Dynamic History–Dependent Variational–Hemivariational Inequalities with Applications

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Outline of the talk

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Mathematical tool: convex subdifferential

Let E be a Banach space and E^* be its dual.

Definition (convex subdifferential)

Let $\varphi\colon E \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The (convex) subdifferential of φ at x, and is defined by

$$\partial\varphi(x) = \{\, x^* \in E^* \mid \varphi(v) \geq \varphi(x) + \langle x^*, v - x\rangle_{E^* \times E} \ \text{ for all } \ v \in E \,\}.$$

Sometimes we refer to $\partial\varphi$ as the subdifferential of φ in the sense of convex analysis. Observe that if $\varphi(x) = +\infty$, then $\partial\varphi(x) = \emptyset$.

The Clarke subgradient (1983)

Definition (Clarke subgradient)

Let $h\colon E \to \mathbb{R}$ be a locally Lipschitz function on a Banach space E.

- The generalized directional derivative of h at $x \in E$ in the direction $v \in E$ is defined by

$$h^0(x; v) = \limsup_{y \to x,\ t \downarrow 0} \frac{h(y + tv) - h(y)}{t}.$$

- The generalized subgradient of h at x is given by

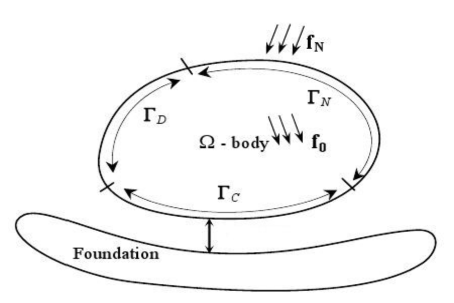
$$\partial h(x) = \{\, \zeta \in E^* \mid h^0(x; v) \geq \langle \zeta, v \rangle_{E^* \times E} \text{ for all } v \in E \,\}.$$

The locally Lipschitz function h is called regular (in the sense of Clarke) at $x \in E$ if for all $v \in E$ the one-sided directional derivative $h'(x; v)$ exists and satisfies $h^0(x; v) = h'(x; v)$ for all $v \in E$.

Part 1

Dynamic frictional nonsmooth contact problem

Physical setting of the contact model



A viscoelastic body occupies an open, bounded and connected set $\Omega \subset \mathbb{R}^d$, $d = 2, 3$.
The boundary Γ is Lipschitz continuous and $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_C$ with mutually disjoint measurable parts, $m(\Gamma_D) > 0$.
We denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ.
The part Γ_C the potential contact surface.

Basic notation

We suppose the body is clamped on Γ_D, volume forces of density $\boldsymbol{f}_0$ act in Ω and surface tractions of density $\boldsymbol{f}_N$ are applied on Γ_N.

The notation $\mathbb{S}^d$ stands for the space of second order symmetric tensors on $\mathbb{R}^d$. On $\mathbb{R}^d$ and $\mathbb{S}^d$ we use the inner products and the Euclidean norms defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \ \|\boldsymbol{u}\| = (\boldsymbol{u} \cdot \boldsymbol{u})^{1/2} \quad \text{for all } \boldsymbol{u} = (u_i), \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \ \|\boldsymbol{\sigma}\| = (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})^{1/2} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}), \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d,$$

respectively.

Basic notation for the contact problem

Given a vector field $\boldsymbol{u}$, notation u_ν and $\boldsymbol{u}_\tau$ represent its normal and tangential components on the boundary defined by

$$u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu} \quad \text{and} \quad \boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}.$$

For a tensor $\boldsymbol{\sigma}$, the symbols σ_ν and $\boldsymbol{\sigma}_\tau$ denote its normal and tangential components on the boundary, i.e.,

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu} \quad \text{and} \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}.$$

Sometimes, we omit the explicit dependence on $\boldsymbol{x} \in \Omega \cup \Gamma$.

Problem (Classical model for the contact process)

Find a displacement field $\boldsymbol{u}\colon \Omega \times (0,T) \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma}\colon \Omega \times (0,T) \to \mathbb{S}^d$ *such that for all* $t \in (0,T)$*,*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\boldsymbol{u}'(t)) + \mathcal{B}\varepsilon(\boldsymbol{u}(t)) + \int_0^t \mathcal{C}(t-s)\varepsilon(\boldsymbol{u}'(s))\,ds \qquad \text{in } \Omega,$$

$$\rho\,\boldsymbol{u}''(t) = \mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) \qquad \text{in } \Omega,$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \qquad \text{on } \Gamma_D,$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_N(t) \qquad \text{on } \Gamma_N,$$

$$-\sigma_\nu(t) \in k(u_\nu(t))\,\partial j_\nu(u_\nu'(t)) \qquad \text{on } \Gamma_C,$$

$$\|\boldsymbol{\sigma}_\tau(t)\| \le F_b\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big),$$

$$-\boldsymbol{\sigma}_\tau(t) = F_b\big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\big)\frac{\boldsymbol{u}_\tau'(t)}{\|\boldsymbol{u}_\tau'(t)\|} \quad \text{if } \boldsymbol{u}_\tau'(t) \neq \boldsymbol{0} \qquad \text{on } \Gamma_C,$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \boldsymbol{u}'(0) = \boldsymbol{w}_0 \qquad \text{in } \Omega.$$

Comments on the model

- Relation

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\boldsymbol{u}'(t)) + \mathcal{B}\varepsilon(\boldsymbol{u}(t)) + \int_0^t \mathcal{C}(t-s)\varepsilon(\boldsymbol{u}'(s))\,ds \ \text{ in } \ \Omega$$

represents the viscoelastic constitutive law in which $\mathcal{A}$ is the viscosity operator, $\mathcal{B}$ is the elasticity operator, $\mathcal{C}$ is the relaxation tensor, and $\varepsilon(\boldsymbol{u})$ denotes the linearized strain tensor defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}) \ \text{ in } \Omega.$$

- Equation

$$\rho\,\boldsymbol{u}''(t) = \mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) \ \text{ in } \ \Omega$$

represents the equation of motion, where ρ denotes the density of mass, and $\boldsymbol{f}_0$ denotes the density of the time-dependent volume forces. For simplicity, we take $\rho = 1$.

Comments on the model

- Condition $\boldsymbol{u}(t) = \mathbf{0}$ on Γ_D is the displacement homogeneous boundary condition which means that the body is fixed on Γ_D.
- Condition $\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_N$ on Γ_N is the traction boundary condition with surface tractions of density $\boldsymbol{f}_N$ acting on Γ_N.
- Relation

  $$-\sigma_\nu(t) \in k(u_\nu(t))\, \partial j_\nu(u'_\nu(t)) \quad \text{on} \quad \Gamma_C$$

  represents the multivalued contact condition with nonmonotone normal damped response in which ∂j_ν denotes the Clarke subgradient of a given function j_ν and k is a damper coefficient which is allowed to depend on the normal displacement.

Comments on the model

- Condition

$$\|\boldsymbol{\sigma}_\tau(t)\| \le F_b\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big),$$

$$-\boldsymbol{\sigma}_\tau(t) = F_b\big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\big)\frac{\boldsymbol{u}'_\tau(t)}{\|\boldsymbol{u}'_\tau(t)\|} \ \text{ if } \boldsymbol{u}'_\tau(t) \neq \mathbf{0} \ \text{ on } \ \Gamma_C$$

represents a version of the Coulomb law of dry friction in which F_b is a given positive function, the friction bound. The friction bound may depend on the quantity

$$S(\boldsymbol{x}, t) = \int_0^t \|\boldsymbol{u}_\tau(\boldsymbol{x}, s)\|\,ds$$

which is the total slip (or the accumulated slip) at the point $\boldsymbol{x} \in \Gamma_3$ over the time period $[0, t]$. Considering such a dependence is reasonable from the physical point of view, since it incorporates the changes on the contact surface resulting from sliding.

A remark

It is well known that a variant of the Coulomb law of dry friction on Γ_C can be equivalently formulated as follows

$$-\boldsymbol{\sigma}_\tau(t) \in F_b\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\, ds\Big)\, \partial \|\boldsymbol{u}'_\tau(t)\|_{\mathbb{R}^d} \quad \text{on} \quad \Gamma_C,$$

where

$$\partial\, \|\boldsymbol{x}\| = \begin{cases} \overline{B}(\mathbf{0}, 1), & \text{if } \boldsymbol{x} = \mathbf{0} \\ \dfrac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, & \text{if } \boldsymbol{x} \neq \mathbf{0} \end{cases}$$

for $\boldsymbol{x} \in \mathbb{R}^d$, $\overline{B}(\mathbf{0}, 1)$ denotes the unit closed ball in $\mathbb{R}^d$.

A remark

J. J. Telega, Variational methods in contact problems in mechanics, Advances in Mechanics **10** (1987), 3-95.

"Generalizations of subdifferential boundary conditions

$$-\sigma_\nu \in \partial j_\nu(u_\nu), \qquad -\boldsymbol{\sigma}_\tau \in \partial j_\tau(\boldsymbol{u}_\tau) \quad \text{on} \quad \Gamma_C$$

to the case of nonmonotone relations, with the use of mathematical results of Rockafellar (1981), are presented by Panagiotopoulos (1983-1985). These generalizations do not have practical applications so far. This follows from the fact that the existing notion of the generalized subdifferential to the case of nonconvex functions is quite complex".

Variational formulation of the contact problem

We use the spaces

$$V = \{\, \boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D \,\}, \quad H = L^2(\Omega;\mathbb{R}^d), \ \mathcal{H} = L^2(\Omega;\mathbb{S}^d).$$

We denote by $\boldsymbol{v}$ the trace on the boundary of an element $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d)$. On V we consider the inner product and the corresponding norm given by

$$(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad \|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \ \text{ for all } \ \boldsymbol{u},\, \boldsymbol{v} \in V.$$

Since $meas(\Gamma_D) > 0$, it follows that V is a Hilbert space. The trace operator is denoted by $\gamma\colon V \to L^2(\Gamma_C;\mathbb{R}^d)$. The space $\mathcal{H}$ is a Hilbert space endowed with the inner product

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}(\boldsymbol{x})\tau_{ij}(\boldsymbol{x})\,dx,$$

and the associated norm $\|\cdot\|_{\mathcal{H}}$.

Variational formulation of the contact problem

Define a space of fourth order tensor fields

$$\mathbf{Q}_\infty = \{\ \mathcal{E} = (\mathcal{E}_{ijkl}) \mid \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega),\ \ 1 \le i,j,k,l \le d\ \}.$$

This is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \sum_{0 \le i,j,k,l \le d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

Time dependent spaces

Given $0 < T < +\infty$, let $\mathcal{V} = L^2(0, T; V)$ and $\mathcal{W} = \{w \in \mathcal{V} \mid w' \in \mathcal{V}^*\}$, where the time derivative $w' = \partial w/\partial t$ is understood in the sense of vector-valued distributions and $\mathcal{V}^* = L^2(0, T; V^*)$.
It is known that the space $\mathcal{W}$ endowed with the graph norm

$$\|w\|_{\mathcal{W}} = \|w\|_{\mathcal{V}} + \|w'\|_{\mathcal{V}^*}$$

is a separable and reflexive Banach space. We identify $\mathcal{H} = L^2(0, T; H)$ with its dual and obtain the continuous embeddings

$$\mathcal{W} \subset \mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*.$$

The embedding

$$\mathcal{W} \subset C(0, T; H)$$

is continuous, $C(0, T; H)$ being the space of continuous functions on $[0, T]$ with values in H.

Hypotheses on the viscosity operator

$\mathcal{A}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$
\left\{
\begin{array}{l}
\text{(a) there exists } L_{\mathcal{A}} > 0 \text{ such that} \\
\qquad \|\mathcal{A}(\boldsymbol{x}, \varepsilon_1) - \mathcal{A}(\boldsymbol{x}, \varepsilon_2)\| \le L_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\| \\
\qquad\quad \text{for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(b) there exists } m_{\mathcal{A}} > 0 \text{ such that} \\
\qquad (\mathcal{A}(\boldsymbol{x}, \varepsilon_1) - \mathcal{A}(\boldsymbol{x}, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \ge m_{\mathcal{A}} \, \|\varepsilon_1 - \varepsilon_2\|^2 \\
\qquad\quad \text{for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) the mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \varepsilon) \text{ is measurable on } \Omega, \\
\qquad \text{for all } \varepsilon \in \mathbb{S}^d. \\
\text{(d) } \mathcal{A}(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{0} \text{ a.e. } \boldsymbol{x} \in \Omega.
\end{array}
\right.
$$

Hypotheses on the elasticity operator

$\mathcal{B}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} \text{(a) there exists } L_{\mathcal{B}} > 0 \text{ such that} \\ \qquad \|\mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{B}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\ \qquad\quad \text{for all } \ \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega. \\ \text{(b) the mapping } \ \boldsymbol{x} \mapsto \mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\ \qquad\quad \text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(c) } \mathcal{B}(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{0} \text{ a.e. } \boldsymbol{x} \in \Omega. \end{cases}$$

Hypotheses on the relaxation tensor and the damper coefficient

The relaxation tensor satisfies

$$\mathcal{C} \in C(0, T; \mathbf{Q}_\infty).$$

$k\colon \Gamma_C \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$\left\{\begin{array}{l}
\text{(a) the mapping } \boldsymbol{x} \mapsto k(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_C, \\
\qquad \text{for any } r \in \mathbb{R}. \\
\text{(b) there are constants } k_1,\ k_2 \text{ such that} \\
\qquad 0 < k_1 \le k(\boldsymbol{x}, r) \le k_2 \ \text{ for all } r \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_C. \\
\text{(c) there exists } L_k > 0 \text{ such that} \\
\qquad |k(\boldsymbol{x}, r_1) - k(\boldsymbol{x}, r_2)| \le L_k |r_1 - r_2| \\
\qquad\quad \text{for all } r_1, r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_C.
\end{array}\right.$$

Hypotheses on the potential function

$j_\nu \colon \Gamma_C \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\left\{ \begin{array}{l} \text{(a) } j_\nu(\cdot, r) \text{ is measurable on } \Gamma_C \text{ for all } r \in \mathbb{R} \text{ and there} \\ \qquad \text{exists } \overline{e} \in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot, \overline{e}(\cdot)) \in L^1(\Gamma_C). \\ \text{(b) } j_\nu(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_C. \\ \text{(c) } |\partial j_\nu(\boldsymbol{x}, r)| \le \overline{c}_0 \text{ for a.e. } \boldsymbol{x} \in \Gamma_C, \\ \qquad \text{for all } r \in \mathbb{R} \text{ with } \overline{c}_0 \ge 0. \\ \text{(d) } j_\nu^0(\boldsymbol{x}, r_1; r_2 - r_1) + j_\nu^0(\boldsymbol{x}, r_2; r_1 - r_2) \le \overline{\beta}\, |r_1 - r_2|^2 \\ \qquad \text{for a.e. } \boldsymbol{x} \in \Gamma_C, \text{ all } r_1,\, r_2 \in \mathbb{R} \text{ with } \overline{\beta} \ge 0. \end{array} \right.$$

Hypotheses on the friction bound, densities of body forces, surface tractions and initial data

$F_b \colon \Gamma_C \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$\left\{ \begin{array}{l} \text{(a) the mapping } \boldsymbol{x} \mapsto F_b(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_C, \\ \qquad \text{for any } r \in \mathbb{R}. \\ \text{(b) there exists } L_{F_b} > 0 \text{ such that} \\ \qquad |F_b(\boldsymbol{x}, r_1) - F_b(\boldsymbol{x}, r_2)| \le L_{F_b} |r_1 - r_2| \\ \qquad\quad \text{for all } \; r_1, r_2 \in \mathbb{R}, \; \text{a.e. } \boldsymbol{x} \in \Gamma_C. \\ \text{(c) the mapping } \boldsymbol{x} \mapsto F_b(\boldsymbol{x}, 0) \text{ belongs to } L^2(\Gamma_C). \end{array} \right.$$

$$\boldsymbol{f}_0 \in L^2(0, T; L^2(\Omega; \mathbb{R}^d)), \quad \boldsymbol{f}_N \in L^2(0, T; L^2(\Gamma_N; \mathbb{R}^d)), \quad \boldsymbol{u}_0, \, \boldsymbol{w}_0 \in V.$$

Example 1 of the normal damped response potential

Consider the function $p\colon \mathbb{R} \to \mathbb{R}$ defined by

$$p(r) = \begin{cases} 0 & \text{if} \quad r < 0, \\ r & \text{if} \quad 0 \le r < 1, \\ 2 - r & \text{if} \quad 1 \le r < 2, \\ \sqrt{r-2} + r - 2 & \text{if} \quad 2 \le r < 6, \\ r & \text{if} \quad 6 \le r < 7, \\ 7 & \text{if} \quad r \ge 7. \end{cases}$$

The function p is continuous and nonconvex, and it is neither monotone, nor Lipschitz continuous. Consider the function $j_\nu\colon \mathbb{R} \to \mathbb{R}$ defined by

$$j_\nu(r) = \int_0^r p(s)\, ds \quad \text{for} \quad r \in \mathbb{R}.$$

Then the Clarke sugradient ∂j_ν is single-valued and

$$\partial j_\nu(r) = j_\nu'(r) = p(r) \quad \text{for all} \quad r \in \mathbb{R}.$$

The function j_ν satisfies the hypotheses (a)-(d).

Example 2 of the normal damped response potential

Let $p\colon \mathbb{R} \to \mathbb{R}$ be the discontinuous function given by

$$p(r) = \begin{cases} 0 & \text{if} \quad r < 0, \\ e^{-r} + a & \text{if} \quad r \geq 0, \end{cases}$$

where $a \geq 0$. Then the function j_ν has the form

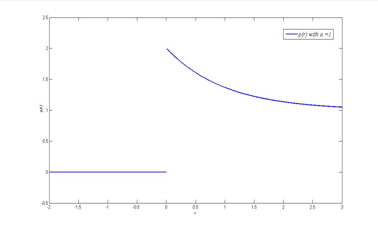
$$j_\nu(r) = \int_0^r p(s)\,ds = \begin{cases} 0 & \text{if} \quad r < 0, \\ -e^{-r} + ar + 1 & \text{if} \quad r \geq 0, \end{cases}$$

its subdifferential is multivalued, and is given by

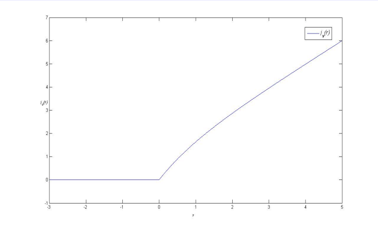
$$\partial j_\nu(r) = \begin{cases} 0 & \text{if} \quad r < 0, \\ [0, 1 + a] & \text{if} \quad r = 0, \\ e^{-r} + a & \text{if} \quad r > 0 \end{cases}$$

and j_ν satisfies the hypotheses (a)-(d).

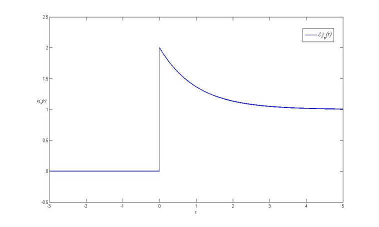
Figures of function p, j_ν and ∂j_ν in Example 2



function p



function j_ν



Clarke subgradient ∂j_ν

Problem (Variational-hemivariational form of the contact problem)

Find a displacement field $\boldsymbol{u}\colon (0,T) \to V$ *such that for a.e.* $t \in (0,T)$,

$$\int_\Omega \boldsymbol{u}''(t)\cdot(\boldsymbol{v}-\boldsymbol{u}'(t))\,dx + (\mathcal{A}\varepsilon(\boldsymbol{u}'(t)),\varepsilon(\boldsymbol{v}-\boldsymbol{u}'(t)))_{\mathcal{H}}$$

$$+(\mathcal{B}\varepsilon(\boldsymbol{u}(t)),\varepsilon(\boldsymbol{v}-\boldsymbol{u}'(t)))_{\mathcal{H}} + \Big(\int_0^t \mathcal{C}(t-s)\varepsilon(\boldsymbol{u}'(s))\,ds, \varepsilon(\boldsymbol{v}-\boldsymbol{u}'(t))\Big)_{\mathcal{H}}$$

$$+\int_{\Gamma_C} F_b\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big)\,(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}'_\tau(t)\|)\,d\Gamma$$

$$+\int_{\Gamma_C} k(u_\nu(t))\,j_\nu^0(u'_\nu(t); v_\nu - u'_\nu(t))\,d\Gamma \ge \langle \boldsymbol{f}(t), \boldsymbol{v}-\boldsymbol{u}'(t)\rangle_{V^*\times V},$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{u}'(0) = \boldsymbol{w}_0.$$

Here $\boldsymbol{f}\colon (0,T) \to V^*$ is defined, for all $\boldsymbol{v} \in V$ and $t \in (0,T)$, by

$$\langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V^*\times V} = (\boldsymbol{f}_0(t), \boldsymbol{v})_H + (\boldsymbol{f}_N(t), \gamma\boldsymbol{v})_{L^2(\Gamma_N;\mathbb{R}^d)}.$$

Unique solvability of the variational-hemivariational inequality

Theorem (1)

Assume the above hypotheses on the data. If

$$m_{\mathcal{A}} > \overline{\beta}\, k_2 \|\gamma\|^2,$$

then the variational-hemivariational inequality has a unique solution with regularity $\boldsymbol{u} \in \mathcal{V}$ and $\boldsymbol{u}' \in \mathcal{W}$.

Conclusion

Under the above hypotheses, the contact problem has a unique weak solution $(\boldsymbol{u}, \boldsymbol{\sigma})$ with the regularity

$$\boldsymbol{u} \in \mathcal{V}, \quad \boldsymbol{u}' \in \mathcal{W}, \quad \boldsymbol{\sigma} \in L^2(0, T; \mathcal{H}), \quad \mathrm{Div}\,\boldsymbol{\sigma} \in \mathcal{V}^*.$$

Idea of the proof: a reformulation in terms of velocity

Let $Y = Z = L^2(\Gamma_3)$. We introduce operators $A\colon (0,T) \times V \to V^*$, $\mathcal{R}_1\colon \mathcal{V} \to \mathcal{V}^*$, $\mathcal{R}\colon \mathcal{V} \to L^2(0,T;Y)$ and $\mathcal{S}\colon \mathcal{V} \to L^2(0,T;Z)$ as follows:

$$\langle A\boldsymbol{w}, \boldsymbol{v}\rangle_{V^*\times V} = (\mathcal{F}\varepsilon(\boldsymbol{w}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}} \quad \text{for all } \boldsymbol{w}, \boldsymbol{v} \in V,\ t \in (0,T),$$

$$\langle (\mathcal{R}_1\boldsymbol{w})(t), \boldsymbol{v}\rangle_{V^*\times V} = \Big(\mathcal{B}\Big(\int_0^t \varepsilon(\boldsymbol{w}(s))\,ds + \boldsymbol{u}_0\Big), \varepsilon(\boldsymbol{v})\Big)_{\mathcal{H}}$$

$$+ \Big(\int_0^t \mathcal{C}(t-s)\varepsilon(\boldsymbol{w}(s))\,ds, \varepsilon(\boldsymbol{v})\Big)_{\mathcal{H}} \quad \text{for all } \boldsymbol{w} \in \mathcal{V},\ \boldsymbol{v} \in V,\ t \in (0,T),$$

$$(\mathcal{R}\boldsymbol{w})(t) = \int_0^t \Big\| \int_0^s \boldsymbol{w}_\tau(r)\,dr + \boldsymbol{u}_{0\tau} \Big\|\,ds \quad \text{for all } \boldsymbol{w} \in \mathcal{V},\ t \in (0,T),$$

$$(\mathcal{S}\boldsymbol{w})(t) = \int_0^t w_\nu(s)\,ds + u_{0\nu} \quad \text{for all } \boldsymbol{w} \in \mathcal{V},\ t \in (0,T).$$

Idea of the proof: a reformulation in terms of velocity

Define $J\colon (0,T)\times Z\times V\to\mathbb{R}$ and $\varphi\colon (0,T)\times Y\times V\to\mathbb{R}$ by

$$J(t,z,\boldsymbol{v}) = \int_{\Gamma_3} k(z)\, j_\nu(v_\nu)\, d\Gamma \quad \text{for all } z\in Z,\ \boldsymbol{v}\in V,\ t\in(0,T),$$

$$\varphi(t,y,\boldsymbol{v}) = \int_{\Gamma_3} F_b(y)\, \|\boldsymbol{v}_\tau\|\, d\Gamma \quad \text{for all } y\in Y,\ \boldsymbol{v}\in V,\ t\in(0,T).$$

We introduce the velocity field $\boldsymbol{w}=\boldsymbol{u}'$.

Idea of the proof: a reformulation in terms of velocity

With the notation above we consider the following problem in terms of the velocity.

Problem (V-HVI in terms of velocity)

Find $\boldsymbol{w} \in \mathcal{W}$ *such that*

$$\begin{cases} \langle \boldsymbol{w}'(t) + A(t, \boldsymbol{w}(t)) + (\mathcal{R}_1 \boldsymbol{w})(t) - \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{w}(t) \rangle_{V^* \times V} \\ \qquad + J^0(t, (\mathcal{S}\boldsymbol{w})(t), \boldsymbol{w}(t); \boldsymbol{v} - \boldsymbol{w}(t)) \\ \qquad + \varphi(t, (\mathcal{R}\boldsymbol{w})(t), \boldsymbol{v}) - \varphi(t, (\mathcal{R}\boldsymbol{w})(t), \boldsymbol{w}(t)) \geq 0 \\ \qquad\qquad \text{for all } \boldsymbol{v} \in V, \ \text{a.e. } t \in (0, T), \\ \boldsymbol{w}(0) = \boldsymbol{w}_0. \end{cases}$$

Idea of the proof

- We prove that the V-HVI in terms of velocity has a unique solution $\boldsymbol{w} \in \mathcal{W}$.
- Then, we define a function $\boldsymbol{u} \colon (0, T) \to V$ by

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{w}(s)\, ds + \boldsymbol{u}_0 \quad \text{for all } t \in (0, T).$$

  Hence, the variational-hemivariational inequality has a unique solution with regularity $\boldsymbol{u} \in \mathcal{V}$ and $\boldsymbol{u}' \in \mathcal{W}$.
- In what follows, it is enough to prove an existence and uniqueness result for the problem in terms of velocity.

Part 2

Abstract first order variational-hemivariational inequality with history-dependent operators

Abstract h.-d. variational-hemivariational inequality

Let Y and Z be two Banach spaces.

Problem (Abstract h.-d. inequality)

Find $w \in \mathcal{W}$ such that

$$\begin{cases} \langle w'(t) + A(t, w(t)) + (\mathcal{R}_1 w)(t) - f(t), v - w(t)\rangle_{V^* \times V} \\ \qquad + J^0(t, (\mathcal{S}w)(t), w(t); v - w(t)) \\ \qquad + \varphi(t, (\mathcal{R}w)(t), v) - \varphi(t, (\mathcal{R}w)(t), w(t)) \geq 0 \\ \qquad\qquad \text{for all } \ v \in V, \ \text{a.e. } \ t \in (0, T) \\ w(0) = w_0. \end{cases}$$

Here, $J\colon (0, T) \times Z \times V \to \mathbb{R}$, $\varphi\colon (0, T) \times Y \times V \to \mathbb{R}$, and $\mathcal{R}_1 \colon \mathcal{V} \to \mathcal{V}^*$, $\mathcal{R}\colon \mathcal{V} \to L^2(0, T; Y)$ and $\mathcal{S}\colon \mathcal{V} \to L^2(0, T; Z)$ are h.-d. operators.

Hypotheses (1)

$A\colon (0, T) \times V \to V^*$ satisfies

(1) $A(\cdot, v)$ is measurable on $(0, T)$ for all $v \in V$.

(2) $A(t, \cdot)$ is demicontinuous on V for a.e. $t \in (0, T)$.

(3) $\|A(t, v)\|_{V^*} \le a_0(t) + a_1 \|v\|_V$ for all $v \in V$, a.e. $t \in (0, T)$ with $a_0 \in L^2(0, T)$, $a_0 \ge 0$ and $a_1 \ge 0$.

(4) $A(t, \cdot)$ is strongly monotone for a.e. $t \in (0, T)$, i.e., for a constant $m_A > 0$,

$$\langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle_{V^* \times V} \ge m_A \|v_1 - v_2\|_V^2$$

for all v_1, $v_2 \in V$, a.e. $t \in (0, T)$.

Hypotheses (2)

$J\colon (0,T) \times Z \times V \to \mathbb{R}$ satisfies

(1) $J(\cdot, z, v)$ is measurable on $(0,T)$ for all $z \in Z$, $v \in V$.

(2) $J(t, \cdot, v)$ is continuous on Z for all $v \in V$, a.e. $t \in (0,T)$.

(3) $J(t, z, \cdot)$ is locally Lipschitz on V for all $z \in Z$, a.e. $t \in (0,T)$.

(4) $\|\partial J(t,z,v)\|_{V^*} \le c_{0J}(t) + c_{1J}\|z\|_Z + c_{2J}\|v\|_V$ for all $z \in Z$, $v \in V$, a.e. $t \in (0,T)$ with $c_{0J} \in L^2(0,T)$, c_{0J}, c_{1J}, $c_{2J} \ge 0$.

(5) $J^0(t, z_1, v_1; v_2 - v_1) + J^0(t, z_2, v_2; v_1 - v_2) \le$ $\overline{m}_{2J}\,\|z_1 - z_2\|_Z\|v_1 - v_2\|_V + m_{2J}\,\|v_1 - v_2\|_V^2$ for all $z_i \in Z$, $v_i \in V$, $i = 1, 2$, a.e. $t \in (0,T)$ with $\overline{m}_{2J} \ge 0$, $m_{2J} \ge 0$.

Hypotheses (3)

$\varphi \colon (0, T) \times Y \times V \to \mathbb{R}$ satisfies

(1) $\varphi(\cdot, y, v)$ is measurable on $(0, T)$ for all $y \in Y$, $v \in V$.

(2) $\varphi(t, \cdot, v)$ is continuous on Y for all $v \in V$, a.e. $t \in (0, T)$.

(3) $\varphi(t, y, \cdot)$ is convex and l.s.c. on V for all $y \in Y$, a.e. $t \in (0, T)$.

(4) $\|\partial\varphi(t, y, v)\|_{V^*} \leq c_{0\varphi}(t) + c_{1\varphi}\|y\|_Y + c_{2\varphi}\|v\|_V$ for all $y \in Y$, $v \in V$, a.e. $t \in (0, T)$ with $c_{0\varphi} \in L^2(0, T)$, $c_{0\varphi}$, $c_{1\varphi}$, $c_{2\varphi} \geq 0$.

(5) $\varphi(t, y_1, v_2) - \varphi(t, y_1, v_1) + \varphi(t, y_2, v_1) - \varphi(t, y_2, v_2) \leq \beta_\varphi \|y_1 - y_2\|_Y \|v_1 - v_2\|_V$ for all $y_i \in Y$, $v_i \in V$, $i = 1, 2$, a.e.

$t \in (0, T)$ with $\beta_\varphi \geq 0$.

Hypotheses (4)

$\mathcal{R}_1 \colon \mathcal{V} \to \mathcal{V}^*$, $\mathcal{R} \colon \mathcal{V} \to L^2(0, T; Y)$ and $\mathcal{S} \colon \mathcal{V} \to L^2(0, T; Z)$ satisfy

(1) $\|(\mathcal{R}_1 v_1)(t) - (\mathcal{R}_1 v_2)(t)\|_{V^*} \le c_{R_1} \displaystyle\int_0^t \|v_1(s) - v_2(s)\|_V \, ds$

for all $v_1, v_2 \in \mathcal{V}$, a.e. $t \in (0, T)$ with $c_{R_1} > 0$.

(2) $\|(\mathcal{R} v_1)(t) - (\mathcal{R} v_2)(t)\|_{Y} \le c_{R} \displaystyle\int_0^t \|v_1(s) - v_2(s)\|_V \, ds$

for all $v_1, v_2 \in \mathcal{V}$, a.e. $t \in (0, T)$ with $c_{R} > 0$.

(3) $\|(\mathcal{S} v_1)(t) - (\mathcal{S} v_2)(t)\|_{Z} \le c_{S} \displaystyle\int_0^t \|v_1(s) - v_2(s)\|_V \, ds$

for all $v_1, v_2 \in \mathcal{V}$, a.e. $t \in (0, T)$ with $c_{S} > 0$.

Hypotheses (5)

Smallness condition: $m_A > \max\{m_{2J}, c_{2J}, c_{2\varphi}\}$.

Regularity of the data: $f \in \mathcal{V}^*$, $w_0 \in V$.

Theorem

Under hypotheses (1)–(5), the abstract history-dependent variational-hemivariational inequality has a unique solution.

Two main ingredients of the proof

Theorem (Existence and uniqueness for evolutionary inclusion)

Find $w \in \mathcal{W}$ such that

$$\begin{cases} w'(t) + A(t, w(t)) + \partial\psi(t, w(t)) \ni f(t) \ \text{ a.e. } t \in (0, T), \\ w(0) = w_0. \end{cases}$$

Lemma (A fixed point result)

Let E be a Banach space and $0 < T < \infty$. Let $\Lambda \colon L^2(0, T; E) \to L^2(0, T; E)$ be an operator such that

$$\|(\Lambda\eta_1)(t) - (\Lambda\eta_2)(t)\|_E^2 \le c \int_0^t \|\eta_1(s) - \eta_2(s)\|_E^2 \, ds$$

for all η_1, $\eta_2 \in L^2(0, T; E)$, a.e. $t \in (0, T)$ with a constant $c > 0$. Then Λ has a unique fixed point in $L^2(0, T; E)$.

Remarks

From the abstract result, similar existence and uniqueness results can be obtained for dynamic contact problems with

(1) other constitutive laws

(2) other boundary conditions, for example,

$$\left.\begin{aligned} &|\sigma_\nu(t)| \le F\Big(\int_0^t u_\nu^+(s)\,ds\Big), \\ &\sigma_\nu(t) = \begin{cases} 0 \quad \text{if} \quad u_\nu < 0 \\ F\Big(\int_0^t u_\nu^+(s)\,ds\Big) \quad \text{if} \quad u_\nu > 0, \end{cases} \end{aligned}\right\} \quad \text{on } \Gamma_C,$$

$$-\boldsymbol{\sigma}_\tau(t) \in \mu(u_\nu(t))\partial j_\tau(\boldsymbol{u}'_\tau(t)) \qquad \text{on } \Gamma_C.$$

Remarks

(3) other boundary conditions, for example,

$$-\sigma_\nu(t) = p_\nu(u_\nu(t)) \qquad \text{on } \Gamma_C,$$

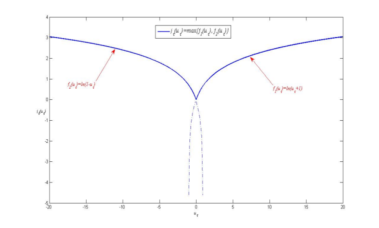
$$\|\boldsymbol{\sigma}_\tau(t)\| \le \mu|\sigma_\nu|, \quad -\boldsymbol{\sigma}_\tau = \mu|\sigma_\nu|\frac{\boldsymbol{u}'_\tau(t)}{\|\boldsymbol{u}'_\tau(t)\|} \ \text{ if } \ \boldsymbol{u}'_\tau(t) \ne \boldsymbol{0} \qquad \text{on } \Gamma_C.$$

Conclusions

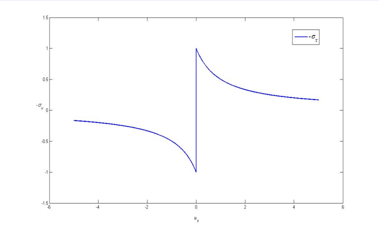
We have provided results on

- the existence and uniqueness of solution to the abstract variational-hemivariational inequality,
- the existence and uniqueness of weak solutions to the dynamic frictional contact problem in nonlinear viscoelasticity,
- applications of the abstract result to other contact problems.

Example: nonconvex friction law

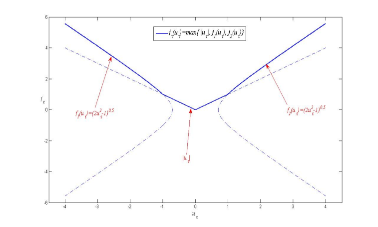


$$j(\boldsymbol{\xi}) = \max\{f_1(\boldsymbol{\xi}),\, f_2(\boldsymbol{\xi})\}$$

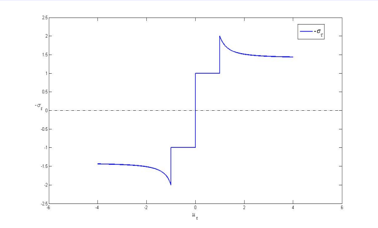


Clarke subgradient ∂j

Example: nonconvex friction law

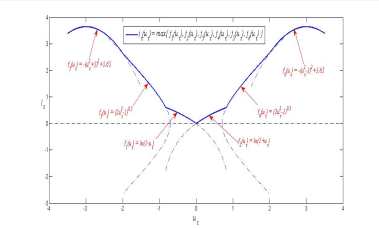


$$j(\xi) = \max\{a\|\xi\|,\, f_1(\xi),\, f_2(\xi)\}$$

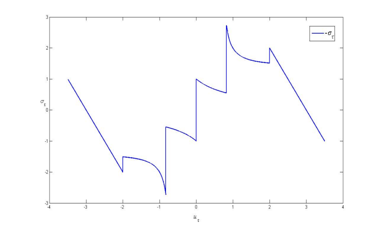


Clarke subgradient ∂j

Example: zig-zag friction law

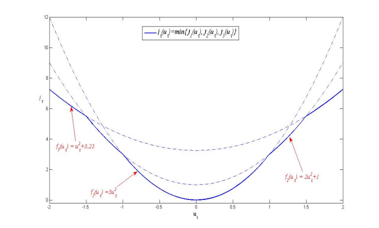


$$j(\xi) = \max\{f_1(\xi), f_2(\xi), f_3(\xi), f_1'(\xi), f_2'(\xi), f_3'(\xi)\}$$

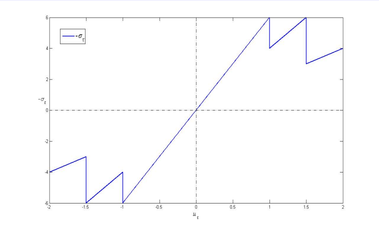


Clarke subgradient ∂j

Example: zig-zag friction law



$$j(\xi) = \min\{f_1(\xi),\ f_2(\xi),\ f_3(\xi)\}$$



Clarke subgradient ∂j

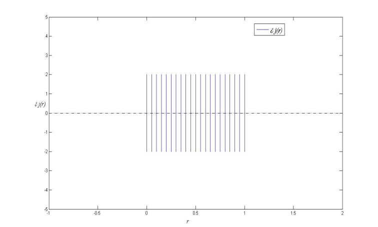
Example: infinite number of jumps

Let I be an open subset of the real line $\mathbb{R}$ and let M be a measurable subset of I such that for every open and nonempty subset I of I, $\mathrm{meas}(I \cap M) > 0$ and $\mathrm{meas}(I \cap (I \setminus M)) > 0$.
Let

$$g(s) = \begin{cases} b_1 & \text{if } s \in M \\ -b_2 & \text{if } s \notin M \end{cases}$$

and $j(r) = \displaystyle\int_0^r g(\theta)\, d\theta$. Then the nonconvex potential j is locally Lipschitz and

$$\partial j(r) = [-b_2, b_1] \quad \text{for every} \quad r \in I.$$



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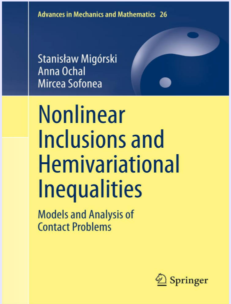
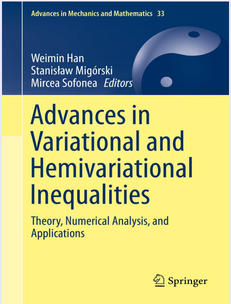
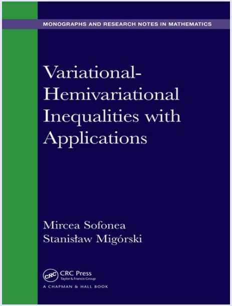
Monographs



2013



2015



2017

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