Sweeping Process and Optimal Control

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Outline of the Talk

- Sweeping Process: Examples
- Minimum Time Function for the Controlled Sweeping Process
  - Dynamic Programming
  - Invariance Principle
  - Hamilton-Jacobi equation
  - A Toy Example
- Mayer Problem for the Controlled Sweeping Process
  - Necessary Conditions (work in progress)
- Conclusions and Open Questions

Sweeping Process

The problem

$$\dot{x}(t) \in -N_{C(t)}(x(t)), \qquad x(0) \in C(0)$$

is known as Sweeping Process. Here $N_{C(t)}(x)$ is a **Normal Cone** such that

$$N_{C(t)}(x) = \begin{cases} \{0\} & x \in \operatorname{int} C(t) \\ \emptyset & x \notin C(t) \end{cases}.$$

The **(unique) solution** $x(.)$ ceases to exist when $x(t) \notin C(t)$!!

Same remark holds true when the Perturbed Sweeping Process is considered

$$\dot{x}(t) \in -N_{C(t)}(x(t)) + g(x(t)), \qquad x(0) \in C(0)$$

We consider a control problem

$$(*) \qquad \dot{x}(t) \in -N_{C(t)}(x(t)) + G(x(t)), \qquad x(0) \in C(0),$$

where,

$$G(x) := \{g(x,u) : \quad u \in U\}.$$

Remarks:

- $(*)$ as control problem is **well-posed**!
- $C(t)$ can be regarded as a **state constraint** for problem $(*)$;
- The dynamics is **not** Lipschitz continuous w.r.t. x and is not autonomous!

Application 1: Electric Networks with Diodes.

An **ideal diode** is an electronic component which has *infinite* resistance in one direction and *zero* resistance in another direction.
Electric networks can be modeled by a 'Linear Complementarity System':

$$(LCS) \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + \lambda(t), & u(t) \in U \\ w(t) = Cx(t) \geq 0 \\ w(t) \perp \lambda(t) \qquad t \in [0, T] \end{cases}$$

Here, $\lambda(t)$ is the diode effect, which can be considered as a selection of

$$\lambda(t) \in -N_K(x(t)), \qquad t \in [0, T]$$

where $K = \{Cx : \ Cx \geq 0, \ x \in \mathbb{R}^n\}$.

Application 2: Hysteresis

The **Play Operator** with *absolutely continuous inputs* can be modeled as follows: given the input $u(.)$ and $z_0 \in Z$ we look for the output $z(t)$ such that

$$(H) \quad \begin{cases} z(t) = w(t) + v(t), & z(t) \in Z \\ < \dot{w}(t), \xi - z(t) > \geq 0 & \forall \xi \in Z \\ \dot{v}(t) = f(z(t), u(t)) & u(t) \in U \end{cases}$$

This formulation is equivalent to

$$\dot{z}(t) \in f(z(t), u(t)) - N_Z(z(t)), \qquad z(0) = z_0 \in Z.$$

Other Applications

- Parameter Estimations **(B. Acary, O. Bonnefon, B. Brogliato, 2011)**;
- Crowd Motion **(B. Maury, A. Roudne-Chupin, F. Santambrogio, J. Venel, 2011)**;
- Soft-robotic applications to Crawling Motion **(A. De Simone, P. Gidoni, in progress)**
- Control Problems with *active* constraints.

Minimum Time Problem with Controlled Sweeping Process

$$(SP) \quad \begin{cases} \text{Minimize } T \\ \text{over } x \in W^{1,1}([t_0, T]; \mathbb{R}^n), \; T > 0 \\ \text{satisfying} \\ \dot{x}(t) \in G(x(t)) - N_{C(t)}(x(t)) =: F(t, x(t)) \text{ a.e.} \\ x(t) \in C(t) \qquad \forall t \in [t_0, T], \\ x(t_0) = x_0 \in C(t_0), \quad x(T) \in S \end{cases}$$

Data: $C : \mathbb{R} \rightsquigarrow \mathbb{R}^n$, $G : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ **multifunctions**.

$S \subset \mathbb{R}^n$ is the target (**closed set**).

Compatibility Condition: $\exists\ \bar{t} > 0$ such that $C(\bar{t}) \cap S \neq \emptyset$.

Minimum Time Function:

$$T(t, x) = \inf\{T > 0 | \exists\ F\text{-}traj.\ x(.) \text{ s.t. } x(t) = x,\ x(t + T) \in S\}$$

Hypothesis on C(.) (H_C)

- there exists $L_C > 0$ such that

$$C(t) \subset C(s) + L_C \mathbb{B}|t-s|$$

for all $s, t \in [t_0, T]$. (Lipschitz continuous).

- $C(.)$ takes values compact sets.

- $C(.)$ is uniformly prox-regular, that is:
$\exists\ r > 0$ such that

$$\xi \cdot (y-x) \leq \frac{1}{2r} ||\xi||\ ||y-x||^2$$

for all $x, y \in C(t)$, for all $\xi \in N_{C(t)}(x)$, for every $t \in [t_0, T]$.

Hypothesis on G(.) (H_G)

Standing Hypothesis (SH)

- $\mathrm{Gr}\, G := \{(x, v) | \, v \in G(x)\}$ is **closed**.
- for each $x \in \mathbb{R}^n$, $G(x)$ is **nonempty**, **convex**, **compact**.

Lipschitz Continuity (LC)

- there exists $L_G > 0$ such that

$$G(x) \subset G(y) + L_G \mathbb{B} |x - y|$$

for all $x, y \in \mathbb{R}^n$.

Invariance Principles

K is a closed set, $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ a multifunction.

Definition: (F, K) is weakly invariant if, for every $x_0 \in K$, there exist $T > 0$ and $x : [0, T] \to \mathbb{R}^n$ such that

$$x(0) = x_0, \qquad x(t) \in K \quad \forall\, t \in [0, T].$$

Definition: (F, K) is strongly invariant if, for every $x_0 \in K$, $T > 0$ and $x : [0, T] \to \mathbb{R}^n$ such that $x(0) = x_0$, we have

$$x(t) \in K \quad \forall\, t \in [0, T].$$

Dynamic Programming for the Controlled SP

Assume $T(.,.)$ continuous. Then both

$$\text{epi}\ T = \{(t,x,\alpha)|\,(t,x)\in \text{Gr}\, C,\ \ T(t,x)\le \alpha\}$$

and

$$\text{hypo}\ T = \{(t,x,\alpha)|\,(t,x)\in \text{Gr}\, C,\ \ T(t,x)\ge \alpha\}$$

are closed.
The dynamic programming for (SP) principle is:

Proposition 1: $(\{1\}\times\{G-N_C\}\times\{-1\}, \text{epi}\ T)$ is weakly invariant (easy Hamiltonian characterization!).

Proposition 2: $(\{1\}\times\{G-N_C\}\times\{1\}, \text{hypo}\ T)$ is strongly invariant (not trivial Hamiltonian characterization!).

Strong Invariance Characterization for Sweeping Process

Theorem: Assume (H_G), (H_C) and take $\mathcal{K} \subset \mathrm{Gr}\, C$ **closed**.

$(\{1\} \times \{G - N_C\}, \mathcal{K})$ is strongly invariant

$$\Longleftrightarrow$$

for every $(\tau, x) \in \mathcal{K}$

$$\min_{v \in \{0\} \times \{-N_{C(\tau)}(x) \cap (L_C + M_G)\mathbb{B}\}} v \cdot p + \max_{v \in \{1\} \times \{G(x)\}} v \cdot p \leq 0$$

for every $p \in N^P_{\mathcal{K}}(\tau, x)$.

Remark: Monotonicity of the normal cone plays a crucial role!

HJ inequalities for SP (Colombo-P., '16)

Theorem: Assume (H_G) and (H_C) and $T(.,.)$ continuous. Then $T(.,.)$ is the unique (bilateral) viscosity solution of

$$\frac{\partial T}{\partial t}(t,x) + \min_{v \in G(x)} v \cdot \frac{\partial T}{\partial x}(t,x) = 0$$

such that:

$$T(t,x) > 0 \qquad \forall\,(t,x) \in \mathrm{Gr}\, C \qquad \text{for which} \quad x \notin S,$$

$$T(t,x) = 0 \qquad \forall\,(t,x) \in \mathrm{Gr}\, C \qquad \text{for which} \quad x \in S,$$

and satisfying other non-standard boundary conditions.

Remark: A-priori Petrov-like conditions involving S and $G(.)$ can be given for $T(.,.)$ being **continuous**.

Non-standard Boundary Conditions

Define Lower and Upper Hamiltonians:

$$H_{-}(\tau, x, \lambda, p) := \min_{\{0\}\times\{-N_{C(\tau)}(x)\cap(L_C+M_G)\mathbb{B}\}\times\{0\}} v\cdot p + \min_{v\in\{1\}\times\{G(x)\}\times\{-1\}} v\cdot p,$$

$$H_{+}(\tau, x, \lambda, p) := \min_{\{0\}\times\{-N_{C(\tau)}(x)\cap(L_C+M_G)\mathbb{B}\}\times\{0\}} v\cdot p + \max_{v\in\{1\}\times\{G(x)\}\times\{-1\}} v\cdot p,$$

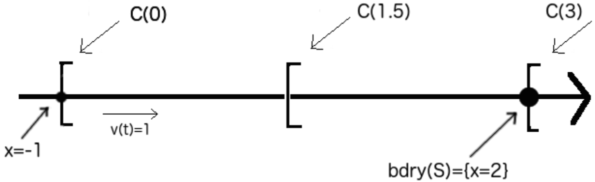
Then, for every $(\tau, x) \in \mathrm{Gr}\,\partial C$:

$$H_{-}(\tau, x, T(\tau, x), p) \leq 0, \quad \forall\, x \notin S,\ \forall\, p \in N^{P}_{\mathrm{epi}\ T}(\tau, x, T(\tau, x)),$$

$$H_{+}(\tau, x, T(\tau, x), p) \leq 0 \qquad \forall\, p \in N^{P}_{\mathrm{hypo}\ T}(\tau, x, T(\tau, x)).$$

A Toy Example

$$G(x) = x + [-1, 1], \quad C(t) = \{x \in \mathbb{R} : \ -1 + t \leq x \leq 2\}, \quad S = \{x \geq 2\}.$$



A computation shows:

$$T(t, x) := \begin{cases} 1 + \log 3 - t & -1 + t \leq x \leq -1 + e^{t-1}, \quad 0 \leq t \leq 1, \\ \log 3 - \log(1 + x) & -1 + e^{t-1} < x \leq 2, \quad 1 \leq t \leq 3. \end{cases}$$

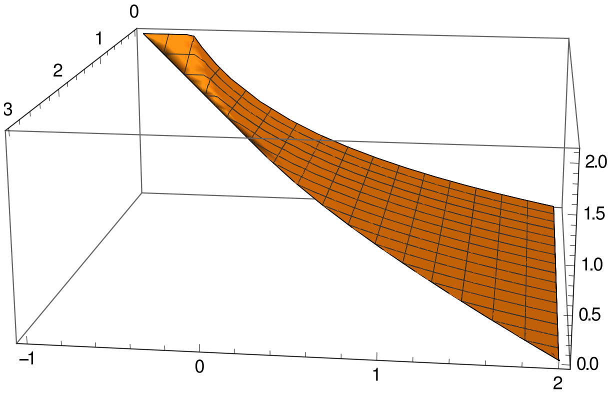


Figure: graph (T) .

Mayer Problem

Consider the Optimal Control Problem

$$(M)\quad \begin{cases} \text{Minimize } h(x(T)) \\ \text{over } x \in W^{1,1}([t_0,T];\mathbb{R}^n),\ \ T>0 \\ \text{satisfying} \\ \dot{x}(t) \in G(x(t)) - N_{C(t)}(x(t)) =: F(t,x(t)) \text{ a.e.} \\ x(t) \in C(t) \qquad \forall t \in [t_0,T], \\ x(t_0) = x_0 \in C(t_0). \end{cases}$$

Data: $C:\mathbb{R} \rightsquigarrow \mathbb{R}^n$, $G:\mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ **multifunctions**.

$h:\mathbb{R}^n \to \mathbb{R}$ is the objective function **(Lipschitz Continuous).**

Non-degenerate Necessary Conditions

We aim at improving the result in **(Arroud-Colombo, 2017)**, providing non-degenerate necessary conditions.

Main ingredients are the following:

i) a localized (around the minimizer $\bar{x}(.)$) version of the Moreau-Yosida approximation dynamics;

ii) use of a partial modification of the constraint $C(t)$:

- $C(t)$ is *inactive* when an **outward** pointing condition holds true.
- $C(t)$ is *active* otherwise.

ii) will permit to the adjoint multipliers to jump at the time in which $\bar{x}(t)$ hits $\partial C(t)$.

(Work in Progress with G. Colombo.)

Conclusions and Open Questions

- Minimum Time Function $T(.,.)$ is characterised as the unique **continuous** viscosity *(bilateral)* solution for (SP).
- Does such a characterization hold true for lower semicontinuous Minimum Time Functions? **Open question!**
- Also, the question whether a Hamilton-Jacobi characterization holds true for the Fully Controlled Sweeping Process

$$\dot{x}(t) \in -N_{C(v(t))}(x(t)) + G(x(t)), \qquad v \in \mathcal{V},$$

has **never been studied.**
- Furthermore, several other questions remain open for what concerns Necessary Conditions.

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