Mean Field Games with state constraints

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CONTROL OF STATE CONSTRAINED DYNAMICAL SYSTEMS

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Organized by G. Colombo, M. Motta, F. Rampazzo, and V. Recupero

joint Work with

R. Capuani (Rome-TV and Paris-D) and P. Cardaliaguet (Paris-Dauphine)

Outline

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1. Introduction to Mean Field Games
2. The MFG problem with state constraints
   - Relaxed solutions to CMFG problem
3. Existence and uniqueness of relaxed solutions to CMFG
   - Existence of relaxed equilibria
   - A uniqueness result for relaxed solutions
4. Regularity of relaxed solutions to CMFG
   - Necessary conditions and smoothness of minimizers
   - Lipschitz relaxed solutions to CMFG problem
   - Sensitivity relations and semiconcavity
5. Outline of future work

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Motivation of MFG theory

MFG theory has grown massively starting from the work by

- Lasry and Lions (2006, 2007)
- Huang, Caines, and Malhamé (2007)

Many authors have collaborated to the development of this theory

Goal

To describe equilibria in collective behaviour of large population of rational agents

- large population $\rightsquigarrow$ infinite number (a continuum) of players
- rational agents $\rightsquigarrow$ each agent is controlling his/her dynamical own state

The Lasry-Lions approach

The idea

To export the principle of statistical mechanics to interactions within rational particles by introducing a macroscopic description through a mean field model

Given $m(t)$ = agent distribution the generic agent at $x \in \overline{\Omega}$ aims to solve

$$\inf_{\gamma(0)=x} \left\{ \int_0^T \left[L(\gamma(t), \dot{\gamma}(t)) + F(\gamma(t), m(t)) \right] dt + G(\gamma(T), m(T)) \right\}$$

The simplest form of the macroscopic model is the PDE system

$$\begin{cases} -u_t + H(x, \nabla_x u) = F(x, m) \\ m_t - \operatorname{div}(m \nabla_p H(x, \nabla_x u)) = 0 \end{cases} \quad [0, T] \times \mathbb{R}^n \quad \begin{cases} u(T, x) = G(x, m(T)) \\ m(0, dx) = m_0(dx) \end{cases}$$

where $H(x, p) := \sup_{v \in \mathbb{R}^n} \{ -\langle p, v \rangle - L(x, v) \}$

- first equation solved by value of minimization problem of a generic agent
- second equation models agent distribution according to optimal feedback
- m_0 initial distribution of players

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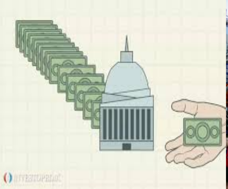
Impact of MFG theory

- MFG system allows for a huge simplification
- solution to the macroscopic MFG system provides approximate Nash equilibria
- Great potential for applications
  - finance, market economics (oil producers, carbon markets...)
  - engineering (smart grids...)
  - crowd dynamics, socio-politics (learning, opinion formation etc...)



Introducing state constraints into MFG

Solution of MFG system in absence of state constraints
(Notes on Mean Field Games by P. Cardaliaguet)

- by vanishing viscosity
- by fixed point argument

$$\mu \longrightarrow u_\mu \begin{cases} -u_t + H(x, \nabla_x u) = F(x,\mu) \\ u(T,x) = G(x, \mu(T)) \end{cases} \longrightarrow m_\mu \begin{cases} m_t - \mathrm{div}(m \nabla_p H_p(x, \nabla_x u_\mu)) = 0 \\ m(0, dx) = m_0(x) dx \end{cases}$$

Our goal To study MFGs with state constraints $(x \in \overline{\Omega})$

Difficulty
Agent distribution may concentrate
on small sets

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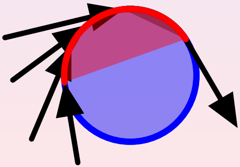

Notation

- $\Omega \subset \mathbb{R}^n$ bounded domain with boundary of class $\mathcal{C}^2$
- $\mathcal{P}(\overline{\Omega})$ Borel probability measures on $\overline{\Omega}$ with

Katorovich-Rubinstein distance

$$d_1(m_1, m_2) = \sup\left\{ \int_{\overline{\Omega}} f\, dm_1 - \int_{\overline{\Omega}} f\, dm_2 \ : \ |f(x) - f(y)| \leqslant |x-y| \right\}$$

- constrained arcs

$$\begin{aligned} \Gamma &= \left\{ \gamma \in AC([0,T];\mathbb{R}^n) \ : \ \gamma(t) \in \overline{\Omega},\ \forall t \in [0,T] \right\} \quad \text{with} \quad \|\cdot\|_\infty \\ \Gamma[x] &= \left\{ \gamma \in \Gamma \ : \ \gamma(0) = x \right\} \quad (x \in \overline{\Omega}) \end{aligned}$$

- $\mathcal{P}(\Gamma)$ Borel probability measures on Γ with d_1 metric
- evaluation map $e_t : \Gamma \to \overline{\Omega}$ ($t \in [0,T]$) defined by $e_t(\gamma) = \gamma(t)$
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Assumptions

$F, G : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ continuous functions

$L : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ continuous such that

- $v \mapsto L(x, v)$ convex $\quad \oplus \quad L \geqslant \ell |v|^2 - \ell_0 \quad (\ell > 0)$
- $|\nabla_x L| + |\nabla_v L| \leqslant C(1 + |v|)$

For any $\eta \in \mathcal{P}(\Gamma)$ define

Associted functional

$$J_\eta[\gamma] = \int_0^T \big[L(\gamma(t), \dot\gamma(t)) + F(\gamma(t), e_t \sharp \eta)\big]\, dt + G(\gamma(T), e_T \sharp \eta) \qquad \forall \gamma \in \Gamma$$

and minimizing arcs at $x \in \overline{\Omega}$

$$\Gamma^\eta[x] = \Big\{\gamma \in \Gamma[x] \ : \ J_\eta[\gamma] = \min_{\Gamma[x]} J_\eta \Big\}$$

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Relaxed equilibria of CMFG

Lagrangian approach Benamou JD., Carlier G., Santambrogio F. (2017)

Let $m_0 \in \mathcal{P}(\overline{\Omega})$

Definition

$\eta \in \mathcal{P}_{m_0}(\Gamma)$ *is called a relaxed CMFG equilibrium for* m_0 *if*

$$\mathrm{spt}(\eta) \subseteq \bigcup_{x \in \overline{\Omega}} \Gamma^{\eta}[x]$$

Equivalently, for η−a.e. $\overline{\gamma} \in \Gamma$,

$$J_\eta[\overline{\gamma}] = \min_{\gamma \in \Gamma[\overline{\gamma}(0)]} J_\eta[\gamma]$$

where

$$J_\eta[\gamma] = \int_0^T \left[L(\gamma(t), \dot{\gamma}(t)) + F(\gamma(t), e_t \sharp \eta)\right] dt + G(\gamma(T), e_T \sharp \eta)$$

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Relaxed solutions to CMFG problem

Let $m_0 \in \mathcal{P}(\overline{\Omega})$

Definition

$(u, m) \in \mathcal{C}([0, T] \times \overline{\Omega}) \times \mathcal{C}\big([0, T]; \mathcal{P}(\overline{\Omega})\big)$ *is a relaxed solution to the CMFG problem if*

$$m(t) = e_t \sharp \eta \qquad \forall t \in [0, T]$$

for some relaxed CMFG equilibrium $\eta \in \mathcal{P}_{m_0}(\Gamma)$ *and*

$$u(t, x) = \min_{\gamma \in \Gamma, \gamma(t) = x} \left\{ \int_t^T \big[L(\gamma(s), \dot{\gamma}(s)) + F(\gamma(s), m(s))\big] \, dt + G(\gamma(T), m(T)) \right\}$$

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Existence results

Theorem

For any $m_0 \in \mathcal{P}(\overline{\Omega})$ there is at least one relaxed CMFG equilibrium

Corollary

For any $m_0 \in \mathcal{P}(\overline{\Omega})$ there is at least one relaxed solution (u, m) to the CMFG problem

Proof of theorem via construction of a fixed point of $E : \mathcal{P}_{m_0}(\Gamma) \rightrightarrows \mathcal{P}_{m_0}(\Gamma)$

$$E(\eta) = \{\mu \in \mathcal{P}_{m_0}(\Gamma) \;:\; \mathrm{spt}(\mu_x) \subseteq \Gamma^\eta[x] \text{ for } m_0\text{–a.e. } x \in \overline{\Omega}\}$$

where $\{\mu_x\}_{x\in\overline{\Omega}} \subset \mathcal{P}(\Gamma)$ is the family of probability measures which disintegrates μ

- $\mu = \int_{\overline{\Omega}} \mu_x \, dm_0(x)$
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Indeed

$$\eta \in \mathcal{P}_{m_0}(\Gamma) \text{ relaxed CMFG equilibrium} \quad \Longleftrightarrow \quad \eta \in E(\eta)$$

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Construction of a fixed point

Kakutan'si fixed-point theorem

- $S \neq \varnothing$ compact convex subset of a locally convex Hausdorff space
- $\phi : S \rightrightarrows S$ nonempty convex-valued with closed graph

$\implies \quad \phi$ has a fixed point.

technical points to check

$\forall \eta \in \mathcal{P}_{m_0}(\Gamma)$

- $E(\eta)$ is nonempty, convex, compact
  The space Γ has to be restricted to
  $$\Gamma_M := \{\gamma \in \Gamma \ : \ \|\dot{\gamma}\|_{L^2(0,T)} \leqslant M\} \quad \text{for a suitable} \quad M > 0$$
  without loss of generality owing to superlinear growth
- $E : \mathcal{P}_{m_0}(\Gamma_M) \rightrightarrows \mathcal{P}_{m_0}(\Gamma_M)$ has closed graph

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Kakutan'si fixed-point theorem

- $S \neq \varnothing$ compact convex subset of a locally convex Hausdorff space
- $\phi : S \rightrightarrows S$ nonempty convex-valued with closed graph

$\implies \quad \phi$ has a fixed point.

technical points to check

$\forall \eta \in \mathcal{P}_{m_0}(\Gamma)$

- $E(\eta)$ is nonempty, convex, compact
  The space Γ has to be restricted to

$$\Gamma_M := \left\{\gamma \in \Gamma \ : \ \|\dot{\gamma}\|_{L^2(0,T)} \leqslant M\right\} \quad \text{for a suitable} \quad M > 0$$

  without loss of generality owing to superlinear growth
- $E : \mathcal{P}_{m_0}(\Gamma_M) \rightrightarrows \mathcal{P}_{m_0}(\Gamma_M)$ has closed graph

Outline

Quasi-uniqueness of the relaxed CMFG solution

Theorem

Assume monotonicity conditions: for any $m_1, m_2 \in \mathcal{P}(\overline{\Omega})$

$$\begin{cases} \displaystyle\int_{\overline{\Omega}} (G(x,m_1) - G(x,m_2))d(m_1 - m_2)(x) \geqslant 0 \\ \displaystyle\int_{\overline{\Omega}} (F(x,m_1) - F(x,m_2))d(m_1 - m_2)(x) > 0 \quad \text{if } F(\cdot,m_1) \neq F(\cdot,m_2) \end{cases}$$

Let (u_1, m_1) *and* (u_2, m_2) *be relaxed solutions to the CMFG problem. Then* $u_1 \equiv u_2$

F satisfies the strict monotonicity condition if $F : \overline{\Omega} \times P(\overline{\Omega}) \to \mathbb{R}$ is of the form

$$F(x,m) = \int_{\Omega} f(y, (\phi \star m)(y))\phi(x-y)\,dy$$

where $\phi : \mathbb{R}^d \to \mathbb{R}$ is a smooth even kernel with compact support and

$$f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \text{ is smooth and } \quad f(x,\cdot) \text{ is strictly increasing}$$

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Proof of uniqueness

Let (u_i, m_i), $i = 1, 2$, be solutions of the constrained MFG system:

- $m_i(t) := e_t \sharp \eta_i$ where $\eta_i \in \mathcal{P}_{m_0}(\Gamma_M)$ are fixed-points of E
- $u_i(t,x) = \min_{\gamma \in \Gamma,\, \gamma(t)=x} \left\{ \int_t^T \left[L(\gamma, \dot\gamma) + F(\gamma, m_i) \right] ds + G(\gamma(T), m_i(T)) \right\}$

If γ is optimal for $u_1(0, \gamma(0))$, then

$$\begin{aligned} u_1(0,\gamma(0)) &= \int_0^T \left[L(\gamma,\dot\gamma) + F(\gamma, m_1) \right] ds + G(\gamma(T), m_1(T)) \\ u_2(0,\gamma(0)) &\leqslant \int_0^T \left[L(\gamma,\dot\gamma) + F(\gamma, m_2) \right] ds + G(\gamma(T), m_2(T)) \end{aligned}$$

Therefore

$$\begin{aligned} & G(\gamma(T), m_1(T)) - G(\gamma(T), m_2(T)) \\ & \quad \leqslant u_1(0,\gamma(0)) - u_2(0,\gamma(0)) + \int_0^T \left[F(\gamma, m_2) - F(\gamma, m_1) \right] ds \end{aligned}$$

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Since η_1 is supported by optimal trajectories for u_1, integrating the above inequality over Γ with respect to η_1 we obtain

Proof of uniqueness (continued)

$$\int_\Gamma \left[G(\gamma(T), m_1(T)) - G(\gamma(T), m_2(T))\right] d\eta_1(\gamma)$$
$$\leqslant \int_\Gamma \left[u_1(0,\gamma(0)) - u_2(0,\gamma(0))\right] d\eta_1(\gamma) + \int_\Gamma \int_0^T \left[F(\gamma, m_2) - F(\gamma, m_1)\right] ds d\eta_1(\gamma)$$

Since $m_1(t) := e_t \sharp \eta_1$, by the change of variable $e_t(\gamma) = \gamma(t)$ we derive

$$\int_{\overline{\Omega}} \left[G(x, m_1(T)) - G(x, m_2(T))\right] m_1(T, dx)$$
$$\leqslant \int_{\overline{\Omega}} \left[u_1(0,x) - u_2(0,x)\right] m_1(0, dx) + \int_0^T \int_{\overline{\Omega}} \left[F(x, m_2) - F(x, m_1)\right] \; m_1(s, dx) ds$$

Similarly

$$\int_{\overline{\Omega}} \left[G(x, m_2(T)) - G(x, m_1(T))\right] m_2(T, dx)$$
$$\leqslant \int_{\overline{\Omega}} \left[u_2(0,x) - u_1(0,x)\right] m_2(0, dx) + \int_0^T \int_{\overline{\Omega}} \left[F(x, m_1) - F(x, m_2)\right] \; m_2(s, dx) ds$$

Proof of uniqueness (completed)

By adding the above inequalities

$$0 \leqslant \int_{\overline{\Omega}} \left[G(x, m_1(T)) - G(x, m_2(T))\right] \left(m_1(T, dx) - m_2(T, dx)\right)$$
$$\leqslant \int_{\overline{\Omega}} \left[u_1(0, x) - u_2(0, x)\right] \left(\underbrace{m_1(0, dx) - m_2(0, dx)}_{=0}\right)$$
$$+ \int_0^T \underbrace{\int_{\overline{\Omega}} \left[F(x, m_2) - F(x, m_1)\right] \left(m_1(s, dx) - m_2(s, dx)\right) ds}_{\leqslant 0}$$

Therefore

$$\int_0^T \underbrace{\int_{\overline{\Omega}} \left[F(x, m_2) - F(x, m_1)\right] \left(m_1(s, dx) - m_2(s, dx)\right)}_{\leqslant 0} ds = 0$$

Since F is stricytly monotone, $\boxed{F(\cdot, m_1) = F(\cdot, m_2)}$ and so $\boxed{u_1 \equiv u_2}$

More notation and assumptions

Recall $\Omega \subset \mathbb{R}^n$ is bounded with $\partial\Omega \in C^2$. Consequently

- distance $\boxed{d_\Omega(x) = \min_{y\in\overline{\Omega}} |x-y|}$

  of class $\mathcal{C}^2(\Omega_\delta^+)$ for some $\delta > 0$ with $\Omega_\delta^+ = \{x \in \mathbb{R}^n \setminus \Omega \ : \ d_\Omega(x) < \delta\}$

- oriented boundary distance $\boxed{b_\Omega(x) = d_\Omega(x) - d_{\mathbb{R}^n\setminus\Omega}(x)}$

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We strengthen the smoothness assumptions on H (and L)

- $H \in \mathcal{C}^2(\overline{\Omega} \times \mathbb{R}^n)$ with

$$\kappa I \leqslant \nabla_p^2 H \leqslant \kappa^{-1} I \quad (\kappa > 0) \quad \text{and} \quad |\nabla_{xp}^2 H| \leqslant C(1+|p|)$$

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Outline

Necessary conditions for smooth state constraints

Theorem

Given $x \in \overline{\Omega}$ *let* γ *minimize over* $\Gamma[x]$ *the functional*

$$\gamma \mapsto \int_0^T \left[L(\gamma(s), \dot{\gamma}(s)) + f(s, \gamma(s))\right] dt + g(\gamma(T))$$

where $g \in \mathcal{C}^1(\overline{\Omega})$ *and* $f : [0, T] \times \overline{\Omega} \to \mathbb{R}$ *satisfies* $|f_t| + |\nabla_x f| \leq C$

Then there exist

- $\nu \in \mathbb{R}$ *and* $\Lambda : [0, T] \to \mathbb{R}$ *measurable with* $|\nu| + \|\Lambda\|_\infty \leqslant C(\Omega, H, f, g)$
- $p : [0, T] \to \mathbb{R}^n$ *Lipschitz*

such that

$$\begin{cases} \dot{\gamma}(t) = -\nabla_p H(\gamma(t), p(t)) \\ \dot{p}(t) = \nabla_x H(\gamma(t), p(t)) - \nabla_x f(t, \gamma(t)) - \Lambda(t) \mathbf{1}_{\partial\Omega}(\gamma(t)) \nabla b_\Omega(\gamma(t)) & \forall t \in [0, T] \\ p(T) = \nabla g(\gamma(T)) + \nu \mathbf{1}_{\partial\Omega}(\gamma(T)) \nabla b_\Omega(\gamma(T)) \end{cases}$$

Consequently, $\gamma \in C^1_{Lip}([0, T]; \mathbb{R}^n)$ *and* $\|\dot{\gamma}\|_{Lip} \leqslant C(\Omega, H, f, g)$

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Outline

Existence of Lipschitz solutions

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Let $m_0 \in \mathcal{P}(\overline{\Omega})$ and suppose

$$|F(x_1, m_1) - F(x_2, m_2)| + |G(x_1, m_1) - G(x_2, m_2)| \leqslant C\Big(|x_1 - x_2| + d_1(m_1, m_2)\Big)$$

Then there exists at least one relaxed solution of CMFG problem (u, m) such that

$$u \in Lip([0, T] \times \overline{\Omega}) \quad \text{and} \quad m \in Lip([0, T]; \mathcal{P}(\overline{\Omega}))$$

Such a solution will be called a Lipschitz relaxed solution of the CMFG problem

The proof applies necessary conditions to construct a relaxed CMFG equilibrium

$$\eta \in \mathcal{P}_{m_0}(\Gamma) \text{ such that } m(t) := e_t \sharp \eta \text{ belongs to } Lip([0, T]; \mathcal{P}(\overline{\Omega}))$$

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Outline

Sensitivity relations

Given

- *a Lipschitz relaxed solution* (u, m) *of the CMFG problem*
- $(t, x) \in [0, T[\times\overline{\Omega}$ *and a solution* $\gamma^* \in \Gamma$ *to*

$$\min_{\gamma\in\Gamma,\gamma(t)=x}\left\{\int_t^T\big[L(\gamma(s),\dot\gamma(s))+F(\gamma(s),m(s))\big]\,dt+G(\gamma(T),m(T))\right\}$$

- *the adjoint state* $p : [t, T] \to \mathbb{R}^n$ *associated with* γ^*

we have that

$$\Big(H(\gamma^*(s),p(s))-F(\gamma^*(s),m(s))\,,\,p(s)\Big)\in D^+u(s,\gamma^*(s))\quad\forall s\in[t,T[$$

and $\forall\rho\in]0,T[$ *there exists* $C_\rho\geqslant 0$ *such that* $\forall\, t, t+\tau\in[0,T-\rho]$ *and all* $x+h\in\overline{\Omega}$

$$u(t+\tau,x+h)-u(t,x)-\tau\big(H(x,p(t))-F(x,m(t))\big)-\langle p(t),h\rangle \leqslant C_\rho(|\tau|+|h|)^{3/2}$$

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and $\forall \rho \in]0, T[$ *there exists* $C_\rho \geqslant 0$ *such that* $\forall\, t, t + \tau \in [0, T - \rho]$ *and all* $x + h \in \overline{\Omega}$

$$u(t + \tau, x + h) - u(t, x) - \tau\big(H(x, p(t)) - F(x, m(t))\big) - \langle p(t), h\rangle \leqslant C_\rho(|\tau| + |h|)^{3/2}$$

Proof of sensitivity relation for $\tau = 0$

We want to show that $\forall\, t \in [0, T-\rho]$ and all $x + h \in \overline{\Omega}$

$$u(t, x+h) - u(t, x) - \langle p(t), h\rangle \leqslant C_\rho |h|^{3/2}$$

Let $0 < \sigma \leqslant \rho$ to be fixed later and define for all $s \in [t, T]$

$$\begin{aligned}\gamma_h(s) &= \overline{\gamma}(s) + \left(1 + \frac{t-s}{\sigma}\right)_+ h \\ \widehat{\gamma}_h(s) &= \gamma_h(s) - d_{\overline{\Omega}}\big(\gamma_h(s)\big)\, Dd_{\partial\Omega}\big(\gamma_h(s)\big)\end{aligned}$$

By dynamic programming

$$\begin{aligned} u(t, x+h) - u(x,t) - \langle p(t), h\rangle &\leqslant \int_t^{t+\sigma} \left[L(\widehat{\gamma}_h, \dot{\widehat{\gamma}}_h) - L(\overline{\gamma}, \dot{\overline{\gamma}})\right] ds \\ &+ \int_t^{t+\sigma} \left[F(\widehat{\gamma}_h, m) - F(\overline{\gamma}, m)\right] ds - \langle p(t), h\rangle \qquad (1)\end{aligned}$$

We want to relate $\langle p(t), h\rangle$ so we expand

$$-\langle p(t), h\rangle = -\langle p(t+\sigma), \underbrace{\widehat{\gamma}_h(t+\sigma) - \overline{\gamma}(t+\sigma)}_{=0}\rangle + \int_t^{t+\sigma} \frac{d}{ds}\langle p, \widehat{\gamma}_h - \overline{\gamma}\rangle\, ds$$

$$= \int_t^{t+\sigma} \langle \dot{p}, \widehat{\gamma}_h - \overline{\gamma}\rangle\, ds + \int_t^{t+\sigma} \langle p, \dot{\widehat{\gamma}}_h - \dot{\overline{\gamma}}\rangle\, ds$$

Proof of sensitivity relation (continued)

By appealing to PMP to represent $\langle \dot p, \widehat\gamma_h - \overline\gamma\rangle$ and $\langle p, \dot{\widehat\gamma}_h - \dot{\overline\gamma}\rangle$ we obtain

$$
\begin{aligned}
& u(t,x+h) - u(x,t) - \langle p(t), h\rangle \\
& \leqslant \int_t^{t+\sigma} \big[F(\widehat\gamma_h, m) - F(\overline\gamma, m) - \langle D_x F(\overline\gamma, m), \widehat\gamma_h - \overline\gamma\rangle\,\big]\,ds \\
& + \int_t^{t+\sigma} \big[L(\widehat\gamma_h, \dot{\widehat\gamma}_h) - L(\overline\gamma, \dot{\widehat\gamma}_h) - \langle D_x L(\overline\gamma, \dot{\widehat\gamma}_h), \widehat\gamma_h - \overline\gamma\rangle\big]\,ds \\
& + \int_t^{t+\sigma} \big[L(\widehat\gamma_h, \dot{\overline\gamma}) - L(\overline\gamma, \dot{\overline\gamma}) - \langle D_v L(\overline\gamma, \dot{\overline\gamma}), \dot{\widehat\gamma}_h - \dot{\overline\gamma}\rangle\big]\,ds \\
& + \int_t^{t+\sigma} \big[\langle D_x L(\overline\gamma, \dot{\widehat\gamma}_h) - D_x L(\overline\gamma, \dot{\overline\gamma}), \widehat\gamma_h - \overline\gamma\rangle\big]\,ds - \int_t^{t+\sigma} \lambda\langle Dd(\overline\gamma), \widehat\gamma_h - \overline\gamma\rangle\,ds \\
& \leqslant C\int_t^{t+\sigma} |\widehat\gamma_h - \overline\gamma|^2\,ds + C\int_t^{t+\sigma} |\dot{\widehat\gamma}_h - \dot{\overline\gamma}|^2\,ds - \int_t^{t+\sigma} \lambda\langle Dd(\overline\gamma), \widehat\gamma_h - \overline\gamma\rangle\,ds \\
& \leqslant C\int_t^{t+\sigma} |\widehat\gamma_h - \overline\gamma|^2\,ds + C\int_t^{t+\sigma} |\dot{\widehat\gamma}_h - \dot{\overline\gamma}|^2\,ds + C\int_t^{t+\sigma} |\widehat\gamma_h - \overline\gamma|\,ds
\end{aligned}
$$

Proof of sensitivity relation (completed)

Recalling

$$\begin{cases} \gamma_h(s) = \overline{\gamma}(s) + \left(1 + \frac{t-s}{\sigma}\right)_+ h \\ \widehat{\gamma}_h(s) = \gamma_h(s) - d_{\overline{\Omega}}\big(\gamma_h(s)\big) Dd_{\partial\Omega}\big(\gamma_h(s)\big) \end{cases}$$

we have that

$$|\widehat{\gamma}_h(s) - \overline{\gamma}(s)| \leq 2|h| \quad \forall s \in [t, t+\sigma]$$

Using the regularity of the distance functions one can also prove (technical)

$$\int_t^{t+\sigma} |\dot{\widehat{\gamma}}_h(s) - \dot{\overline{\gamma}}(s)|^2 \, ds \leqslant C \frac{|h|^2}{\sigma} + C|h|\sigma$$

Therefore

$$u(t, x+h) - u(x,t) - \langle p(t), h \rangle \leqslant C|h| \left(\frac{|h|}{\sigma} + \sigma \right) \leqslant 2C|h|^{3/2}$$

by taking $\sigma = |h|^{1/2}$

Semiconcavity

Theorem

Any Lipschitz relaxed solution (u, m) *of CMFG problem is locally semiconcave on* $[0, T[\times\overline{\Omega}$ *with a fractional modulus:*

$\forall \rho \in]0, T[$ *there exists* $C_\rho \geq 0$ *such that*

$$u(t+\tau, x+h) + u(t-\tau, x-h) - 2u(t, x) \leqslant C_\rho(|\tau| + |h|)^{3/2}$$

for all $t, t \pm \tau \in [0, T-\rho]$ *and* $x, x \pm h \in \overline{\Omega}$

Poitwise solutions of the MFG system

Given a Lipschitz relaxed solution (u, m) to CMFG problem we would like to show:

(I) u is a constrained viscosity solution of

$$\begin{cases} -u_t + H(x, \nabla_x u) = F(x, m) & \text{in } [0, T[\times\overline{\Omega} \\ u(x, T) = G(x, m(T)) & \forall x \in \overline{\Omega} \end{cases}$$

(II) $\exists!$ continuous vector field $V : \mathrm{spt}(m) \cap (]0, T[\times\overline{\Omega}) \to \mathbb{R}^n$ such that

(i) $\forall (t, x) \in \mathrm{spt}(m)$

$$V(t, x) \in -\nabla_p H(x, \nabla_x^+ u(t, x))$$

(ii) m satisfies the continuity equation

$$\begin{cases} m_t + div(mV) = 0 & \text{in }]0, T[\times\overline{\Omega} \\ m(0) = m_0 \end{cases} \tag{2}$$

in the sense of distributions

$$\int_0^T \int_{\overline{\Omega}} (\phi_t + \langle V, \nabla_x \phi \rangle)\, dm(s, x) ds = 0 \qquad \forall \phi \in C_c^\infty(]0, T[\times\overline{\Omega})$$

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Thank you for your attention!