac{1}{2} \int_{\Omega} (y(\cdot, T) - y_d)^2 \, dx + \frac{\beta}{2} \int_{\Omega_T} u^2 \, dx \, dt$$

subject to $y = Su$, that is

$$y_t - \Delta y = w + u \quad \text{in } \Omega_T = \Omega \times (0, T)$$
$$w = W[y] \quad \text{in } \Omega_T$$

$$y(\cdot, 0) = y_0 \text{ on } \Omega, \quad y = 0 \text{ on } \Gamma_T$$
Optimality condition

Reduced cost functional \( j(u) = J(Su, u) \)

At a minimizer \( u \): for the directional derivative

\[
j'(u; h) \geq 0 \quad \text{for all } h
\]

\[
j'(u; h) = \int_{\Omega} (y(\cdot, T) - y_d) d(\cdot, T) \, dx + \beta \int_{\Omega_T} uh \, dx \, dt
\]

where \( d \) solves the first order problem

\[
d_t - \Delta d = h + p \quad \text{in } \Omega_T
\]

\[
p = \mathcal{W}'[y; d] \quad \text{in } \Omega_T
\]

\[
d(\cdot, 0) = 0 \text{ on } \Omega, \quad d = 0 \text{ on } \Gamma_T
\]