State-dependent sweeping processes

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... discretization is simple (catching-up algorithm)

$$u_n(t) = u_{n,i+1} = \mathrm{prox}(u_{n,i}, C(t_{n,i+1}))$$

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... deal with time-dependent domains via an implicit tangency condition

Variations on the SP

- Perturbed sweeping process

$$-\frac{du}{dt}(t) \in N_{C(t)}(u(t)) + F(t, u(t))$$

- Bounded variation cases: a priori bound on the variation of the set or nonempty interior assumptions

Variations on the SP (2)

- Second order problems: inelastic shocks

$$\dot{q} = u, du - f(t, q(t))dt \in N_{C(t)}(u(t))$$

- Extensions (nonconvex sets, degenerate sweeping processes, ...) and applications

State-dependent sweeping processes

A basic state-dependent sweeping process in a Hilbert space may be written in short as

$$-\frac{du}{dt}(t) \in N_{C(t,u(t))}(u(t)),$$

where $u : I = [0, T] \to H$ is abs. continuous, $C(t, u) \subset H$ and $N_{C(t,u)}(x)$ is the outward normal cone to $C(t, u)$ at x. Implicitly $u(t) \in C(t, u(t))$, for all t, including for the initial value. The r.h.s. may also contain standard $f = f(t, u)$ terms.

Existence results

In [1], in their simplest form, sets $C(t,u)$ are closed and convex and the dependence $(t,u) \mapsto C(t,u)$ is Lipschitz-continuous w.r.t. Hausdorff distance h

$$h(C(t,u), C(s,v)) \leq L_1 |t-s| + L_2 |u-v|_H,$$

with $L_2 < 1$. In infinite-dimensional settings, compactness assumptions may be added, for technical reasons.

[1] Kunze, MMM, On parabolic quasi-variational inequalities and state-dependent sweeping processes, *Top. Methods Nonlinear Anal.* 12 (1998) 179-191.

Existence results (2)

- The sets may be 'not far from convex', say prox-regular or phi-convex, as in
[3] Chemetov, N, Monteiro Marques, MDP, Non-convex quasi-variational sweeping processes, *Set-Valued Analysis* 15 (2007) 209-221.

Existence results (3)

It is also possible to work in ordered Hilbert spaces:
[4] Chemetov, N, Monteiro Marques, MDP and Stefanelli, U, Ordered non-convex quasi-variational sweeping processes, *J Convex Analysis* 15 (2008) 201-214.

Example

An example of application is given in
[2] Kunze, MMM, A note on Lipschitz continuous solutions of a parabolic quasi-variational inequality, in *Nonlinear evolution equations and their applications (Macau, 1998)*, 109-115, World Sci. Publ., 1999.

State-dependent evolution problems

If the normal cone to $C(t, u)$ is replaced by a maximal monotone operator $A(t, u)$, the problem is to find more generally $u : I \to H$ such that

$$-\frac{du}{dt}(t) \in A(t, u(t))(u(t))$$

meaning that $u(t) \in D(A(t, u(t)))$ and that, for all $v \in D(A(t, u(t)))$ and $z \in A(t, u(t))v$, one has

$$\langle \frac{du}{dt}(t) + z, v - u(t) \rangle \geq 0.$$

Assuming that the dependence of the m.m.o. on the state is measured by Vladimirov's pseudo-distance, one extends the previous study.

Concluding remarks

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- The S.P. is a good testing ground for more involved problems, notably with respect to the variation of domains.
- Many other people have worked in the S. P. or its variants or more generally monotone problems. I will forget many important contributions, if I make a list, but...: Castaing, Valadier, Attouch, Thibault, Benabdellah; Colombo, Goncharov, Ricupero; Frankowska, Krejci, Vladimirov, Makarenkov, Brokate, Venel and Adly et al et al and so and so on....

Conclusions

- In comparison, have the control problems with SP been given the same level of attention?

Looking forward to hear from you...
THANK YOU for your attention.