Optimal Control for a Controlled Sweeping Process with Applications to the Crowd Motion Model

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The International Conference on
Control of State-Constrained Dynamical Systems
September 25-29, 2017 at Padua, Italy

Outline

Optimal Control of Nonconvex Sweeping Process

Given a terminal cost function φ and a running cost ℓ, consider the optimal control problem (P): minimize

$$J[x,u,a] := \varphi\big(x(T)\big) + \int_0^T \ell\big(t, x(t), u(t), a(t), \dot{x}(t), \dot{u}(t), \dot{a}(t)\big)\ , dt \quad (1)$$

over $z(\cdot) := (x(\cdot), u(\cdot), a(\cdot)) \in W^{1,2}$ satisfying:

$$\begin{cases} -\dot{x}(t) \in N(x(t); C(t)) + f(x(t), a(t)) \text{ a.e. } t \in [0,T], \\ x(0) := x_0 \in C(0) \subset \mathbb{R}^n, \\ C(t) := C + u(t) \text{ with} \\ C := \{x \in \mathbb{R}^n |\ g_i(x) \geq 0\ \forall\ i = 1, \ldots, m\} \end{cases} \quad (2)$$

with the state constraints:

$$0 < r_1 \leq \|u(t)\| \leq r_2 \text{ and } g_i(x(t) - u(t)) \geq 0\ \forall t \in [0,T], i = 1, \ldots, m. \quad (3)$$

where g_i are convex $\mathcal{C}^2$-smooth functions, the trajectory $x(t)$ and control $u(t) = (u_1(t), \ldots, u_n(t)$, $a(t) = (a_1(t), \ldots, a_n(t))$ functions are absolutely continuous on the fixed interval $[0, T]$

The normal cone in the nonconvex sweeping process is understood as the proximal one defined via the projections

$$N_P(\bar{x};\Omega) := \left\{ v \in \mathbb{R}^n \middle| \, \exists\, \alpha > 0 \ \text{ s.t. } \ \bar{x} \in \Pi(\bar{x} + \alpha v; \Omega) \right\}, \ \bar{x} \in \Omega$$

with $N_P(\bar{x};\Omega) := \emptyset$ for $\bar{x} \notin \Omega$. However, all the major normal cones agree under the assumptions made ensuring the uniform prox-regularity (or "positive reach") of the sweeping sets $C(t)$.

Generalized Differentiation

See **[Mor06,RW98]**

Normal Cone to a closed set $\Omega \subset \mathbb{R}^n$ at $\bar{x} \in \Omega$

$$N(\bar{x};\Omega) := \left\{ v \middle| \, \exists x_k \to \bar{x}, w_k \in \Pi(x_k;\Omega), \alpha_k \geq 0, \alpha_k(x_k - w_k) \to v \right\}$$

Subdifferential of an l.s.c. function $\varphi\colon \mathbb{R}^n \to (-\infty,\infty]$ at $\bar{x}$

$$\partial\varphi(\bar{x}) := \left\{ v \middle| \, (v,-1) \in N((\bar{x},\varphi(\bar{x}));\mathrm{epi}\,\varphi) \right\}, \quad \bar{x} \in \mathrm{dom}\,\varphi$$

Coderivative of a set-valued mapping F

$$D^*F(\bar{x},\bar{y})(u) := \left\{ v \middle| \, (v,-u) \in N((\bar{x},\bar{y});\mathrm{gph}\,F) \right\}, \quad \bar{y} \in F(\bar{x})$$

Generalized Hessian of φ at $\bar{x}$

$$\partial^2\varphi(\bar{x}) := D^*(\partial\varphi)(\bar{x},\bar{v}), \quad \bar{v} \in \partial\varphi(\bar{x})$$

Enjoy **FULL CALCULUS** and **COMPLETELY CALCULATED** in terms of the given data of (P)

Necessary Optimality Conditions

For simplicity consider the case of smooth costs φ, ℓ

THEOREM Let $\bar{z}(\cdot)$ be a strong local minimizer for (P). Then there exist a multiplier $\lambda \geq 0$, an adjoint arc $p(t) = (p^x, p^u, p^a)(t) \in W^{1,2}$, subgradient functions $w(t) = (w^x, w^u, w^a) \in L^2$ and $v(t) = (v^x, v^u, v^a) \in L^2$ such that

$$\big(w(t), v(t)\big) \in \operatorname{co} \partial \ell\big(t, \bar{z}(t), \dot{\bar{z}}(t)\big) \text{ a.e.}$$

and Borel measures $\gamma \in C^*$, $\xi^1 \in C^*_+$, $\xi^2 \in C^*_-$ satisfying

- **Primal-Dual Dynamic Relationships**

$$\dot{\bar{x}}(t) + f\big(\bar{x}(t), \bar{a}(t)\big) = \sum_{i=1}^{m} \eta_i(t) \nabla g_i\big(\bar{x}(t) - \bar{u}(t)\big) \text{ a.e.}$$

with the uniquely defined $\eta(t) \in L^2$ and

$$\dot{p}(t) = \lambda w(t) + \Big(\nabla_x f\big(\bar{x}(t), \bar{a}(t)\big)^* \big(\lambda v^x(t) - q^x(t)\big), 0, \nabla_b f\big(\bar{x}(t), \bar{a}(t)\big)^* \big(\lambda v^x(t) - q^x(t)\big)\Big)$$

$$q^u(t) = \lambda \nabla_{\dot{u}} \ell\big(t, \dot{\bar{u}}(t)\big),\ q^a(t) \in \lambda \partial_{\dot{a}} \ell\big(t, \dot{\bar{a}}(t)\big) \ \text{ a.e.}$$

where $q(t) = (q^x, q^u, q^a)$ is of bounded variation given by

$$q(t) := p(t) - \int_{[t,T]} \big(-d\gamma(s), 2\bar{u}(s) d(\xi^1(s) + \xi^2(s)) + d\gamma(s), 0\big)$$

Moreover, we have the implications

$$\begin{cases} g_i\big(\bar{x}(t) - \bar{u}(t)\big) > 0 \Rightarrow \eta_i(t) = 0, \\ \eta_i(t) > 0 \Rightarrow \langle \nabla g_i\big(\bar{x}(t) - \bar{u}(t), \lambda v^x(t) - q^x(t)\big)\rangle = 0 \end{cases}$$

- **Transversality Conditions**

$$-p^x(T) + \sum_{i \in I(\bar{x}(T)-\bar{u}(T))} \eta_i(T) \nabla g_i\big(\bar{x}(T) - \bar{u}(T)\big) \in \lambda \partial \varphi(\bar{x}(T))$$

$$p^u(T) - \sum_{i \in I(\bar{x}(T)-\bar{u}(T))} \eta_i(T) \nabla g_i\big(\bar{x}(T) - \bar{u}(T)\big) \in$$

$$-2\bar{u}(T)\left(N_{[0,r_2]}\big(\|\bar{u}(T)\|\big) + N_{[r_1,\infty)}\big(\|\bar{u}(T)\|\big)\right)$$

$$p^a(T) = 0$$

where $I(y) \subset \{1, \ldots, m\}$ is the set of active constraint indices

- **Nontriviality Conditions**

$$\lambda + \|q^u(0)\| + \|p(T)\| + \|\xi^1\| + \|\xi^2\| > 0$$

Furthermore we have the implications

$$\big[g_i(x_0 - \bar{u}(0)) > 0,\ i = 1, \ldots, m\big] \Rightarrow \big[\lambda + \|p(T)\| + \|\xi^1\| + \|\xi^2\| > 0\big]$$

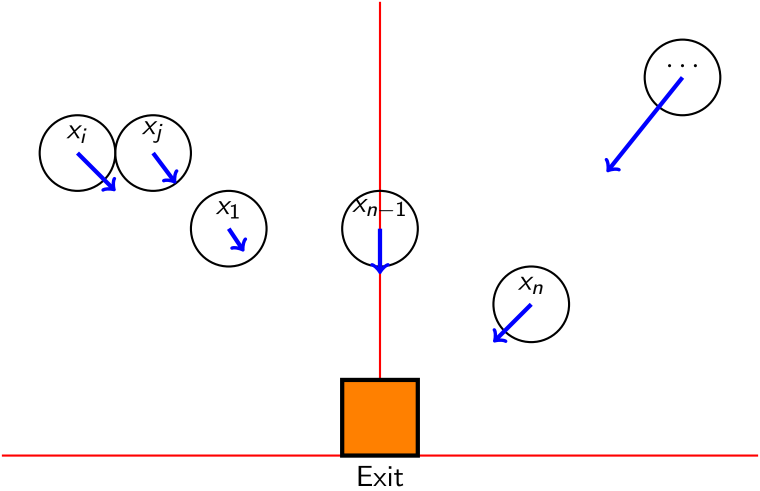
$$\big[g_i(\bar{x}(T) - \bar{u}(T)) > 0,\ r_1 < \|\bar{u}(T)\| < r_2,\ i = 1, \ldots, m\big] \Rightarrow$$
$$\big[\lambda + \|q^u(0)\| + \|\xi^1\| + \|\xi^2\| > 0\big]$$

Applications to the Crowd Motion Model

We can apply our necessary optimality condition derived in theory to solve the controlled crowd motion model in the planar.
The dynamic description of this model as a sweeping process was developed by Maury and Venel **[MauryVenel11]**.
The crowd motion model is designed to deal with local interactions between participants to describe the dynamics of pedestrian traffic.
This microscopic model for crowd motion rests on two principles.

- A spontaneous velocity is the velocity that each participant would like to have in the absence of others.
- The actual velocity is the projection of the spontaneous velocity onto the set of admissible velocities.

Consider n participants ($n \geq 2$) identified with rigid disks of the same radius R in a planar.



Applications to the Crowd Motion Model

- Non-overlapping condition:

$$Q_0 := \{x \in \mathbb{R}^{2n} |\ D_{ij}(x) \geq 0\ \ \forall i \neq j\},$$

where $D_{ij}(x) := \|x_i - x_j\| - 2R$.

- The spontaneous velocity of participants is

$$U(x) = \big(U_0(x_1), \ldots, U_0(x_n)\big)\ \text{ for }\ x \in Q_0,$$

- The set of feasible velocities is

$$C_x := \{v = (v_1, \ldots, v_n) \in \mathbb{R}^{2n} |\ \forall i < j\ \ D_{ij}(x) = 0 \Longrightarrow \langle G_{ij}(x), v\rangle \geq 0\},$$

with

$$G_{ij}(x) = \nabla D_{ij}(x) = (0, \ldots, 0, -e_{ij}(x), 0, \ldots, 0, e_{ij}(x), 0, \ldots, 0) \in \mathbb{R}^{2n}$$

and $e_{ij}(x) = \dfrac{x_j - x_i}{\|x_j - x_i\|}$.

Specify our data to fit this model:

- The set C is

$$C := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^{2n} |\ g_{ij}(x) \geq 0,\ \forall i \neq j, i, j = 1, \ldots, n\}$$

with $g_{ij}(x) := D_{ij}(x) = \|x_i - x_j\| - 2R$.

- As all the participants exhibit the same behavior and want to reach the exit by the shortest path, their spontaneous velocities are

$$\mathbf{U}(x) = (U_0(x_1), \ldots, U_0(x_n)) \ \text{ with } \ U_0(x_i) = -s_i \nabla D(x_i)$$

where $D(x_i)$ stands for the distance between the position x_i and the exit positioned at the origin, and where the scalar $s_i \geq 0$ denotes the speed.
In this case $D(x_i) = \|x_i\|$ so $\nabla D(x_i) = \dfrac{x_i}{\|x_i\|}$ and thus $s_i = \|U_0(x_i)\|$.

The perturbation is

$$f(x) = \left(-\frac{s_1}{\|x_1\|}x_1, \ldots, -\frac{s_n}{\|x_n\|}x_n\right) \in \mathbb{R}^{2n} \text{ for all } x = (x_1, \ldots, x_n) \in Q_0,$$

So

$$f(x) = (-s_1 \cos\theta_1, -s_1 \sin\theta_1, \ldots, -s_n \cos\theta_n, -s_n \sin\theta_n)$$

where θ_i denotes the direction of x_i.

We involve $a(\cdot) = (a_1(\cdot), \ldots, a_n(\cdot))$ into perturbations to control the speed of participants:

$$f(x, a) = (s_1 a_1 \cos\theta_1, s_1 a_1 \sin\theta_1, \ldots, s_n a_n \cos\theta_n, s_n a_n \sin\theta_n)\,.$$

Applications to the Crowd Motion Model

- Define the vector function $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n) : [0, T] \to \mathbb{R}^{2n}$ by

$$\bar{u}_{i+1}(t) = \bar{u}_i(t) = \left(\frac{r}{\sqrt{2n}}, \frac{r}{\sqrt{2n}} \right)$$

where r is a number such that $r_1 \leq r \leq r_2$.

- The **controlled crowd motion dynamics** is

$$\begin{cases} -\dot{x}(t) \in N\big(x(t); C(t)\big) + f\big(x(t), a(t)\big) & \text{for a.e. } t \in [0, T], \\ C(t) := C + \bar{u}(t), \ \|\bar{u}(t)\| = r \in [r_1, r_2] & \text{on } [0, T], \\ x(0) = x_0 \in C(0), \end{cases}$$

- Consider the Bolza problem:

$$\text{minimize} \quad J[x, a] := \frac{1}{2}\left(\|x(T)\|^2 + \int_0^T \|a(t)\|^2 dt \right)$$

We would like to *minimize* the **distance** of all the participants to the exit together with the **energy** of the feasible controls $a(\cdot)$.

Applications to the Crowd Motion Model

Since all the participants would like to reach the exit by the shortest path and when two participants i and j are in contact they tend to adjust their velocities (speeds and directions) and maintain their new constant velocities until reaching someone or the end of the process at time $t = T$, then the trajectory x_i admits the following representations:

$$\bar{x}_i(t) = (\|\bar{x}_i(t)\| \cos\theta_i(t), \|\bar{x}_i(t)\| \sin\theta_i(t)) \quad \text{for all} \quad i = 1, \ldots, n,$$

where θ_i denotes the piecewise-constant direction of participant i (each participant only switches his/her direction when he/she is in contact).

Necessary Optimality Conditions

(1) $w(t) = (0, \bar{a}(t)), v(t) = (0,0)$ for a.e. $t \in [0, T]$;

(2)

$$\dot{\bar{x}}(t) + (s_1\bar{a}_1(t)\cos\theta_1(t), s_1\bar{a}_1(t)\sin\theta_1(t), \ldots,$$
$$s_n\bar{a}_n(t)\cos\theta_n(t), s_n\bar{a}_n(t)\sin\theta_n(t)) = \sum_{i<j}\eta_{ij}(t)\nabla g_{ij}(\bar{x}(t) - \bar{u}(t))$$
$$= \Big(-\sum_{j>1}\eta_{1j}(t)\frac{\bar{x}_j(t)-\bar{x}_1(t)}{\|\bar{x}_j(t)-\bar{x}_1(t)\|}, \ldots, \sum_{i<j}\eta_{ij}(t)\frac{\bar{x}_j(t)-\bar{x}_i(t)}{\|\bar{x}_j(t)-\bar{x}_i(t)\|}$$
$$-\sum_{i>j}\eta_{ji}(t)\frac{\bar{x}_i(t)-\bar{x}_j(t)}{\|\bar{x}_i(t)-\bar{x}_j(t)\|}, \ldots, \sum_{j<n}\eta_{jn}(t)\frac{\bar{x}_n(t)-\bar{x}_j(t)}{\|\bar{x}_n(t)-\bar{x}_j(t)\|}\Big)$$

(3) $\|\bar{x}_i(t) - \bar{x}_j(t)\| > 2R \Longrightarrow \eta_{ij}(t) = 0$ for all $i<j$ and a.e. $t \in [0, T]$;

(4) $\eta_{ij}(t) > 0 \Longrightarrow \Big\langle q_j^x(t) - q_i^x(t), \bar{x}_j(t) - \bar{x}_i(t)\Big\rangle = 0$ for all $i<j$ and a.e. $t \in [0, T]$;

$$(5)\quad \begin{cases} \dot{p}(t) = \big(0, \lambda\bar{a}_1(t) - s_1(\cos\theta_1(t)q^x_{11}(t) + \sin\theta_1(t)q^x_{12}(t)) \\ , \ldots, \lambda\bar{a}_n(t) - s_n(\cos\theta_n(t)q^x_{n1}(t) + \sin\theta_n(t)q^x_{n2}(t))\big) \end{cases}$$

$$(6)\quad q^x(t) = p^x(t) + \gamma([t, T]) \text{ for a.e. } t \in [0, T];$$

$$(7)\quad q^a(t) = p^a(t) = 0 \text{ for a.e. } t \in [0, T];$$

$$(8)\quad \begin{cases} p^x(T) + \lambda\bar{x}(T) = \Big(-\sum_{j>1} \eta_{1j}(T)\dfrac{\bar{x}_j(T) - \bar{x}_1(T)}{\|\bar{x}_j(T) - \bar{x}_1(T)\|}, \ldots, \\ \sum_{i<j} \eta_{ij}(T)\dfrac{\bar{x}_j(T) - \bar{x}_i(T)}{\|\bar{x}_j(T) - \bar{x}_i(T)\|} - \sum_{i>j} \eta_{ji}(T)\dfrac{\bar{x}_i(T) - \bar{x}_j(T)}{\|\bar{x}_i(T) - \bar{x}_j(T)\|}, \\ \ldots, \sum_{j<n} \eta_{jn}(T)\dfrac{\bar{x}_n(T) - \bar{x}_j(T)}{\|\bar{x}_n(T) - \bar{x}_j(T)\|}\Big); \end{cases}$$

$$(9)\quad p^a(T) = 0;$$

$$(10)\quad \lambda + \|p^x(T)\| > 0.$$

The Crowd Motion Problem with Two Participants.

- Let t_1 be the first time that two participants are in contact, i.e., $\|\bar{x}_1(t_1) - \bar{x}_2(t_1)\| = 2R$.
- The velocities of two participants (before and after t_1) are given by

$$\begin{cases} \dot{\bar{x}}_1(t) = (-s_1\bar{a}_1(t)\cos\theta_1(0), -s_1\bar{a}_1(t)\sin\theta_1(0)), \\ \dot{\bar{x}}_2(t) = (-s_2\bar{a}_2(t)\cos\theta_2(0), -s_2\bar{a}_2(t)\sin\theta_2(0)) \end{cases}$$

  and

$$\begin{cases} \dot{\bar{x}}_1(t) = -s_1\bar{a}_1(t)(\cos\theta_1(t_1), \sin\theta_1(t_1)) - \dfrac{\eta_{12}(t)}{2R}(\bar{x}_2(t) - \bar{x}_1(t)), \\ \dot{\bar{x}}_2(t) = -s_2\bar{a}_2(t)(\cos\theta_2(t_1), \sin\theta_2(t_1)) + \dfrac{\eta_{12}(t)}{2R}(\bar{x}_2(t) - \bar{x}_1(t)) \end{cases}$$

- After two participants are in contact, they switch their directions:

$$\theta_1(t_1) = \theta_2(t_1) = \theta.$$

- Since the speeds are constant, it is to suppose that the functions $\bar{a}_i(\cdot)$ are constant $\bar{a}_i$ on $[0, T]$ for all $i = 1, 2$. Thus the vector function $\eta_{12}(\cdot)$ is piecewise constant on $[0, T]$:

$$\eta_{12}(t) = \begin{cases} \eta_{12}(0) = 0 & \text{a.e. } t \in [0, t_1) \text{ including } t = 0 \\ \eta_{12}(t_1) & \text{a.e. } t \in [t_1, 6] \text{ including } t = t_1. \end{cases}$$

- The trajectories are

$$\begin{cases} \bar{x}_1(t) = (\bar{x}_{11}(0), \bar{x}_{12}(0)) + (-s_1\bar{a}_1 \cos\theta_1(0)t, -s_1\bar{a}_1 \sin\theta_1(0)t) \\ \bar{x}_2(t) = (\bar{x}_{21}(0), \bar{x}_{22}(0)) + (-s_2\bar{a}_2 \cos\theta_2(0)t, -s_2\bar{a}_2 \sin\theta_2(0)t) \end{cases} \tag{4}$$

for all $t \in [0, t_1)$ and

$$
\begin{cases}
\bar{x}_1(t) = (\bar{x}_{11}(0), \bar{x}_{12}(0)) + (-s_1\bar{a}_1\cos\theta_1(0)t_1 \\
+(-s_1\bar{a}_1 + \eta_{12}(t_1))\cos\theta(t-t_1), -s_1\bar{a}_1\sin\theta_1(0)t_1 \\
+(-s_1\bar{a}_1 + \eta_{12}(t_1))\sin\theta(t-t_1)) \\
\\
\bar{x}_2(t) = (\bar{x}_{21}(0), \bar{x}_{22}(0)) + (-s_2\bar{a}_2\cos\theta_2(0)t_1 \\
+(-s_2\bar{a}_2 - \eta_{12}(t_1))\cos\theta(t-t_1), -s_2\bar{a}_2\sin\theta_2(0)t_1 \\
+(-s_2\bar{a}_2 - \eta_{12}(t_1))\sin\theta(t-t_1))
\end{cases} \tag{5}
$$

for all $t \in [t_1, T]$.

- Since two participants have the same velocities as they have the same speeds and directions after t_1, then $\dot{x}_1(t) = \dot{x}_2(t)$ for all $t \in [t_1, T]$, which implies

$$\eta_{12}(t_1) = \frac{s_1 \bar{a}_1 - s_2 \bar{a}_2}{2}.$$

- Moreover, using the fact that $\|\bar{x}_2(t_1) - \bar{x}_1(t_1)\| = 2R$ allows us to calculate the time t_1 as follows:

$$\left\{[s_1 \bar{a}_1 \cos\theta_1(0) - s_2 \bar{a}_2 \cos\theta_2(0)]^2 + [s_1 \bar{a}_1 \sin\theta_1(0) - s_2 \bar{a}_2 \sin\theta_2(0)]^2\right\} t_1^2$$
$$+ 2\left\{[\bar{x}_{21}(0) - \bar{x}_{11}(0)][s_1 \bar{a}_1 \cos\theta_1(0) - s_2 \bar{a}_2 \cos\theta_2(0)]\right.$$
$$\left.+ [\bar{x}_{22}(0) - \bar{x}_{12}(0)][s_1 \bar{a}_1 \sin\theta_1(0) - s_2 \bar{a}_2 \sin\theta_2(0)]\right\} t_1$$
$$+ [\bar{x}_{21}(0) - \bar{x}_{11}(0)]^2 + [\bar{x}_{22}(0) - \bar{x}_{12}(0)]^2 - 4R^2 = 0$$

- If $\eta_{12}(t_1) = 0$ then $s_1\bar{a}_1 = s_2\bar{a}_2$.
- If $\eta_{12}(t_1) > 0$ then $s_2\bar{a}_1 = s_1\bar{a}_2$ due to (4), (5), and (7).
  Hence, in both cases we can express the cost functional in terms of $\bar{a}_1, \bar{a}_2$, and θ and can solve the optimization problem completely.

- Specify the data as follows:

$$n = 2,\ T = 6,\ s_1 = 6,\ s_2 = 3,$$
$$x_{01} = \left(-48 - \tfrac{6}{\sqrt{2}}, 48 + \frac{6}{\sqrt{2}}\right),\ x_{02} = (-48, 48),\ R = 3.$$

- We have $t_1 = 0$ (two participants are in contact at the initial time), $\theta_1(\cdot) = \theta_2(\cdot) = 135^\circ$ ($\theta_1(\cdot)$ and $\theta_2(\cdot)$ are constant on the interval $[0, T]$).
- Consider two cases:
  Case 1: $\eta_{12}(t_1) = \eta_{12}(0) = 0.$ Then the cost functional is

$$J[x, a] = 1311\bar{a}_1^2 - 36(96\sqrt{2} + 6)\bar{a}_1 + \left(48 + \frac{6}{\sqrt{2}}\right)^2 + 48^2.$$

So J attains its minimum at $\bar{a}_1 = \dfrac{(96\sqrt{2}+6)18}{1311} \approx 1.95$ and thus $\bar{a}_2 = 2\bar{a}_1 \approx 3.9$.
The minimum cost in this case is $J \approx 66.49$.
Also, we can compute the trajectory as follows:

$$\begin{cases} \bar{x}_1(t) = \left(-48 - \dfrac{6}{\sqrt{2}} + 8.27t, 48 + \dfrac{6}{\sqrt{2}} - 8.27t\right) \\ \bar{x}_2(t) = (-48 + 8.27t, 48 - 8.27t) \end{cases}$$

The spontaneous velocities are $(8.27, -8.27)$ and $(8.27, -8.27)$.
Case 2: $\eta_{12}(t_1) = \eta_{12}(0) > 0$. In this case, we have $\bar{a}_1 = 2\bar{a}_2$ and thus $\eta_{12}(t_1) = \frac{9}{2}\bar{a}_2$. Then the cost functional is

$$J[x,a] = 2040\bar{a}_2^2 - 45(96\sqrt{2}+6)\bar{a}_2 + \left(48 + \frac{6}{\sqrt{2}}\right)^2 + 48^2$$

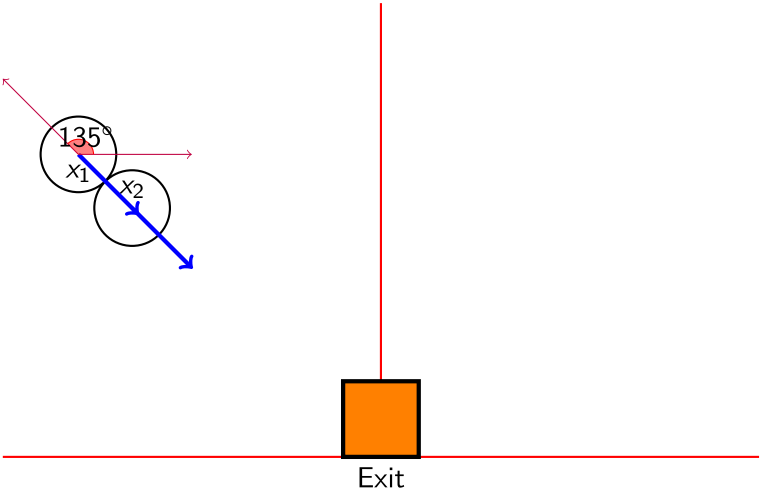
So J attains its minimum at $\bar{a}_2 = \dfrac{45(96\sqrt{2}+6)}{4080} \approx 1.56$ and hence $\bar{a}_1 = 2\bar{a}_2 \approx 3.12$.

The minimum cost in this case is $J \approx 45.9$.

Comparing two above cases, we conclude that the optimal solution is

$$\begin{cases} (\bar{a}_1, \bar{a}_2) = (3.12, 1.56) \\ \bar{x}_1(t) = \left(-48 - \dfrac{6}{\sqrt{2}} + 8.27t, 48 + \dfrac{6}{\sqrt{2}} - 8.27t\right) \\ \bar{x}_2(t) = (-48 + 8.27t, 48 - 8.27t) \end{cases}$$

The spontaneous velocities are $(13.24, -13.24)$ and $(3.31, -3.31)$.


135°
x_1
x_2
Exit

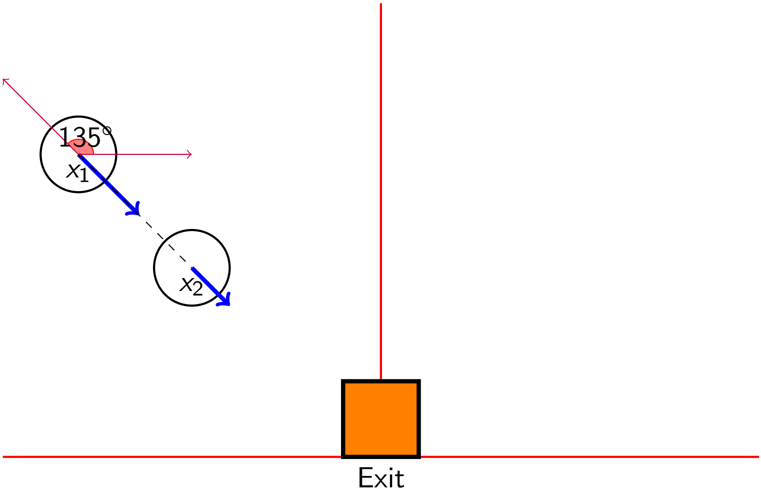
- Specify the data as follows:
  $n = 2, T = 6, s_1 = 6, s_2 = 3, x_{01} = (-60, 60), x_{02} = (-48, 48), R = 3.$
- In this case we have $t_1 > 0$ and $\theta_1(0) = \theta_2(0) = 135^\circ$.
- The optimal solution is

$$(\bar{a}_1, \bar{a}_2) = (3.36, 1.68),$$

$$\bar{x}_1(t) = \begin{cases} (14.26t - 60, -14.26t + 60) & \text{for } t \in [0, 0.73) \\ (8.91t - 56.12, -8.91t + 56.12) & \text{for } t \in [0.73, 6] \end{cases}$$

and

$$\bar{x}_2(t) = \begin{cases} (3.56t - 48, -3.56t + 48) & \text{for } t \in [0, 0.73) \\ (8.91t - 51.88, -8.91t + 51.88) & \text{for } t \in [0.73, 6]. \end{cases}$$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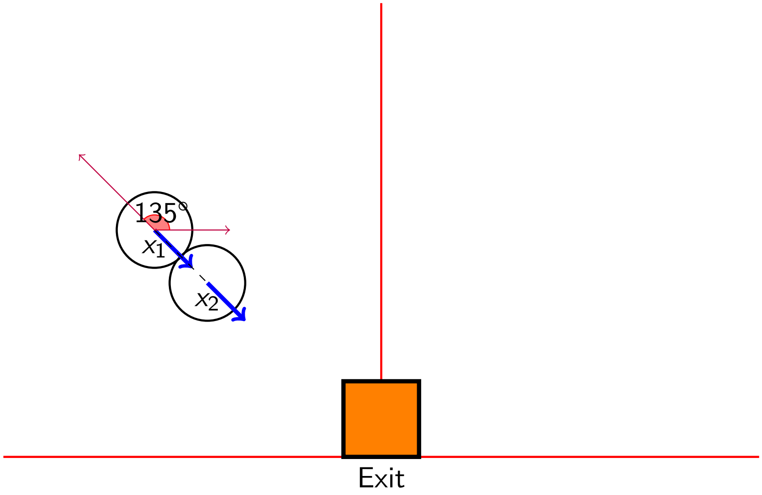
x_1
x_2
Exit


135°
x_1
x_2
Exit

- Specify the data as follows:

$$n = 2, T = 6, s_1 = 6, s_2 = 3, x_{01} = (-60, 60)\,,\ x_{02} = (-48, 54), R = 3.$$

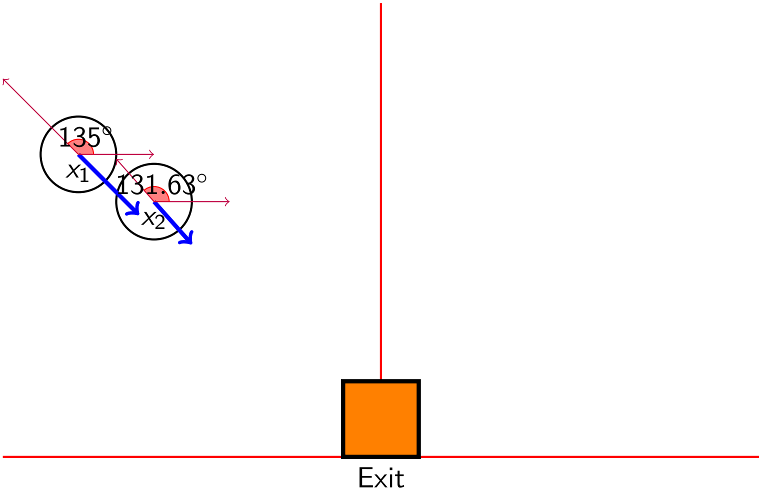
- In this case we have $t_1 > 0$, $\theta_1(0) = 135^\circ$, and $\theta_2(0) = 131.63^\circ$.
- The optimal solution is

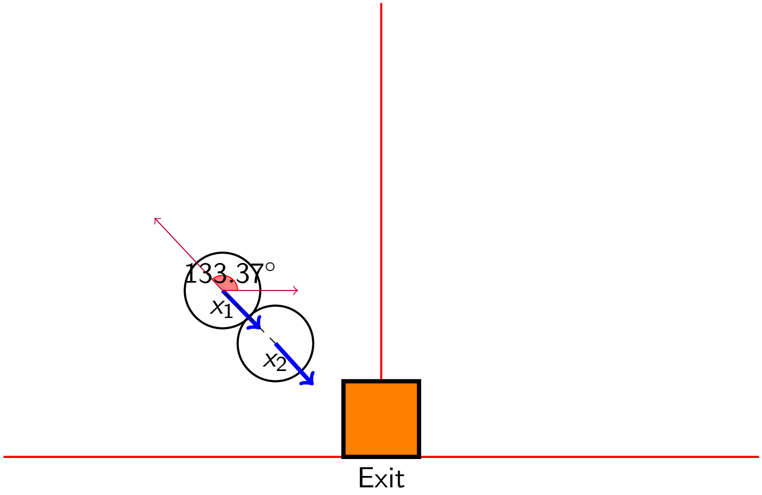
$$(\bar{a}_1, \bar{a}_2) = (2.62, 1.31),$$

$$\bar{x}_1(t) = \begin{cases} (-60 + 11.12t, 60 - 11.12t) & \text{for} \quad t \in [0, 0.702) \\ (-56.93 + 6.75t, 57.21 - 7.14t) & \text{for} \quad t \in [0.702, 6] \end{cases}$$

and

$$\bar{x}_2(t) = \begin{cases} (-48 + 2.61t, 54 - 2.94t) & \text{for} \quad t \in [0, 0.702) \\ (-50.91 + 6.75t, 56.96 - 7.14t) & \text{for} \quad t \in [0.702, 6] \end{cases}$$


135°
x_1
131.63°
x_2
Exit


133.37°
x1
x2
Exit

Some References

A. V. Arutyunov and S. M. Aseev. Investigation of the degeneracy phenomenon of the maximum principle for optimal control problems with state constraints, *SIAM J. Control Optim.*, 35: 930–952, 1997.

T. H. Cao and B. S. Mordukhovich. Optimal control of a perturbed sweeping process via discrete approximations. *Disc. Cont. Dyn. Syst. Ser. B,* 21 (2016), No. 10; DOI 10.3934/dcdsb.2016100.

T. H. Cao and B. S. Mordukhovich. Optimality conditions for a controlled sweeping process with applications to the crowd motion model. *Disc. Cont. Dyn. Syst., Ser B,* 21 (2017), 267-306; DOI:10.3934/dcdsb.2017014.

F. H. Clarke, Yu. S Ledyaev, R. J. Stern and P. R. Wolenski. *Nonsmooth Analysis and Control Theory*. Springer, 1998.

G. Colombo, R. Henrion, N. D. Hoang and B. S. Mordukhovich. Optimal control of the sweeping process over polyhedral controlled sets. *J. Diff. Eqs.* 260 (2016), 3397-3447.

Some References

- R. Henrion, B. S. Mordukhovich and N. M. Nam. Second-order analysis of polyhedral systems in finite and infinite dimensions with applications to robust stability of variational inequalities. *SIAM J. Optim.*, 20:2199–2227, 2010.
- B. Maury and J. Venel. A mathematical framework for a crowd motion model. *C. R. Acad. Sci. Paris Ser. I*, 346:1245–1250, 2008.
- B. Maury and J. Venel. Handling of contacts in crowd motion simulations. In C. Appert-Rolland et al., editors, *Traffic and Granular Flow '07*, pages 171–180. Springer, 2009.
- B. S. Mordukhovich. Discrete approximations and refined Euler-Lagrange conditions for differential inclusions. *SIAM J. Control Optim.*, 33:882–915, 1995.
- B. S. Mordukhovich. *Variational Analysis and Generalized Differentiation, I: Basic Theory*. Springer, 2006.

Some References

- R. T. Rockafellar and R. J-B. Wets. *Variational Analysis*. Springer, 2004.
- A. A. Tolstonogov. *Control sweeping process*, J. Convex Anal, **23** (2016), 1099–1123
- R. B. Vinter. *Optimal Control*, Birkhaüser, 2000.

Thank you!