Value Function and Optimal Trajectories for some State Constrained Control Problems

Hasnaa Zidani

ENSTA ParisTech, Univ. of Paris-saclay

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- For a given non-empty compact subset U of $\mathbb{R}^k$, define the set of admissible controls as:

$$\mathcal{U} := \Big\{ \mathbf{u} : (0,T) \to \mathbb{R}^k, \text{ measurable}, \mathbf{u}(t) \in U \text{ a.e } \Big\}.$$

- Consider the following control system:

$$(1) \qquad \begin{cases} \dot{\mathbf{y}}(s) := f(\mathbf{y}(s), \mathbf{u}(s)), & \text{a.e } s \in [t,T], \\ \mathbf{y}(t) := x, \end{cases}$$

where $f : \mathbb{R}^d \times U \to \mathbb{R}^d$ is continuous, and Lipschitz continuous w.r.t x.

- The set of trajectories:

$$S_{[t,T]}(x) := \{\mathbf{y}^{\mathbf{u}}_{t,x} \in W^{1,1}(t,T;\mathbb{R}^d), \;\; \mathbf{y}^{\mathbf{u}}_{t,x} \text{ satisfies (1) for some } \mathbf{u} \in \mathcal{U}\},$$

The multi-application: $x \rightsquigarrow \mathcal{S}_{[t,T]}(x)$ is Lipschitz continuous.

State constrained optimal control problems

$$\begin{aligned} \inf \quad & \Phi(\mathbf{y}^{\mathbf{u}}_{t,x}(T)) \\ \text{s.t.} \quad & \mathbf{u} \in \mathcal{U}, \\ & \mathbf{y}^{\mathbf{u}}_{t,x}(s) \in \mathcal{K} \ \ \forall s \in [t,T]. \end{aligned}$$

- $\mathcal{K}$ is a closed sub-set of $\mathbb{R}^d$.
- The final cost $\Phi : \mathbb{R}^d \to \mathbb{R}$ is a Lipschitz continuous function

Outline

1. Characterization of the value function under some controllability assumptions
2. A general case where the controllability assumption is not satisfied
3. A numerical example
4. End-point constrained control problem

Outline

Set of constrained trajectories

- Assume that $f(x, U) := \{f(x, u),\ u \in U\}$ is a convex set.
  Then, by Filippov's theorem, the set of trajectories $\mathcal{S}_{[t,T]}(x)$ is a compact set of $W^{1,1}([t, T])$ endowed with the C^0-topology.

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- The set of feasible trajectories:

$$\mathcal{S}^{\mathcal{K}}_{[t,T]}(x) := \{\mathbf{y} \in \mathcal{S}_{[t,T]}(x) \mid \mathbf{y}(s) \in \mathcal{K}\ \ \forall s \in [t, T]\}$$

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- Inward pointing (IP) condition: Assume $\overline{\mathring{\mathcal{K}}} = \mathcal{K}$ and

$$\exists \beta > 0, \quad \forall x \in \partial\mathcal{K}, \quad \min_{u \in U} f(x,u) \cdot \eta_x < -\beta.$$

  Then, for $x \in \mathcal{K}$, $\mathcal{S}^{\mathcal{K}}_{[t,T]}(x) \neq \emptyset$, and $x \longmapsto \mathcal{S}^{\mathcal{K}}_{[t,T]}(x)$ is Lipschitz.

  Ref: Arutyunov'84, Soner'86, Rampazzo-Vinter'99, Vinter-Frankowska'00, Clarke-Rifford-Stern'02 ...

A State constrained control problem

$$\vartheta(t,x) := \min\left\{ \Phi(\mathbf{y}(T)) \;\middle|\; \mathbf{y} \in \mathcal{S}^{\mathcal{K}}_{[t,T]}(x) \right\},$$

- In general, ϑ is only l.s.c. on $\mathcal{K}$.

- Under *suitable controllability assumptions*, ϑ is the unique *constrained viscosity* solution of:

$$\partial_t \vartheta(t,x) + \mathcal{H}(x, D_x\vartheta(t,x)) = 0 \quad \text{on } (0,T)\times\mathcal{K},$$
$$\vartheta(0,x) = \Phi(x) \quad \text{on } \mathcal{K}.$$

- Under IP condition, the value function ϑ is Lipschitz continuous on $\mathcal{K}$. Soner'86, Motta'95, Ishii-Koike'96, Frankowska-et-al.'00-, ...
- If $\mathcal{K}$ is convex, $f(x,u) = Ax + Bu$ and $\exists(\bar{x},\bar{v}), A\bar{x} + B\bar{u} = 0$ Hermosilla-Vinter-HZ'17
- If $\mathcal{K}$ has a stratified structure + *a local controllability assumption*: Hermosilla-HZ'15, Hermosilla-Wolenski-HZ'17

How can we characterize (and compute) the value function for problems lacking controllability assumptions?

Outline

An other alternative ...

- Assume $\mathcal{K}$ is a closed nonempty set (no additional requirement)
- Consider a function $g : \mathbb{R}^d \to \mathbb{R}$, Lipschitz continuous, such that

$$\forall x \in \mathbb{R}^d, \quad g(x) \leq 0 \Leftrightarrow x \in \mathcal{K}.$$

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- In particular,

$$[\mathbf{y}(s) \in \mathcal{K}, \ \forall s \in [t, T]] \Longleftrightarrow \max_{s \in [t,T]} g(\mathbf{y}(s)) \leq 0.$$

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$$[\mathbf{y}(s) \in \mathcal{K},\ \forall s \in [t, T]] \Longleftrightarrow \max_{s \in [t,T]} g(\mathbf{y}(s)) \leq 0.$$
- Assume that, for every $x \in \mathbb{R}^d$, $f(x, U) := \{f(x, u),\ u \in U\}$ is a convex set.

An auxiliary control problem (Altarovici-Bokanowski-HZ'13)

- Consider the following auxiliary control problem ($z \in \mathbb{R}$):

$$w(t,x,z) \quad := \quad \inf_{\mathbf{y} \in \mathcal{S}^{\mathcal{K}}_{[t,T]}(x)} \left\{ \left(\Phi(\mathbf{y}^{\mathbf{u}}_{t,x}(T)) - z\right) \bigvee \max_{s \in [t,T]} g(\mathbf{y}^{\mathbf{u}}_{t,x}(s)) \right\}.$$

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Improvement function (Mifflin'77, Solodov-Sagastizábal'04, Apkarian-et-al.'08, ...)

$$(\mathcal{P}) \qquad \min_{G(X)\le 0} F(X)$$

- The auxiliary optimization problem:

$$(\widehat{\mathcal{P}}) \qquad \min_X \left\{ (F(X) - z) \bigvee G(X) \right\}$$

- Under *Slater condition*:

$$\bar{X} \text{ is optimal for } (P) \Leftrightarrow \begin{cases} \bar{X} \text{ solution of } (\widehat{\mathcal{P}}) \text{ for } \bar{z} = F(\bar{X}), \\ \min_X \left\{ (F(X) - \bar{z}) \bigvee G(X) \right\} = 0. \end{cases}$$

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Theorem

For every $x \in \mathcal{K}$, we have:

$$(i) \qquad \mathcal{E}pi\, \vartheta(t,\cdot) = \left\{ (x,z) : w(t,x,z) \leq 0 \right\},$$

$$(ii) \qquad \vartheta(t,x) = \min \left\{ z \in \mathbb{R}\,,\ w(t,x,z) \leq 0 \right\},$$

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(ii) $\quad \vartheta(t,x) = \min \left\{ z \in \mathbb{R}\,,\ w(t,x,z) \leq 0 \right\},$

(iii) $\quad$ *Under IP condition: for every* $x \in \overset{\circ}{\mathcal{K}}$ *we have:*

$$\vartheta(t,x) = z, \quad s.t.\ w(t,x,z) = 0.$$

- Define the Hamiltonian as:

$$\mathcal{H}(x,p) := \max_{u\in U}\big(-f(x,u)\cdot p\big) \qquad \forall x,p \in \mathbb{R}^d.$$

Theorem

The value function w is the unique Lipschitz continuous viscosity solution of the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\min\Big(-\partial_t w(t,x,z) + H(x,D_x w),\ w(t,x,z)-g(x)\Big) = 0 \quad [0,T[\times\mathbb{R}^d\times\mathbb{R},$$

$$w(T,x,z) = (\Phi(x)-z)\bigvee g(x), \qquad \mathbb{R}^d\times\mathbb{R}.$$

A particular choice of function g

- Let $\eta > 0$ and define the following extended set $\mathcal{K}_\eta$:

$$\mathcal{K}_\eta :\equiv \mathcal{K} + \mathbb{B}(0, \eta).$$

- $g(y) := d_{\mathcal{K}}(y)$ the signed distance to $\mathcal{K}$.
- Consider the following auxiliary control problem :

$$w(t,x,z) \quad := \quad \inf_{\mathbf{y} \in S_{[t,T]}(x)} \left[\left(\Phi(\mathbf{y}(T)) - z) \bigvee \max_{s \in [t,T]} g(\mathbf{y}(s)) \right) \bigwedge \eta \right],$$

where $a \wedge b = \min(a, b)$.

Theorem

Let $(t, x, z) \in [0, T] \times \mathcal{K} \times \mathbb{R}$. The following assertions hold:

$$(i) \qquad \vartheta(t,x) - z \leq 0 \Leftrightarrow w(t,x,z) \leq 0,$$

$$(ii) \qquad \vartheta(t,x) = \min \left\{ z \in \mathbb{R} \;,\; w(t,x,z) \leq 0 \right\}.$$

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Theorem

The function w *is the unique Lipschitz continuous viscosity solution of the following HJB equation:*

$$\min\Big(-\partial_t w(t,x,z) + H(x, \nabla_x w),\ w(t,x,z) - g(x) \Big) = 0 \quad [0,T[\times \mathcal{K}_\eta \times \mathbb{R},$$

$$w(T,x,z) = \Psi_\eta(x,z), \qquad \mathcal{K}_\eta \times \mathbb{R},$$

$$w(t,x,z) = \eta, \qquad y \notin \mathcal{K}_\eta,$$

where $\Psi_\eta(x,z) = \Big[(\Phi(x) - z) \bigvee g(y) \Big] \bigwedge \eta.$

- Define the exit time function:

$$\begin{aligned} \mathcal{T}(y,z) &:= \inf\left\{ t \in [0,T] \;\middle|\; \vartheta(t,y) \leq z \right\} \\ &= \inf\left\{ t \in [0,T] \;\middle|\; w(t,y,z) \leq 0 \right\} \end{aligned}$$

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- Link between ϑ and $\mathcal{T}$:

(i) $\quad \mathcal{T}$ is the exit time function for $\mathcal{E}\mathrm{pi}(\Phi) \bigcap (\mathcal{K} \times \mathbb{R}^d)$,

(ii) $\quad \mathcal{T}(y,z) = t \Rightarrow\; w(t,y,z) = 0$,

(iii) $\quad \vartheta(t,y) = \inf\left\{z \;\middle|\; \mathcal{T}(y,z) \leq t\right\}$.

Reconstruction of optimal trajectories

Proposition

Let $x \in \mathcal{K}$ *such that* $\vartheta(t,x) < \infty$. *Define* $z^* := \vartheta(t,x)$.

- *Let* $(\mathbf{y}^*, \mathbf{z}^*)$ *be the optimal trajectory for the auxiliary control problem associated with the initial point* $(x, z^*) \in \mathcal{K} \times \mathbb{R}$. *Then, the trajectory* $\mathbf{y}^*$ *is optimal for the original control problem. The converse is also true.*

Reconstruction of optimal trajectories - **Algorithm A.**

- For $n \geq 1$, consider $(t_0 = 0, t_1, ..., t_{n-1}, t_n = T)$ a uniform partition of $[0, T]$ with $\Delta t = \frac{T}{n}$.
- Let $\{\mathbf{y}^n(\cdot), \mathbf{z}^n(\cdot)\}$ be a trajectory defined recursively on the intervals $(t_{i-1}, t_i]$, with $\mathbf{z}^n(\cdot) := z = \vartheta(0, y)$ and $\mathbf{y}^n(0) = y$.

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- [Step 1] Knowing $y_k^n = \mathbf{y}^n(t_k)$, choose the optimal control at t_k s.t.:

$$u_k^n \in \arg\min_{u \in U} \left(w(t_k, y_k^n + \Delta t f_{\Delta t}(y_k^n, u), z) \right).$$

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- [Step 2] Define $\mathbf{u}^n(t) := u_k^n, \forall t \in (t_k, t_{k+1}]$ and $\mathbf{y}^n(t)$ on $(t_k, t_{k+1}]$ as the solution of

$$\dot{\mathbf{y}}(t) := f(\mathbf{y}(t), \mathbf{u}^n(t)) \text{ a.e } \ t \in (t_k, t_{k+1}],$$

with initial condition $\mathbf{y}^n(t_k)$ at t_k and $\mathbf{z}^n(\cdot) := z$.

Theorem

Let $\{\mathbf{y}^n(\cdot), \mathbf{z}^n(\cdot), \mathbf{u}^n(\cdot)\}$ *be a sequence generated by algorithm A for* $n \geq 1$*. Then, the sequence of trajectories* $\{\mathbf{y}^n(\cdot)\}_n$ *has cluster points with respect to the uniform convergence topology. For any cluster point* $\bar{\mathbf{y}}(\cdot)$ *there exists a control law* $\bar{\mathbf{u}}(\cdot)$ *such that* $(\bar{\mathbf{y}}(\cdot), \bar{\mathbf{z}}(\cdot), \bar{\mathbf{u}}(\cdot))$ *is optimal for the auxiliary control problem.*

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- Let w^Δ be a numerical approximate solution such that,

$$|w^\Delta(t, y, z) - w(t, y, z)| \leq E_1(\Delta t, \Delta y),$$

where $E_1(\Delta t, \Delta y) \to 0$ as $\Delta t, \Delta y \to 0$.

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- Let $\{\mathbf{Y}^n(.), \mathbf{u}^n(.)\}$ be the sequence generated by the algorithm A with w^Δ.

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- Let $\{\mathbf{Y}^n(.), \mathbf{u}^n(.)\}$ be the sequence generated by the algorithm A with w^Δ.

- Then, $(Y^n)_n$ converges to an optimal trajectory for the auxiliary control problem.

Outline

Abort landing problem in presence of windshear

(Miele, Wang and Melvin(1987,1988); Bulirsch, Montrone and Pesch (1991..); Botkin-Turova(2012 ...))

Consider the flight motion of an aircraft in a vertical plane:

$$\begin{cases} \dot{x} = V\cos\gamma + w_x \\ \dot{h} = V\sin\gamma + w_h \\ \dot{V} = \frac{F_T}{m}\cos(\alpha+\delta) - \frac{F_D}{m} - g\sin\gamma - (\dot{w}_x\cos\gamma + \dot{w}_h\sin\gamma) \\ \dot{\gamma} = \frac{1}{V}(\frac{F_T}{m}\sin(\alpha+\delta) + \frac{F_L}{m} - g\cos\gamma + (\dot{w}_x\sin\gamma - \dot{w}_h\cos\gamma)) \end{cases}$$

where

$$\begin{aligned} \dot{w}_x &= \frac{\partial w_x}{\partial x}(V\cos\gamma + w_x) + \frac{\partial w_x}{\partial h}(V\sin\gamma + w_h) \\ \dot{w}_h &= \frac{\partial w_h}{\partial x}(V\cos\gamma + w_x) + \frac{\partial w_h}{\partial h}(V\sin\gamma + w_h) \end{aligned}$$

and

- $F_T := F_T(V)$ is the thrust force
- $F_D := F_D(V,\alpha)$ and $F_L := F_L(V,\alpha)$ are the drag and lift forces
- $w_x := w_x(x)$ and $w_h := w_h(x,h)$ are the wind components
- m, g, and δ are constants.

Controlled system

- Consider the state $\mathbf{y}(.) = (x(.), h(.), V(.), \gamma(.), \alpha(.))$.
- The control variable $\mathbf{u}$ is the angular speed of the angle of attack α.
- Let T be a fixed time horizon and let $\mathcal{U}$ be the set of admissible controls

$$\mathcal{U} := \left\{ \mathbf{u} : (0, T) \to \mathbb{R}, \text{ measurable, } \mathbf{u}(t) \in U \text{ a.e } \right\}$$

  where U is a compact set.
- The controlled dynamics in this case is:

$$\begin{cases} \dot{x} = V\cos\gamma + w_x, \\ \dot{h} = V\sin\gamma + w_h, \\ \dot{V} = \frac{F_T}{m}\cos(\alpha+\delta) - \frac{F_D}{m} - g\sin\gamma - (\dot{w}_x\cos\gamma + \dot{w}_h\sin\gamma), \\ \dot{\gamma} = \frac{1}{V}(\frac{F_T}{m}\sin(\alpha+\delta) + \frac{F_L}{m} - g\cos\gamma + (\dot{w}_x\sin\gamma - \dot{w}_h\cos\gamma)), \\ \dot{\alpha} = \mathbf{u}. \end{cases}$$

Formulation of the optimal control problem

- Aim: Maximize the minimal altitude over a time interval:

$$\min_{\theta \in [0,t]} h(\theta)$$

while the aircraft stays in a given domain $\mathcal{K}$.

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- Consider the following optimal control problem:

$$(\mathbf{P}):\quad \vartheta(t,y) = \inf\left\{ \max_{\theta\in[0,t]} \Phi(\mathbf{y}_y^{\mathbf{u}}(\theta)),\, \middle|\, \mathbf{u}\in\mathcal{U},\ \text{and } \mathbf{y}_y^{\mathbf{u}}(s)\in\mathcal{K},\ \forall s\in[0,t]\right\}$$

where $\Phi(\mathbf{y}_y^u(.)) = H_r - h(.)$, H_r being a reference altitude, and $\mathcal{K}$ is a set of state constraints.

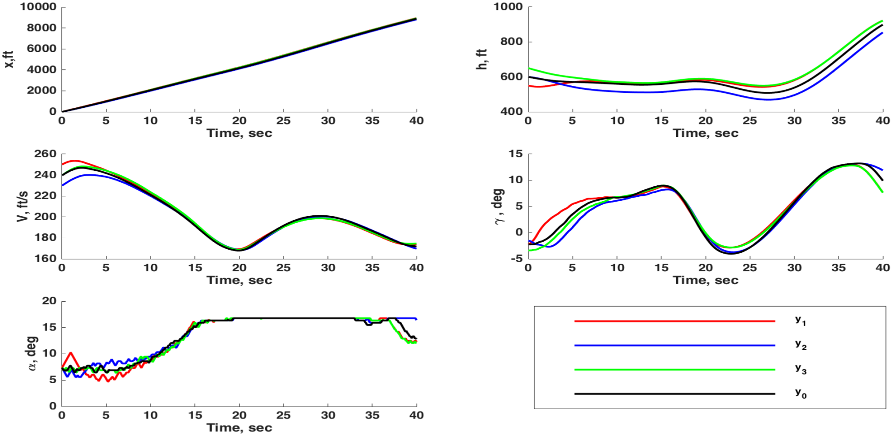

Figure: Optimal trajectories for different initial conditions

Outline

➤ Consider the end-point constrained control problem

$$\begin{aligned} \inf \quad & \Phi(\mathbf{y}^{\mathbf{u}}_{t,x}(T)) \\ \text{s.t.} \quad & \mathbf{u} \in \mathcal{U}, \\ & g(\mathbf{y}^{\mathbf{u}}_{t,x}(T)) \leq 0. \end{aligned}$$

➤ An associated auxiliary control problem ($z \in \mathbb{R}$) can be defined as:

$$w(t,x,z) \quad := \quad \inf_{\mathbf{u}\in\mathcal{U}} \left\{ (\Phi(\mathbf{y}^{\mathbf{u}}_{t,x}(T)) - z) \bigvee g(\mathbf{y}^{\mathbf{u}}_{t,x}(T)) \right\}.$$

➤ Assume that $\Phi : \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ are of C^1

Relationship between the PMP and HJB

- Let $x_o \in \mathbb{R}^d$ and let $z^* := \min\{z : w(0, x_o, z) \leq 0\}$.

- If $z^* < \infty$, then there exists $\mathbf{u}^* \in \mathcal{U}$ and its associated trajectory $\mathbf{y}^* \in \mathcal{S}_{[0,T]}(x_o)$ such that:

$$g(\mathbf{y}^*(T)) \leq 0, \quad \vartheta(0, x_o) = z^* = \Phi(\mathbf{y}^*(T)).$$

Relationship between the PMP and HJB

- Denote $H(x, u, p) = \langle p, f(x, u) \rangle$. There exists $(\mathbf{p}_x^*, \mathbf{p}_z^*)$ satisfying:

$$\begin{aligned} -\dot{\mathbf{p}}_x^*(s) &= \partial_x H(\mathbf{y}^*(s), \mathbf{u}^*(s), \mathbf{p}_x^*(s)) \\ -\dot{\mathbf{p}}_z^*(s) &= 0 \\ \begin{pmatrix} \mathbf{p}_x^*(T) \\ \mathbf{p}_z^*(T) \end{pmatrix} &\in \partial_{x,z} \left\{ (\Phi(\mathbf{y}^*(T)) - z^*) \bigvee g(\mathbf{y}^*(T)) \right\} \end{aligned}$$

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$$\begin{aligned} -\dot{\mathbf{p}}_x^*(s) &= \partial_x H(\mathbf{y}^*(s), \mathbf{u}^*(s), \mathbf{p}_x^*(s)) \\ -\dot{\mathbf{p}}_z^*(s) &= 0 \\ \begin{pmatrix} \mathbf{p}_x^*(T) \\ \mathbf{p}_z^*(T) \end{pmatrix} &= \begin{pmatrix} \lambda_o \nabla \Phi(\mathbf{y}^*(T)) + \lambda \nabla g(\mathbf{y}^*(T)) \\ -\lambda_o \end{pmatrix} \end{aligned}$$

where $\lambda_o, \lambda \in [0,1]$ and $\lambda_0 + \lambda = 1$.

Relationship between the PMP and HJB

- Denote $H(x,u,p) = \langle p, f(x,u)\rangle$. There exists $(\mathbf{p}_x^*, \mathbf{p}_z^*)$ satisfying:

$$\begin{aligned} -\dot{\mathbf{p}}_x^*(s) &= \partial_x H(\mathbf{y}^*(s), \mathbf{u}^*(s), \mathbf{p}_x^*(s)) \\ -\dot{\mathbf{p}}_z^*(s) &= 0 \\ \begin{pmatrix} \mathbf{p}_x^*(T) \\ \mathbf{p}_z^*(T) \end{pmatrix} &= \begin{pmatrix} \lambda_o \nabla\Phi(\mathbf{y}^*(T)) + \lambda\nabla g(\mathbf{y}^*(T)) \\ -\lambda_o \end{pmatrix} \end{aligned}$$

  where $\lambda_o, \lambda \in [0,1]$ and $\lambda_0 + \lambda = 1$.

- For a.e $s \in [0,T]$,

$$\mathcal{H}(\mathbf{y}^*(s), \mathbf{p}_x^*(s)) = H(\mathbf{y}^*(s), \mathbf{u}^*(s), \mathbf{p}_x^*(s)).$$

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- Denote $H(x, u, p) = \langle p, f(x, u)\rangle$. There exists $(\mathbf{p}_x^*, \mathbf{p}_z^*)$ satisfying:

$$
\begin{array}{rcl}
-\dot{\mathbf{p}}_x^*(s) & = & \partial_x H(\mathbf{y}^*(s), \mathbf{u}^*(s), \mathbf{p}_x^*(s)) \\
-\dot{\mathbf{p}}_z^*(s) & = & 0 \\
\begin{pmatrix} \mathbf{p}_x^*(T) \\ \mathbf{p}_z^*(T) \end{pmatrix} & = & \begin{pmatrix} \lambda_o \nabla \Phi(\mathbf{y}^*(T)) + \lambda \nabla g(\mathbf{y}^*(T)) \\ -\lambda_o \end{pmatrix}
\end{array}
$$

  where $\lambda_o, \lambda \in [0, 1]$ and $\lambda_0 + \lambda = 1$.

- For a.e $s \in [0, T]$,

$$\mathcal{H}(\mathbf{y}^*(s), \mathbf{p}_x^*(s)) = H(\mathbf{y}^*(s), \mathbf{u}^*(s), \mathbf{p}_x^*(s)).$$

- Moreover,

$$(\mathbf{p}_x^*(s), \mathbf{p}_z^*(s)) \in \partial_{x,z} w(s, \mathbf{y}^*(s), z^*).$$

- The same framework can be extended to:
  - final state constraints and time-dependent state constraints
  - impulsive control problems (Forcadel-Rao-HZ'13)
  - supremum running cost problems (Assellaou-Bokanowski-Desilles-HZ'17),
  - stochastic control setting (Bokanowski-Picarelli-HZ'16)

- The relationship between the PMP and the auxiliary value function can be (easily) derived for control problems with a finite number of state constraints.

....thanks for your attention.