Mayer and minimum time problem for multi-agent systems

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Control of state constrained dynamical systems

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Introduction

We formulate some control problems in the space of probability measures endowed with the Wasserstein distance as a natural generalization of the classical problems in $\mathbb{R}^d$.

Motivations

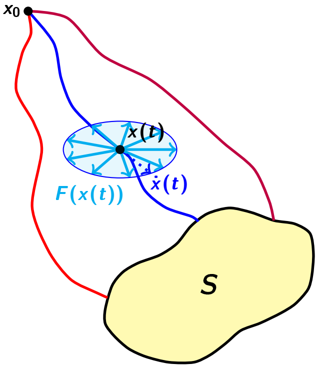
to model situations in which we have only a probabilistic knowledge of the initial state (e.g. noise in the measurements).

to model multi-agent systems, where only a statistical (*macroscopic*) description of the system is available. (e.g. gas/crowd dynamics).

Classical optimal control system

Controlled dynamics: in form of differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(x(t)), & \text{for a.e. } t > 0, \\ x(0) = x_0 \in \mathbb{R}^d. \end{cases}$$



Problem: to minimize a given cost functional $J(\cdot)$ on the set of admissible trajectories.

Hypothesis: $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$, $F(\cdot)$ not empty, convex, compact valued, continuous w.r.t. the Hausdorff metric and with linear growth.

Example: minimum time

Minimum time needed to steer x_0 to a given closed *target set* $\emptyset \neq S \subseteq \mathbb{R}^d$:

$$T(x_0) := \inf\{\bar{t} > 0 : \exists x(\cdot) \text{ sol. of the control system s.t. } x(\bar{t}) \in S\}.$$

Generalized Problem - State and Dynamics

- Initial state: probability measure $\mu_0 \in \mathscr{P}_p(\mathbb{R}^d)$ with $\mathrm{m}_p(\mu_0) := \int_{\mathbb{R}^d} |x|^p \, d\mu_0(x) < +\infty$;
- Trajectory: time-depending probability measure on $\mathbb{R}^d$, $\boldsymbol{\mu} := \{\mu_t\}_{t\in[0,T]}$, $\mu_{|t=0} = \mu_0$, (AC curve in $\mathscr{P}_p(\mathbb{R}^d)$);
- Dynamics: since total mass must be preserved during the evolution, the process will be described by a (controlled) continuity equation

$$\partial_t \mu_t + \mathrm{div}(v_t \mu_t) = 0, \qquad \text{for } 0 < t < T;$$

- Control set: v_t to be chosen in the set of $L^2_{\mu_t}$–selections of F for a.e. $t \in [0, T]$, to respect the classical underlying control problem.

State Space

$\mathscr{P}_p(\mathbb{R}^d)$ endowed with the topology induced by the p-Wasserstein distance $W_p(\cdot,\cdot)$, $p \geq 1$.

Let $\mu_1, \mu_2 \in \mathscr{P}_p(\mathbb{R}^d)$, the p-Wasserstein distance is defined as

$$W_p(\mu_1,\mu_2) := \left(\inf\left\{\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x_1 - x_2|^p \, d\pi \,:\, \pi \in \Pi(\mu_1,\mu_2)\right\}\right)^{1/p}$$

Where the set of admissible transport plans $\Pi(\mu_1,\mu_2)$ is defined by the following

$$\Pi(\mu_1,\mu_2) := \left\{ \pi \in \mathscr{P}_p(\mathbb{R}^d\times\mathbb{R}^d) : \begin{array}{c} \pi(A_1\times\mathbb{R}^d) = \mu_1(A_1), \\ \pi(\mathbb{R}^d\times A_2) = \mu_2(A_2), \\ \forall A_i \ \mu_i\text{-measurable set }, \\ i = 1,2 \end{array} \right\}$$

Generalized Dynamics and control set

Continuity equation:

$$\begin{cases} \partial_t \mu_t(x) + \mathrm{div}(v_t(x)\mu_t(x)) = 0, & \text{for } 0 < t < T,\ x \in \mathbb{R}^d, \\ \mu_{|t=0} = \mu_0 \in \mathscr{P}_p(\mathbb{R}^d). \end{cases} \quad (1)$$

which represents the conservation of the total mass $\mu_0(\mathbb{R}^d)$.

We require the velocity field $v_t(\cdot)$ to satisfy $v_t(x) \in F(x)\ \forall x \in \mathbb{R}^d$.

If $v_t(\cdot)$ is locally Lipschitz in x unif. w.r.t. t, we consider the characteristic system:

$$\begin{cases} \dot\gamma(t) = v_t(\gamma(t)), & \text{for a.e. } t \in (0, T) \\ \gamma(0) = x \end{cases}$$

Let $T_t(x)$ denote the unique solution, then: $\mu_t = T_t \sharp \mu_0$, where

$$T_t \sharp \mu_0(B) := \mu_0(T_t^{-1}(B)), \qquad \forall B \subset \mathbb{R}^d,\ B \text{ Borel set}.$$

Admissible curves

Let $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$, $\tau > 0$, $\alpha, \beta \in \mathscr{P}(\mathbb{R}^d)$. We say that $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,\tau]} \subseteq \mathscr{P}_p(\mathbb{R}^d)$ is an admissible trajectory defined in $[0, \tau]$ joining α and β, if $\exists \boldsymbol{\nu} = \{\nu_t\}_{t\in[0,\tau]} \subseteq \mathscr{M}(\mathbb{R}^d; \mathbb{R}^d)$ a family of Borel vector-valued measures s.t.

- $\boldsymbol{\mu}$ is a narrowly continuous solution of $\partial_t \mu_t + \mathrm{div} \nu_t = 0$, with $\mu_{t=0} = \alpha$, $\mu_{t=\tau} = \beta$.
- $J_F(\boldsymbol{\mu}, \boldsymbol{\nu}) < +\infty$, where

$$J_F(\boldsymbol{\mu}, \boldsymbol{\nu}) := \begin{cases} 0, & \text{if } \nu_t \ll \mu_t \text{ and } \dfrac{\nu_t}{\mu_t}(x) \in F(x) \text{ for a.e. } t \in [0, \tau],\ \mu_t\text{-a.e. } x, \\ +\infty, & \text{otherwise.} \end{cases}$$

In this case, we will shortly say that $\boldsymbol{\mu}$ is driven by $\boldsymbol{\nu}$.

Superposition principle: idea

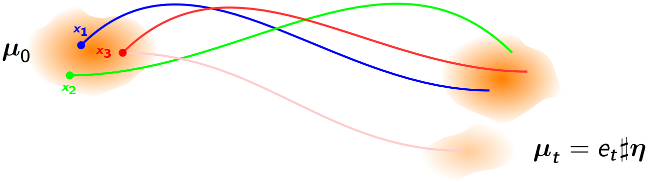
With milder assumptions on v, the (possible not-unique) solution μ_t of the continuity equation can be represented by a superposition of integral solutions of the underlying characteristic system, i.e. of ODEs of the form $\dot{x}(t) = v(x(t))$, or $\dot{x}(t) = v(t, x(t))$.

For this approach, see

L. Ambrosio
The flow associated to weakly differentiable vector fields: recent results and open problems, 2011

and the references therein, where it is also shown that in some cases it is possible to provide conditions on v (assuming for instance Sobolev or BV regularity, and some bounds on the weak derivatives) to recover uniqueness and stability of the solutions in a suitable *smaller class* of measures (Lagrangian flow problem). The representation is not unique.

Superposition principle: idea



For every point $x \in \operatorname{supp} \mu_0$, consider the set of all integral solutions of $\dot\gamma(t) = v_t \circ \gamma(t)$, $\gamma(0) = x$, and define a probability measure $\boldsymbol{\eta}_x$ on it (if there is a unique solution, $\boldsymbol{\eta}_x$ reduces to a Dirac delta). Let $\boldsymbol{\eta} := \mu_0 \otimes \boldsymbol{\eta}_x$ be the product measure, which is a probability measure on $\mathbb{R}^d \times \Gamma_T$, where $\Gamma_T := C^0([0,T];\mathbb{R}^d)$. For any $\gamma \in \Gamma_T$ consider the evaluation operator $e_t(x,\gamma) = \gamma(t)$. Then $t \mapsto \mu_t = e_t \sharp \boldsymbol{\eta}$ is a solution of the continuity equation. Conversely, every solution can be represented in this way for a suitable $\boldsymbol{\eta}$.

Superposition principle: statement

Let $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]}$ be a solution of the continuity equation $\partial_t\mu_t + \mathrm{div}(v_t\mu_t) = 0$ for a suitable Borel vector field $v : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ satisfying

$$\int_0^T \int_{\mathbb{R}^d} \frac{|v_t(x)|}{1+|x|}\, d\mu_t(x)\, dt < +\infty\,.$$

Then there exists a probability measure $\boldsymbol{\eta} \in \mathscr{P}(\mathbb{R}^d \times \Gamma_T)$, with $\Gamma_T = C^0([0,T];\mathbb{R}^d)$ endowed with the sup norm, such that

(i) $\boldsymbol{\eta}$ is concentrated on the pairs $(x,\gamma) \in \mathbb{R}^d \times \Gamma_T$ such that γ is an absolutely continuous solution of

$$\begin{cases} \dot\gamma(t) = v_t(\gamma(t)), & \text{for } \mathscr{L}^1\text{-a.e } t \in (0,T) \\ \gamma(0) = x, \end{cases}$$

(ii) for all $t\in[0,T]$ and all $\varphi \in C^0_b(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \varphi(x) d\mu_t(x) = \iint_{\mathbb{R}^d\times\Gamma_T} \varphi(\gamma(t))\, d\boldsymbol{\eta}(x,\gamma).$$

Conversely, given any $\boldsymbol{\eta}$ satisfying (i) above and defined $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]}$ as in (ii) above, we have that $\partial_t\mu_t + \mathrm{div}(v_t\mu_t) = 0$ and $\mu_{|t=0} = e_0\sharp\boldsymbol{\eta}$.

Superposition Principle for Differential Inclusions

The Superposition Principle deals with the macroscopic velocity vector field v_t. However in many applications the solutions must be constructed by superposition of admissible trajectories for the finite-dimensional differential inclusion that are not a priori solution of a given vector field. To this aim we provide the following result.

Theorem [SP for differential inclusions, Cavagnari-M-Piccoli]

Let $\boldsymbol{\eta} \in \mathscr{P}(\mathbb{R}^d \times \Gamma_T)$ be concentrated on the set of pairs $(\gamma(0), \gamma) \in \mathbb{R}^d \times \Gamma_T$ such that $\gamma \in AC([0,T];\mathbb{R}^d)$ is a Carathéodory solution of the differential inclusion $\dot\gamma(t) \in F(\gamma(t))$. For all $t \in [0,T]$, set $\mu_t := e_t \sharp \boldsymbol{\eta}$, and let $\{\eta_{t,y}\}_{y\in\mathbb{R}^d} \subseteq \mathscr{P}(\mathbb{R}^d \times \Gamma_T)$ be the disintegration of $\boldsymbol{\eta}$ w.r.t. the evaluation operator $e_t : \mathbb{R}^d \times \Gamma_T \to \mathbb{R}^d$, i.e. for all $\varphi \in C_b^0(\mathbb{R}^d \times \Gamma_T)$

$$\iint_{\mathbb{R}^d \times \Gamma_T} \varphi(x,\gamma)\, d\boldsymbol{\eta}(x,\gamma) = \int_{\mathbb{R}^d} \int_{e_t^{-1}(y)} \varphi(x,\gamma)\, d\eta_{t,y}(x,\gamma)\, d\mu_t(y).$$

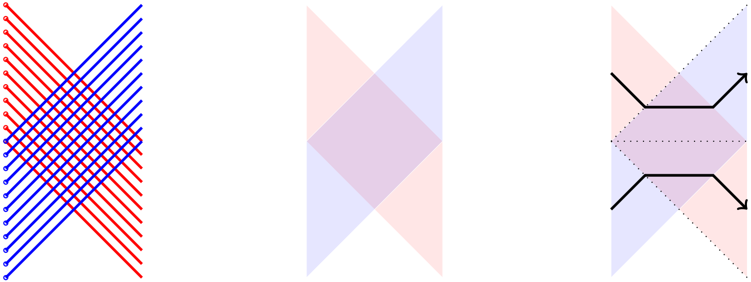
Then if $\mu_0 \in \mathscr{P}_p(\mathbb{R}^d)$, the curve $\boldsymbol{\mu} := \{\mu_t\}_{t\in[0,T]} \subseteq \mathscr{P}_p(\mathbb{R}^d)$, is an admissible trajectory driven by $\boldsymbol{\nu} = \{\nu_t\}_{t\in[0,T]}$, where $\nu_t = v_t \mu_t$ and the vector field

$$v_t(y) = \int_{e_t^{-1}(y)} \dot\gamma(t)\, d\eta_{t,y}(x,\gamma).$$

is well-defined for a.e. $t \in [0,T]$ and μ_t-a.e. $y \in \mathbb{R}^d$.

SP for Differential Inclusions - Comments and an example

We recall that in general there is not an unique η representing a given μ: in particular, the effect in passing from the microscopic point of view encoded in η to the macroscopic description provided by μ, may cause a loss of information (due to averaging). An interesting example of this situation is given below.



We start with some weighted Dirac deltas on the y-axis and made them evolve along the characteristics. We refine the distribution of delats to obtain the 1-dimensional Lebesgue measure restricted to $\{0\} \times [-1, 1]$. The averaged vector field is drawed (dotted characteristics are neglible).

Properties of the set of admissible trajectories - 1

Theorem[Cavagnari-M-Nguyen-Priuli, Cavagnari-M-Piccoli, M-Quincampoix]

Let $a, b, c \in \mathbb{R}$, $a < b < c$, $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be satisfying $\boldsymbol{(F)}$. Recalling that the space $X := C^0([a,b]; \mathscr{P}_p(\mathbb{R}^d))$ with the metric

$$d_X(\boldsymbol{\mu}, \boldsymbol{\nu}) = \sup_{t\in[a,b]} W_p(\mu_t, \nu_t), \text{ for all } \boldsymbol{\mu} = \{\mu_t\}_{t\in[a,b]}, \boldsymbol{\nu} = \{\nu_t\}_{t\in[a,b]},$$

is a complete metric space, we have that

1. the set of admissible trajectories is closed in (X, d_X);
2. if $\{\boldsymbol{\mu}^N\}_{N\in\mathbb{N}}$ is a sequence of admissible trajectories satisfying $\sup_{N\in\mathbb{N}}\{\mathrm{m}_p(\mu_0^N)\} < \infty$, then it admits a d_X-convergent subsequence.
3. given $\mu \in \mathscr{P}_p(\mathbb{R}^d)$, $\boldsymbol{\mu} = \{\mu_t\}_{t\in[a,b]} \in \mathscr{A}^F_{[a,b]}(\mu)$, $\boldsymbol{\nu} = \{\nu_t\}_{t\in[b,c]} \in \mathscr{A}^F_{[b,c]}(\mu_a)$ then the concatenation is an admissible trajectory.

Properties of the set of admissible trajectories - 2

(continued)

4. if $\boldsymbol{\mu} = \{\mu_t\}_{t\in[a,b]}$ is an admissible trajectory, and $\boldsymbol{\eta} \in \mathscr{P}(\mathbb{R}^d \times \Gamma_{[a,b]})$ satisfies $\mu_t = e_t\sharp\boldsymbol{\eta}$ for all $t \in [a,b]$, then for $s_1, s_2 \in [a,b]$ we have

$$\|e_{s_1} - e_{s_2}\|_{L^2_{\boldsymbol{\eta}}} \le Ce^{2(b-a)C}\left(1 + \min_{i=1,2} \mathrm{m}_2^{1/2}(\mu_{s_i})\right)|s_1 - s_2|,$$

where $C = \max_{y\in F(0)} |y| + \mathrm{Lip}(F)$.

5. if $\boldsymbol{\mu} = \{\mu_t\}_{t\in[a,b]}$ is an admissible trajectory, and $\boldsymbol{\eta} \in \mathscr{P}(\mathbb{R}^d \times \Gamma_{[a,b]})$ satisfies $\mu_t = e_t\sharp\boldsymbol{\eta}$ for all $t \in [a,b]$, given $\bar{t} \in [a,b]$, every limit for $i \to +\infty$ of a $L^2_{\boldsymbol{\eta}}$-weak converging sequence $\dfrac{e_t - e_{\bar{t}}}{t - \bar{t}}$ belongs to the set $\{v \circ e_{\bar{t}} : v \in L^2_{\mu_t}, v(x) \in F(x) \text{ for } \mu_{\bar{t}}\text{-a.e. } x \in \mathbb{R}^d\}$.

Proof is based on Superposition Principle and Gronwall estimates.

Properties of the set of admissible trajectories - 3

Proposition [Prescribed Initial velocity of measure trajectories]

Let $a, b \in \mathbb{R}$, $a < b$, $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be satisfying the standing assumptions, $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Then for every $v_a \in L^2_\mu(\mathbb{R}^d)$ such that $v_a(x) \in F(x)$ for μ-a.e. $x \in \mathbb{R}^d$ there exist $\boldsymbol{\eta} \in \mathscr{P}(\mathbb{R}^d \times \Gamma_{[a,b]})$ such that $\boldsymbol{\mu} = \{e_t \sharp \boldsymbol{\eta}\}_{t \in [a,b]} \in \mathscr{A}^F_{[a,b]}(\mu)$ and

$$\lim_{t \to a^+} \int_{\mathbb{R}^d \times \Gamma_{[a,b]}} \langle \varphi \circ e_0(x,\gamma), \frac{e_t(x,\gamma) - e_a(x,\gamma)}{t-a} \rangle \, d\boldsymbol{\eta}(x,\gamma) = $$

$$= \int_{\mathbb{R}^d} \langle \varphi(x), v_a(x) \rangle \, d\mu(x).$$

Proof is based essentialy on the possibility to parametrize F, and on the Filippov's Lemma.

Cost functionals - Overview

- We decribed up to now the macroscopic dynamic of the agents, supposing conservation of the total mass.
- In real-life models, the agents also interact between them, and the interaction can be of local or nonlocal type.
- The effects of these interactions will be encoded in the cost functional that we want to minimize.
- To this aim convexity and lower semicontinuity of functional depending on measures will play a crucial role.
- Extensions to situations where the total mass is not preserved during the evolution (e.g. evacuation problems) are very difficult due to the lack of a general superposition principle allowing us to represent them as superposition of weighted characteristics. Nevertheless, such a representation can be built by hand in many interesting cases.

Extensions - Some natural cost functional

Here we present some natural cost functions.

A functional with a local constraints on velocities and position.

$$\hat{J}_{\mathrm{sys}}(T,\boldsymbol{\mu},\boldsymbol{\nu}) := \begin{cases} \displaystyle\int_0^T \int_{\mathbb{R}^d} L_c^a\left(t, x, \frac{\nu_t}{\mu_t}(x)\right)\, d\mu_t(x)\, dt, & \text{if } \nu_t \ll \mu_t,\ \dfrac{\nu_t}{\mu_t}(x) \in F(x) \\ & \text{for a.e. } t \in [0,T],\ \mu_t - \text{a.e. } x \in \mathbb{R}^d \\ +\infty, & \text{otherwise,} \end{cases} \tag{2}$$

A functional penalizing density concentration w.r.t. a given measure. Given $\sigma \in \mathscr{M}^+(\mathbb{R}^d)$, we define the functional

$$\hat{J}^{\sigma}_{\mathrm{dens}}(T,\boldsymbol{\mu},\boldsymbol{\nu}) := \begin{cases} \displaystyle\int_0^T \int_{\mathbb{R}^d} L_{\mathrm{dens}}\left(t, x, \frac{\mu_t}{\sigma}(x), \frac{\nu_t}{\sigma}(x)\right)\, d\sigma\, dt, & \text{if } \mu_t \ll \sigma \text{ and } |\nu_t| \ll \sigma \\ & \text{for a.e. } t \in [0,T], \\ +\infty, & \text{otherwise,} \end{cases} \tag{3}$$

Extensions - Some natural cost functional, continued

A functional describing an interaction between position and velocities.

$$J_{\text{inter}}(T,\boldsymbol{\eta}) = \int_{X_{\text{inter}}} \int_0^T L_{\text{inter}}\left(t,\gamma_x(t),\gamma_y(t),\dot\gamma_x(t),\dot\gamma_y(t)\right)\,dt\,d\boldsymbol{\eta}(x,\gamma_x)\,d\boldsymbol{\eta}(y,\gamma_y), \tag{4}$$

$$\hat{J}_{\text{inter}}(T,\boldsymbol{\mu},\boldsymbol{\nu}) = \begin{cases} \displaystyle\int_0^T \iint_{\mathbb{R}^d\times\mathbb{R}^d} L_{\text{inter}}\left(t,x,y,\frac{\nu_t}{\mu_t}(x),\frac{\nu_t}{\mu_t}(y)\right)\,d\mu_t(x)\,d\mu_t(y)\,dt, & \text{if } \nu_t \ll \mu_t,\ \dfrac{\nu_t}{\mu_t}(x)\in F(x) \text{ a.e. } t\in[0,T],\ \mu_t-\text{a.e. } x\in\mathbb{R}^d, \\ +\infty, & \text{otherwise,} \end{cases} \tag{5}$$

State of art, so far...

Natural questions

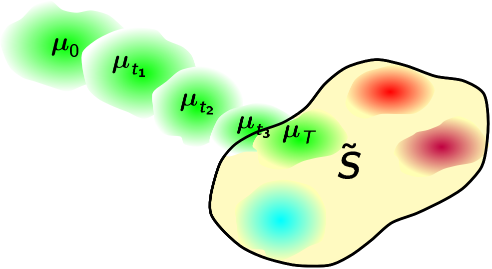
- Existence of optimal trajectories?
- Dynamic programming principle?
- Smoothness of the value function?
- Hamilton-Jacobi-Bellman equation?
- Necessary conditions?

Up to now

- Dynamic programming principle for all the functionals
- Mayer problem with smooth terminal cost function and no interaction
- Minimum time problem with no interaction
- Some cases of problems with mass loss (optimal equipment and evacuation)
- Application to some simple pursuit-evasion games

Generalized Target in Wasserstein space - Overview

- Target set: defined by duality (an observer wants to steer the system into states in which the results of some measurements are below a fixed threshold);
- Minimum time: straightforward generalization of the classical one.



Definition and basic properties of the generalized target

For $p \geq 1$, $\Phi \subseteq C^0(\mathbb{R}^d, \mathbb{R})$ s.t. $\exists x_0 \in \mathbb{R}^d$ with $\phi(x_0) \leq 0$ $\forall \phi \in \Phi$, and for all $\phi \in \Phi$ there exists $D_\phi > 0$ s.t. $\phi(x) \geq -D_\phi$ $\forall x \in \mathbb{R}^d$:

$$\tilde{S}_p^\Phi := \left\{ \mu \in \mathscr{P}_p(\mathbb{R}^d) : \int_{\mathbb{R}^d} \phi(x)\, d\mu(x) \leq 0 \text{ for all } \phi \in \Phi \right\}.$$

We say that Φ satisfies property (T_p) with $p > 0$ if

(T_p) *for all* $\phi \in \Phi$ *there exist* $A_\phi, C_\phi > 0$ *such that* $\phi(x) \geq A_\phi |x|^p - C_\phi$.

We obtain that:

- $\tilde{S}_p^\Phi$ is closed and convex;
- if (T_p) holds, then $\tilde{S}_p^\Phi$ is compact in the W_p-topology (hence in the w^*-topology).

We say that $\tilde{S}_p^\Phi$ admits a classical counterpart if $\exists S \subseteq \mathbb{R}^d$ s.t.

$$\tilde{S}_p^\Phi = \{ \mu \in \mathscr{P}_p(\mathbb{R}^d) \; : \; \operatorname{supp} \mu \subseteq S \}$$

Generalized minimum time

We define the generalized minimum time function
$\tilde{T}_p^\Phi : \mathscr{P}_p(\mathbb{R}^d) \to [0, +\infty]$ as:

$$\tilde{T}_p^\Phi(\mu_0) := \inf \left\{ J_F(\boldsymbol{\mu}, \boldsymbol{\nu}) : \begin{array}{l} \boldsymbol{\mu} \text{ is an admissible curve in } [0, \tau] \\ \text{driven by } \boldsymbol{\nu}, \text{ with } \begin{array}{l} \mu_{|t=0} = \mu_0 \\ \mu_{|t=\tau} \in \tilde{S}_p^\Phi \end{array} \end{array} \right\},$$

where, by convention, $\inf \emptyset = +\infty$.

Given $\mu_0 \in \mathscr{P}_p(\mathbb{R}^d)$, an admissible curve $\boldsymbol{\mu} = \{\mu_t\}_{t \in [0, \tilde{T}_p^\Phi(\mu_0)]} \subseteq \mathscr{P}_p(\mathbb{R}^d)$, driven by a family of Borel vector-valued measures $\boldsymbol{\nu} = \{\nu_t\}_{t \in [0, \tilde{T}_p^\Phi(\mu_0)]}$, s.t. $\mu_{|t=0} = \mu_0$ and $\mu_{|t=\tilde{T}_p^\Phi(\mu_0)} \in \tilde{S}_p^\Phi$ is optimal for μ_0 if

$$\tilde{T}_p^\Phi(\mu_0) = J_F(\boldsymbol{\mu}, \boldsymbol{\nu}).$$

Dynamic programming principle

Theorem

Let $0 \leq s \leq \tau$,
$F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be a set-valued function,
$\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,\tau]}$ be an admissible curve for Σ_F.

Then

$$\tilde{T}_p^\Phi(\mu_0) \leq s + \tilde{T}_p^\Phi(\mu_s).$$

Moreover, if $\tilde{T}_p^\Phi(\mu_0) < +\infty$, then

$$\text{equality holds } \forall s \in [0, \tilde{T}_p^\Phi(\mu_0)] \iff \boldsymbol{\mu} \text{ is optimal for } \mu_0 = \mu_{|t=0}.$$

The proof is based on gluing results for solutions of continuity equation.

Existence theorem

Theorem (Existence of minimizers)

Assume standard assumptions on F.

Let $p > 1$,
$\mu_0 \in \mathscr{P}_p(\mathbb{R}^d)$,
$\Phi \in C^0(\mathbb{R}^d;\mathbb{R})$,
$\tilde{T}_p^\Phi(\mu_0) < \infty$.

Then there exists an admissible curve $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]}$ driven by $\boldsymbol{\nu} = \{\nu_t\}_{t\in[0,T]}$ which is optimal for μ_0, that is $\tilde{T}_p^\Phi(\mu_0) = J_F(\boldsymbol{\mu},\boldsymbol{\nu})$.

The proof is based on the previous result of compactness of admissible trajectories in the space of measures, together with the lower semicontinuity of the minimum time functional J_F.

Comparison results (for $\tilde{S}^{\Phi} = \tilde{S}^{\{d_S\}}$)

Proposition

Under the standard assumptions on F we have

$$\tilde{T}_p(\mu_0) \geq ||T||_{L^\infty_{\mu_0}} \qquad \forall \mu_0 \in \mathscr{P}_p(\mathbb{R}^d);$$
$$\tilde{T}_p(\delta_{x_0}) = T(x_0) \qquad \forall x_0 \in \mathbb{R}^d.$$

Theorem

Assume the standard hypothesis on F.

Let $p > 1$,
$\mu_0 \in \mathscr{P}_p(\mathbb{R}^d)$,
$S \subseteq \mathbb{R}^d$ be a weakly invariant set for the dynamics $\dot{x}(t) \in F(x(t))$.

Then

$$\tilde{T}^{\Phi}_p(\mu_0) = \|T(\cdot)\|_{L^\infty_{\mu_0}}.$$

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$S \subseteq \mathbb{R}^d$ be a weakly invariant set for the dynamics $\dot{x}(t) \in F(x(t))$.

Then

$$\tilde{T}^{\Phi}_p(\mu_0) = \|T(\cdot)\|_{L^\infty_{\mu_0}}.$$

Controllability in the C^1_c case

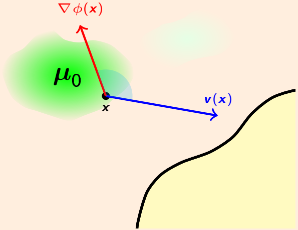
Theorem [Petrov-like condition]

Assume the standard hypothesis on F, $p \geq 1$, $\mu_0 \in \mathscr{P}_p(\mathbb{R}^d)$.

Let $\Phi \subseteq C^1_c(\mathbb{R}^d; \mathbb{R})$.

Assume that

$\exists v : \mathbb{R}^d \to \mathbb{R}^d$ Borel vector field,
$\exists \boldsymbol{\mu} := \{\mu_t\}_{t\in[0,+\infty[} \subseteq \mathscr{P}_p(\mathbb{R}^d)$
adm. traj. driven by $\boldsymbol{\nu}$,
with $\boldsymbol{\nu} = \{\nu_t = v\mu_t\}_{t\in[0,+\infty[}$,
$\mu_{|t=0} = \mu_0$,



such that the following controllability condition holds:

(C_c) for all $\phi \in \Phi$ exists $k^\phi > 0$ s.t. $\langle \nabla\phi(x), v(x)\rangle \leq -k^\phi$ for a.e. $t > 0$ and μ_t-a.e. $x \in \mathbb{R}^d$.

Then we have $$\tilde{T}^\Phi_p(\mu_0) \leq \sup_{\phi\in\Phi} \left\{ \frac{1}{k^\phi} \int_{\mathbb{R}^d} \phi(x)\, d\mu_0(x) \right\}.$$

Extensions of the smooth controllability condition

G. Cavagnari has obtained more refined controllability conditions in

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by weakening the requirements on Φ. In general this operation is highly nontrivial, since - unless we restrict ourselves on particular class of measures, the evolution may be highly sensitive to the singularity set of the functions of Φ.

We are currently investigating so-called *higher order* controllability conditions by defining a proper notion of *commutator for the flow* of the continuity equation, recalling some ideas of Rampazzo-Sussman construction for Lie Bracket of nonsmooth vector fields. Our analysis is complicated by the possibly highly nonsmoothness of the driving vector fields.

Mayer problem

Given a cost function $\mathscr{G} : \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}$ and a time horizon $T > 0$, we will consider the problem of minimizing the cost over all the endpoints of the trajectories in the space of measures that can be represented as a superposition of trajectories defined in $[0, T]$ of a given differential inclusions $\dot{x}(t) \in F(x(t))$, weighted by a probability measure μ on the initial state.

Throughout this section, we will made the following standing assumptions:

(F) $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is a Lipschitz continuous set-valued map with nonempty compact convex values;

($\mathscr{G}$) $\mathscr{G} : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is bounded and Lipschitz continuous w.r.t. W_2 metric.

Value function for the Mayer problem

Given $s \in [0, T]$, $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, we define the *value function* $V : [0, T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ by setting

$$V(s, \mu) = \inf \left\{ \mathscr{G}(\mu_T) : \{\mu_t\}_{t \in [s,T]} \in \mathscr{A}^F_{[s,T]}(\mu) \right\}.$$

We say that $\{\mu_t\}_{t \in [s,T]} \in \mathscr{A}^F_{[s,T]}(\mu)$ is an *optimal trajectory* for $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ if $V(s, \mu) = \mathscr{G}(\mu_T)$.

From the properties of the set of admissible trajectories, since $\mathscr{G}(\cdot)$ is l.s.c., we deduce immediately the existence of optimal trajectories for every $\mu \in \mathscr{P}_2(\mathbb{R}^d)$.

Regularity of the value function

Proposition

Let $T > 0$, $F, \mathscr{G}$ be satisfying **(F)** and **($\mathscr{G}$)**, respectively. Then $V : [0, T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is bounded and for every $K \geq 0$, it is Lipschitz continuous on the set $\{(t, \mu) \in [0, T] \times \mathscr{K},\ \mathrm{m}_2(\mu) \leq K\}$.

The proof differs from the classical one, since the continuity equation in general does not enjoy uniqueness and Lipschitz continuous dependence of the solutions from the initial data: indeed, by means of dynamic transport plans, it is needed to construct a suitable shifted trajectory from an optimal one.

Dynamic Programming Principle for the Mayer problem

Proposition

For all $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ and $\tau \in [0, T]$ we have

$$V(\tau, \mu) = \inf \left\{ V(s, \mu_s) : \{\mu_t\}_{t \in [\tau, T]} \in \mathscr{A}^F_{[\tau, T]}(\mu),\ s \in [\tau, T] \right\},$$

i.e., $V(\tau, \mu_\tau) \leq V(s, \mu_s)$ for all $\tau \leq s \leq T$ and $\{\mu_t\}_{t \in [\tau, T]} \in \mathscr{A}^F_{[\tau, T]}(\mu)$, and $V(\tau, \mu_\tau) = V(s, \mu_s)$ for all $\tau \leq s \leq T$ if and only if $\{\mu_t\}_{t \in [\tau, T]}$ is an optimal trajectory for μ.

The proof is the same of the classical finite-dimensional case.

Viscosity sub/super-differentials

Definition [M.-Quincampoix]

Let $w : [0, T] \times \mathscr{P}_2 \to \mathbb{R}$ be a map, $(\bar{t}, \bar{\mu}) \in]0, T[\times \mathscr{P}_2(\mathbb{R}^d)$, $\delta > 0$. We say that $(p_{\bar{t}}, p_{\bar{\mu}}) \in \mathbb{R} \times L^2_{\bar{\mu}}(\mathbb{R}^d)$ belongs to the viscosity δ-superdifferential of w at $(\bar{t}, \bar{\mu})$ if

i.) there exists $\bar{\nu}$ and $\gamma \in \Pi_o(\bar{\mu}, \bar{\nu})$ such that for all Borel map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ satisfying $\phi \in L^2_\mu(\mathbb{R}^d) \cap L^2_\nu(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \phi(x), x - y \rangle \, d\gamma(x, y) = \int_{\mathbb{R}^d} \langle \phi(x), p^\mu_\gamma(x) \rangle \, d\mu(x).$$

ii.) for all $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ we have

$$w(t, \mu) - w(\bar{t}, \bar{\mu}) \leq p_t(t - \bar{t}) + \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} \langle x_2, x_3 - x_1 \rangle \, d\tilde{\mu}(x_1, x_2, x_3) + \\ + \delta \sqrt{(t - \bar{t})^2 + W^2_{2,\tilde{\mu}}(\bar{\mu}, \mu)} + o(|t - \bar{t}| + W_{2,\tilde{\mu}}(\bar{\mu}, \mu)),$$

for all $\tilde{\mu} \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ satisfying $\pi_{12} \sharp \tilde{\mu} = (\mathrm{Id}_{\mathbb{R}^d}, p_{\bar{\mu}}) \sharp \bar{\mu}$ and $\pi_{13} \sharp \tilde{\mu} \in \Pi(\bar{\mu}, \mu)$.

Hamilton-Jacobi-Bellman Equation

We consider an equation in the form

$$\partial_t w(t,\mu) + \mathscr{H}(\mu, Dw(t,\mu)) = 0, \tag{6}$$

where $\mathscr{H}(\mu,p)$ is defined for any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ and $p \in L^2_\mu(\mathbb{R}^d)$. We say that a function $w : [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is

- a *subsolution* of (6) if w is u.s.c. and there exists a constant $C > 0$ such that
$$p_t + \mathscr{H}(\mu, p_\mu) \geq -C\delta,$$
for all $(t,\mu) \in]0,T[\times \mathscr{P}_2(\mathbb{R}^d)$, $(p_t,p_\mu) \in D^+_\delta w(t_0,\mu_0)$, and $\delta > 0$.
- a *supersolution* of (6) if w is l.s.c. and there exists a constant $C > 0$ such that
$$p_t + \mathscr{H}(\mu, p_\mu) \leq C\delta,$$
for all $(t,\mu) \in]0,T[\times \mathscr{P}_2(\mathbb{R}^d)$, $(p_t,p_\mu) \in D^-_\delta w(t_0,\mu_0)$, and $\delta > 0$.
- a *solution* of (6) if w is both a supersolution and a subsolution.

Comparison principle

Theorem [M.-Quincampoix]

Consider the HJB equation for an Hamiltonian function $\mathscr{H}$ satisfying the following properties

- positive homogenuity: for every $\lambda \geq 0$, $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, $p \in L^2_\mu(\mathbb{R}^d)$ we have $\mathscr{H}(\mu, \lambda p) = \lambda \mathscr{H}(\mu, p)$;
- dissipativity: there exists $k \geq 0$ such that for all $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$, $\gamma \in \Pi_o(\mu, \nu)$, defined $p^\mu_\gamma = \mathrm{Id}_{\mathbb{R}^d} - \mathrm{Bar}_1(\gamma)$, $q^\nu_\gamma = \mathrm{Id}_{\mathbb{R}^d} - \mathrm{Bar}_1(\gamma^{-1})$, we have

$$\mathscr{H}_F(\mu, p_\mu) - \mathscr{H}_F(\nu, q_\nu) \leq kW_2^2(\mu, \nu).$$

Let w_1 be a bounded and Lipschitz continuous subsolution and w_2 be a bounded and Lipschitz continuous supersolution to (6). Then

$$\inf_{(s,\mu)\in[0,T]\times\mathscr{P}_2(\mathbb{R}^d)} w_2(s, \mu) - w_1(s, \mu) = \inf_{\mu\in\mathscr{P}_2(\mathbb{R}^d)} w_2(T, \mu) - w_1(T, \mu).$$

In particular, the equation admits at most one Lipschitz continuous bounded solution.

Proof: based on the doubling variable method and to the analysis of the superdifferential of the Wasserstein squared distance given by Ambrosio-Gigli-Savaré.

Hamiltonian function for the minimum time

Theorem

Assume standard assumptions on F and that $F(\cdot)$ is bounded.
Then $\tilde{T}_2(\cdot)$ is a viscosity solution of $\mathscr{H}_F(\mu, D\tilde{T}_2(\mu)) = 0$,

where the Hamiltonian function is defined by

$$\mathscr{H}_F(\mu, p_\mu) := -1 - \inf \left\{ \int_{\mathbb{R}^d} \langle p_\mu(x), v_\mu(x) \rangle \, d\mu(x) \right\},$$

and the infimum is taken on the Borel maps $v_\mu : \mathbb{R}^d \to \mathbb{R}^d$ satisfying $v_\mu(x) \in F(x)$ for μ-a.e. $x \in \mathbb{R}^d$. In the case f Lipschitz continuity, it is the unique solution.

HJB equation for the Mayer problem

Theorem

Given $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, $p_\mu \in L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)$, we set

$$\mathscr{H}_F(\mu, p_\mu) := \inf \left\{ \int_{\mathbb{R}^d} \langle p_\mu(x), v_\mu(x) \rangle \, d\mu(x) \right\},$$

where the infimum is taken on the Borel maps $v_\mu : \mathbb{R}^d \to \mathbb{R}^d$ satisfying $v_\mu(x) \in F(x)$ for μ-a.e. $x \in \mathbb{R}^d$. Then the value function $V(\cdot)$ is the unique Lipschitz continuous solution of the equation on sets of measures with uniformly bounded second-order moment.

Application to a pursuit-evasion game - I

We consider a two player zero sum game, where the two players are two populations, each of them evolving according to

$$\partial_t \mu_t^i + \mathrm{div}(v_t^i \mu_t^i) = 0, \qquad\qquad i = 1, 2,$$

where for a.e. $t \in [0, T]$ and μ_t^i-a.e. $x \in \mathbb{R}^d$ we have $v_t^i(x) \in F_i(x)$, $i = 1, 2$.

We consider finite horizon $T > 0$, and a bounded Lipschitz terminal cost $\mathscr{G} = \mathscr{G}(\mu_1, \mu_2)$. The objective of the first and of the second player are to minimize and to maximize it, respectively.

Due to the ill-posedness of the continuity equation (since in general the vector field v_t is not Lipschitz continuous), a convenient choice is to define the strategy (with delay) directly on the trajectories.

Application to a pursuit-evasion game

We consider two set-valued map $F, G : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ satisfying $(\boldsymbol{F})$. Given $\mu_a \in \mathscr{P}_2(\mathbb{R}^d)$, the set of admissible trajectories starting from μ_a at time $t = a$ defined on $[a, b]$ for the first player will be $\mathscr{A}^F_{[a,b]}(\mu_a)$, and, similarly, given $\nu_a \in \mathscr{P}_2(\mathbb{R}^d)$, the set of admissible trajectories starting from ν_a at time $t = a$ defined on $[a, b]$ for the second player will be $\mathscr{A}^G_{[a,b]}(\nu_a)$.

Definition 1 (Nonanticipative strategies)

A *strategy* for the first player defined on $[t_0, T]$ will be a map $\alpha : \mathscr{A}^G_{[t_0,T]} \to \mathscr{A}^F_{[t_0,T]}$. A strategy for the first player α defined on $[t_0, T]$ will be called *nonanticipative with delay* τ if there exists $\tau > 0$ such that given $t_0 \le s \le T$, $\boldsymbol{\nu}^i = \{\nu^i_t\}_{t\in[t_0,T]} \in \mathscr{A}^G_{[t_0,T]}$, $i = 1, 2$, satisfying $\nu^1_t = \nu^2_t$ for all $t_0 \le t \le s$, and set $\alpha(\boldsymbol{\nu}^i) = \{\mu^i_t\}_{t\in[t_0,T]}$, $i = 1, 2$, we have $\mu^1_t = \mu^2_t$ for all $t_0 \le t \le \min\{s + \tau, T\}$.

Strategy sets

Definition 2

Given $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$, we define

$$\begin{aligned}
\mathcal{A}_\tau(t_0) &:= \left\{\alpha : \mathscr{A}^G_{[t_0,T]} \to \mathscr{A}^F_{[t_0,T]} : \alpha \text{ is a nonant. strategy w. delay } \tau\right\},\\
\mathcal{A}_\tau(t_0,\mu_0) &:= \left\{\alpha \in \mathcal{A}_\tau(t_0) : \alpha(\mathscr{A}^G_{[t_0,T]}) \subseteq \mathscr{A}^F_{[t_0,T]}(\mu_0)\right\},\\
\mathcal{A}(t_0) &:= \bigcup_{\tau>0} \mathcal{A}_\tau(t_0),\\
\mathcal{A}(t_0,\mu_0) &:= \left\{\alpha \in \mathcal{A}(t_0) : \alpha(\mathscr{A}^G_{[t_0,T]}) \subseteq \mathscr{A}^F_{[t_0,T]}(\mu_0)\right\}.
\end{aligned}$$

By switching the roles of F and G in the previous definitions, we obtain the corresponding definition of strategy and nonanticipative strategy defined on $[t_0, T]$ with delay τ for the second player. The corresponding defined sets are named by $\mathcal{B}_\tau(t_0)$, $\mathcal{B}_\tau(t_0,\nu_0)$, $\mathcal{B}(t_0)$, $\mathcal{B}(t_0,\nu_0)$, respectively, for any given $\nu_0 \in \mathscr{P}_2(\mathbb{R}^d)$.

Normal form

Lemma 3 (Normal form)

Let $t_0 < \tau < T$. *For any* $(\alpha, \beta) \in \mathcal{A}_\tau(t_0) \times \mathcal{B}_\tau(t_0)$ *there is a unique pair* $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathscr{A}^F_{[t_0,b]} \times \mathscr{A}^G_{[t_0,b]}$ *such that* $\alpha(\boldsymbol{\nu}) = \boldsymbol{\mu}$ *and* $\beta(\boldsymbol{\mu}) = \boldsymbol{\nu}$.

Upper and lower value functions

Definition 4

We consider a *payoff function* $\mathcal{G} : \mathscr{P}(\mathbb{R}^d) \times (\mathbb{R}^d) \to \mathbb{R}$ bounded and locally Lipschitz continuous, and we assume that F and G satisfy **(F)**. Given $t_0 \in [0, T]$, $\mu_0, \nu_0 \in \mathscr{P}_2(\mathbb{R}^d)$, $(\alpha, \beta) \in \mathcal{A}(\mu_0, t_0) \times \mathcal{B}(\nu_0, t_0)$ we define

$$J(t_0, \mu_0, \nu_0, \alpha, \beta) = \mathcal{G}\left(\mu_T, \nu_T\right),$$

where $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]} \in \mathscr{A}^F_{[t_0,T]}(\mu_0)$, $\boldsymbol{\nu} = \{\nu_t\}_{t\in[0,T]} \in \mathscr{A}^G_{[t_0,T]}(\nu_0)$, and $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathscr{A}^F_{[t_0,T]}(\mu_0) \times \mathscr{A}^G_{[t_0,T]}(\nu_0)$ is the unique element of $\mathscr{A}^F_{[t_0,T]}(\mu_0) \times \mathscr{A}^G_{[t_0,T]}(\nu_0)$, given by Lemma 3, satisfying $\alpha(\boldsymbol{\nu}) = \boldsymbol{\mu}$ and $\beta(\boldsymbol{\nu}) = \boldsymbol{\mu}$.

The *upper* and *lower* value function $V^{\pm} : [0, T] \times \mathscr{P}_2(\mathbb{R}^d) \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ are defined by setting

$$V^+(t_0, \mu_0, \nu_0) = \inf_{\alpha\in\mathcal{A}(t_0,\mu_0)} \sup_{\beta\in\mathcal{B}(t_0,\nu_0)} J(t_0, \mu_0, \nu_0, \alpha, \beta),$$

$$V^-(t_0, \mu_0, \nu_0) = \sup_{\beta\in\mathcal{B}(t_0,\nu_0)} \inf_{\alpha\in\mathcal{A}(t_0,\mu_0)} J(t_0, \mu_0, \nu_0, \alpha, \beta).$$

Existence of a value and its characterization

Definition 5 (Hamiltonian function for the pursuit-evasion game)

We consider F, G satisfying $\mathbf{(F)}$, and define the following Hamiltonian function for all $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$, $p_\mu \in L^2_\mu(\mathbb{R}^d)$, $p_\nu \in L^2_\mu(\mathbb{R}^d)$

$$\mathscr{H}_{PE}(\mu,\nu,p_\mu,p_\nu) = \inf_{\substack{v(\cdot)\in L^2_\mu(\mathbb{R}^d)\\ v(x)\in F(x)\ \mu\text{-a.e.}x}} \int_{\mathbb{R}^d} \langle p_\mu(x), v(x)\rangle\, d\mu(x) + \\ + \sup_{\substack{w(\cdot)\in L^2_\nu(\mathbb{R}^d)\\ w(x)\in G(x)\ \nu\text{-a.e.}x}} \int_{\mathbb{R}^d} \langle p_\nu(x), w(x)\rangle\, d\nu(x). \quad (7)$$

Theorem 6

Consider F, G satisfying $\mathbf{(F)}$, and a bounded Lipschitz continuous payoff function $\mathcal{G}$. Then the game has a value, i.e., $V^+ = V^- =: V$ and V is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation $\partial_t V + \mathscr{H}_{PE}(\mu,\nu,D_\mu V, D_\nu V) = 0$, $V(T,\mu,\nu) = \mathcal{G}(\mu,\nu)$.

Work in progress

- comparison principle for Hamilton-Jacobi equation under weaker smoothness assumption of the value function;
- Pontryagin maximum principle and necessary conditions;
- more general cost functions;
- application to pedestrian dynamics (evacuation problem, problems with mass sources and sinks).

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Thank you!

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