Differential inclusions and applications

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Outline

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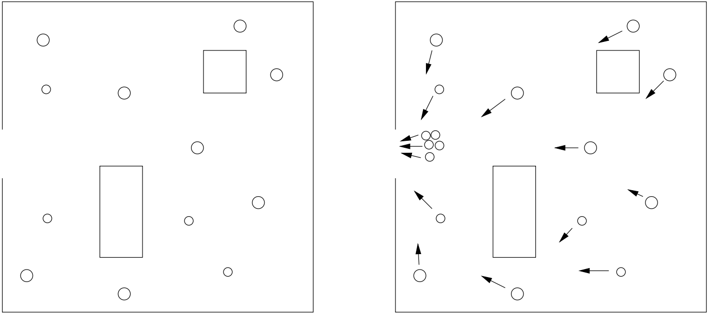
A crowd motion model with several goals

- to deal with emergency evacuation
- to take into account direct contacts between individuals
- to determine the areas where people are crushed

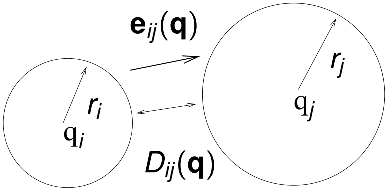
Two principles



Spontaneous velocity

Actual velocity

Notations



$$\mathbf{q} = (q_1, q_2, .., q_N) \in \mathbb{R}^{2N}$$

$$\mathbf{e}_{ij}(\mathbf{q}) = \frac{q_j - q_i}{|q_j - q_i|}$$

Set of feasible configurations

$$Q_0 = \left\{ \mathbf{q} \in \mathbb{R}^{2N},\ \forall\, i < j, \quad D_{ij}(\mathbf{q}) = |q_i - q_j| - r_i - r_j \geq 0 \right\}$$

$$\mathbf{G}_{ij}(\mathbf{q}) = \nabla D_{ij}(\mathbf{q}) = (0 \dots 0, \underbrace{-\mathbf{e}_{ij}(\mathbf{q})}_{i}, 0 \dots 0, \underbrace{\mathbf{e}_{ij}(\mathbf{q})}_{j}, 0 \dots 0)$$

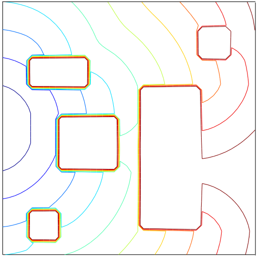
Spontaneous velocity

Notation : $\mathbf{U}(\mathbf{q}) = (\mathrm{U}_1(\mathbf{q}),\ \mathrm{U}_2(\mathbf{q}), \ldots,\ \mathrm{U}_N(\mathbf{q}))$

Example :

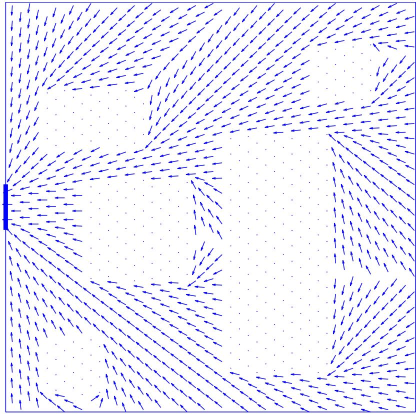
$$\mathrm{U}_i(\mathbf{q}) = -\mathsf{s}_i \nabla \mathcal{D}(\mathrm{q}_i),$$

where $\mathcal{D}(\mathbf{x})$ represents the geodesic distance between $\mathbf{x}$ and the exit.



Contour levels of $\mathcal{D}$

Example of spontaneous velocity



Direction opposite to the gradient of the geodesic distance $\mathcal{D}$.

Actual velocity

To handle the contacts, we define the

cone of admissible velocities

$$\mathcal{C}_{\mathbf{q}} = \left\{ \mathbf{v} \in \mathbb{R}^{2N}, \ \forall\, i < j \quad D_{ij}(\mathbf{q}) = 0 \quad \Rightarrow \quad \mathbf{G}_{ij}(\mathbf{q}) \cdot \mathbf{v} \geq 0 \right\},$$

where $\mathbf{G}_{ij}(\mathbf{q}) = \nabla D_{ij}(\mathbf{q})$.

If $\mathbf{u}$ is the actual velocity of the N pedestrians, the model can be expressed as follows :

$$\begin{cases} \mathbf{q} = \mathbf{q}_0 + \int \mathbf{u}, \\ \mathbf{u} = \mathrm{P}_{\mathcal{C}_{\mathbf{q}}} \mathbf{U}. \end{cases}$$

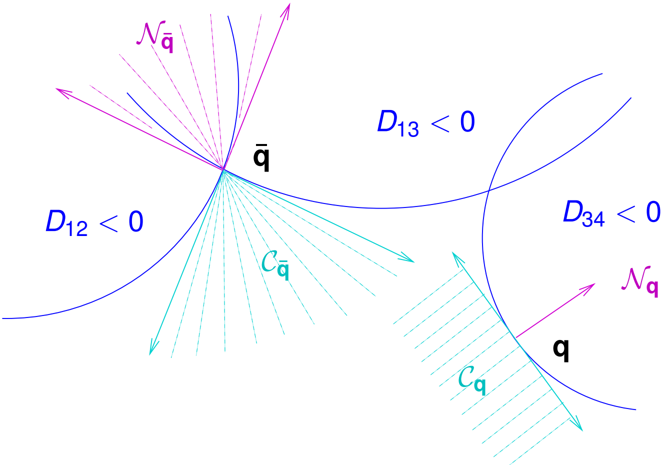
Outline

Cone $\mathcal{N}_{\mathbf{q}}$

Let us define $\mathcal{N}_{\mathbf{q}}$ the polar cone of $\mathcal{C}_{\mathbf{q}}$:

Definition

$$\mathcal{N}_{\mathbf{q}} = \mathcal{C}_{\mathbf{q}}^{\circ} = \{\mathbf{w}\,,\ (\mathbf{w},\mathbf{v}) \leq 0 \quad \forall\, \mathbf{v} \in \mathcal{C}_{\mathbf{q}}\}\,.$$



Cone $\mathcal{N}_{\mathbf{q}}$

Proposition

$$\mathcal{N}_{\mathbf{q}} = \left\{ -\sum \lambda_{ij} \mathbf{G}_{ij}(\mathbf{q})\,,\;\; \lambda_{ij} \geq 0\,,\;\; D_{ij}(\mathbf{q}) > 0 \Longrightarrow \lambda_{ij} = 0 \right\}.$$

Since $\mathcal{C}_{\mathbf{q}}$ and $\mathcal{N}_{\mathbf{q}}$ are mutually polar cones, the following property holds (J.-J. Moreau 62)

Property

$$\mathrm{P}_{\mathcal{C}_{\mathbf{q}}} + \mathrm{P}_{\mathcal{N}_{\mathbf{q}}} = \mathrm{Id}.$$

Differential inclusion

According to the previous property,

$$\dot{\mathbf{q}} = \mathbf{u} = \mathrm{P}_{\mathcal{C}_{\mathbf{q}}}(\mathbf{U}(\mathbf{q})) = \mathbf{U}(\mathbf{q}) - \mathrm{P}_{\mathcal{N}_{\mathbf{q}}}(\mathbf{U}(\mathbf{q})),$$

which is equivalent to

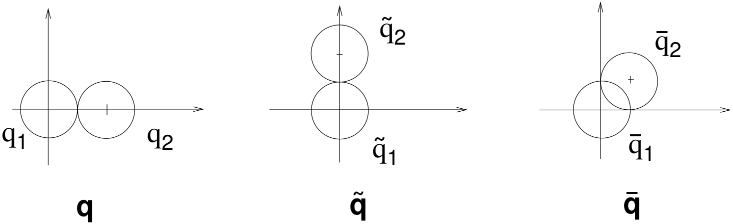
$$\dot{\mathbf{q}} + \mathrm{P}_{\mathcal{N}_{\mathbf{q}}}(\mathbf{U}(\mathbf{q})) = \mathbf{U}(\mathbf{q}).$$

and so the problem can be formulated as a first order differential inclusion .

Model

$$\left\{ \begin{array}{l} \dfrac{d\mathbf{q}}{dt} + \mathcal{N}_{\mathbf{q}} \ni \mathbf{U}(\mathbf{q}), \\ \mathbf{q}(0) = \mathbf{q}_0. \end{array} \right.$$

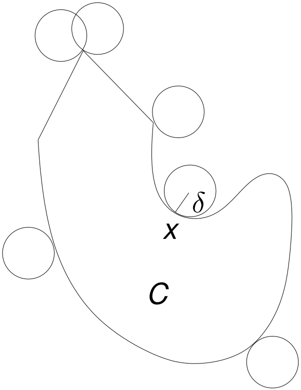
Non-convexity of the feasible set Q_0



$$\text{where } \bar{\mathbf{q}} = \frac{\mathbf{q} + \tilde{\mathbf{q}}}{2}.$$

So Q_0 is not a convex set !

Uniform prox-regularity



Uniformly prox-regular set

Let C be a closed subset of a Hilbert space H,
C is η-prox-regular if the projection on C is single-valued and continuous at any point x satisfying $d_C(x) < \eta$.

H. Federer 59, *positively reached sets*
A. Canino 88, *p-convex sets*
F. Clarke, R. Stern, P. Wolenski 95, *proximally smooth sets*
R. Poliquin, R. Rockafellar, L. Thibault 00, *prox-regular sets*

Prox-regularity of Q_0

Proposition

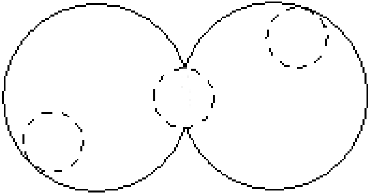
Q_0 is η-prox-regular with $\eta = \eta(N, r_i)$.

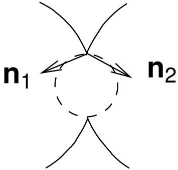
Sketch of the proof :

One constraint's case :

$Q_{ij} = \{\mathbf{q} \in \mathbb{R}^{2N},\ D_{ij}(\mathbf{q}) = |\mathrm{q}_j - \mathrm{q}_i| - (r_j + r_i) \geq 0\}$ is η_{ij}-prox-regular with $\eta_{ij} = \dfrac{r_i + r_j}{\sqrt{2}}$.

Extension to several constraints : $Q_0 = \bigcap_{i<j} Q_{ij}$.





Key point of the proof

A reverse triangle inequality

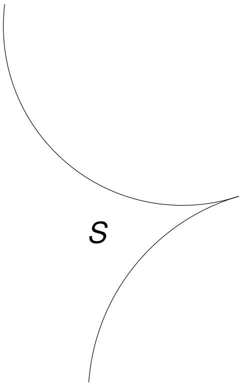
For every $\mathbf{q} \in Q_0$, for every $\lambda_{ij} \geq 0$, there exists $\gamma > 1$ such that

$$\sum_{(i,j)\in I(\mathbf{q})} \lambda_{ij} |\mathbf{G}_{ij}(\mathbf{q})| \leq \gamma \left| \sum_{(i,j)\in I(\mathbf{q})} \lambda_{ij} \mathbf{G}_{ij}(\mathbf{q}) \right|,$$

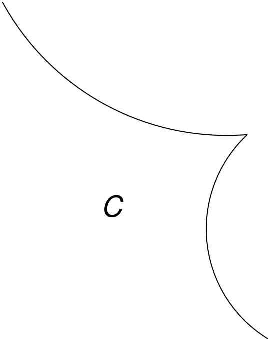
where

$$I(\mathbf{q}) = \{(i,j), i < j, D_{ij}(\mathbf{q}) = 0\}.$$

A second important geometrical assumption



The set S
is not suitable.



The set C
is suitable. No "thin peaks".

Well-posedness

Theorem

Assume that $\mathbf{U}$ is bounded and Lipschitz continuous. Then for any $\mathbf{q}_0$ in Q_0, there is a unique absolutely continuous map $\mathbf{q}$ satisfying

$$\begin{cases} \dfrac{d\mathbf{q}}{dt} + \mathrm{N}(Q_0, \mathbf{q}) \ni \mathbf{U}(\mathbf{q}) & \text{a.e. in } [0, T], \\ \mathbf{q}(0) = \mathbf{q}_0. \end{cases}$$

Numerical scheme

Initialization : $\mathbf{q}^0 = \mathbf{q}_0$

Time-loop : $\mathbf{q}^{\mathrm{n}}$ is known

$$\mathbf{u}^{\mathrm{n}} = \mathrm{P}_{C_{\mathrm{h}}(\mathbf{q}^{\mathrm{n}})}(\mathbf{U}(\mathbf{q}^{\mathrm{n}}))$$

$$\mathbf{q}^{\mathrm{n}+1} = \mathbf{q}^{\mathrm{n}} + \mathrm{h}\,\mathbf{u}^{\mathrm{n}}$$

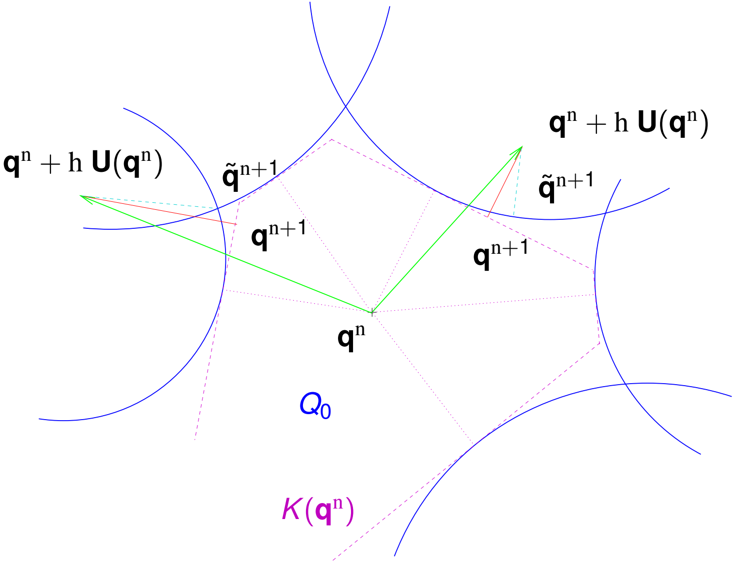
where $C_{\mathrm{h}}(\mathbf{q}^{\mathrm{n}}) = \left\{\mathbf{v} \in \mathbb{R}^{2N}, \forall\, i < j,\ D_{ij}(\mathbf{q}^{\mathrm{n}}) + \mathrm{h}\,\mathbf{G}_{ij}(\mathbf{q}^{\mathrm{n}}) \cdot \mathbf{v} \geq 0\right\}$.

In terms of position, this algorithm can be formulated as follows :

$$\mathbf{q}^{\mathrm{n}+1} = \mathrm{P}_{K(\mathbf{q}^{\mathrm{n}})}(\mathbf{q}^{\mathrm{n}} + \mathrm{h}\,\mathbf{U}(\mathbf{q}^{\mathrm{n}}))$$

with $K(\mathbf{q}^{\mathrm{n}}) = \left\{\mathbf{q} \in \mathbb{R}^{2N}, \forall\, i < j,\ D_{ij}(\mathbf{q}^{\mathrm{n}}) + \mathbf{G}_{ij}(\mathbf{q}^{\mathrm{n}}) \cdot (\mathbf{q} - \mathbf{q}^{\mathrm{n}}) \geq 0\right\}$

Comparison between theoretical and numerical projections



Continuous and discrete problems

Discrete differential inclusion :

$$\mathbf{u}^{n} + \mathrm{N}(K(\mathbf{q}^{n}), \mathbf{q}^{n+1}) \ni \mathbf{U}(\mathbf{q}^{n}).$$

Continuous differential inclusion :

$$\frac{d\mathbf{q}}{dt} + \mathrm{N}(Q_0, \mathbf{q}) \ni \mathbf{U}(\mathbf{q}).$$

Proposition

$$\mathrm{N}(Q_0, \mathbf{q}) = \mathrm{N}(K(\mathbf{q}), \mathbf{q}).$$

Convergence

Let $\mathbf{q}_\mathrm{h}$ be the continous piecewise linear function associated to the numerical scheme

Theorem

Assume that $\mathbf{U}$ is bounded and Lipschitz continous.
Then $\mathbf{q}_\mathrm{h}$ uniformly converges in $[0, T]$ to the map $\mathbf{q}$ satisfying :

$$\begin{cases} \dfrac{d\mathbf{q}}{dt} + \mathrm{N}(Q_0, \mathbf{q}) \ni \mathbf{U}(\mathbf{q}) \quad \text{a.e. in } [0, T], \\ \mathbf{q}(0) = \mathbf{q}_0. \end{cases}$$

Numerical simulations

- Arches

  Movie Pressure

- With individual strategies

  Movie

- Evacuation of a building

  Movie Geodesics Movie Zoom

Outline

Set defined by inequalities

If the moving set is defined by some inequalities :

$$C(t) := \left\{ x \in \mathbb{R}^d,\ g_i(t,x) \geq 0 \right\},$$

what are the assumptions which imply

• the well-posedness of the associated sweeping process

and

• the convergence of the numerical scheme based on a linear approximation of the constraints ?

Sufficient assumptions

So we consider $C(t) := \bigcap_{i=1}^{p} C_i(t) := \left\{ x \in \mathbb{R}^d,\ g_i(t,x) \geq 0 \right\}.$

We define also $\Omega_i := \{(t,x),\ t \in I,\ x \in C_i(t)\}$.

Assume that there exist $\alpha, \beta, M, \kappa > 0$ such that

$g_i \in C^2\,(\Omega + \kappa B(0,1))$ and satisfies in $\Omega_i + \kappa B(0,1)$:

$$\alpha \leq |\nabla_x g_i(t,x)| \leq \beta, \qquad |\partial_t g_i(t,x)| \leq \beta \tag{1}$$

$$|D_x^2 g_i(t,x)|, \qquad |\partial_t^2 g_i(t,x)|, \qquad |\partial_t \nabla_x g_i(t,x)| \leq M. \tag{2}$$

For all $t \in I$, we define for $\rho > 0$

$$I_\rho(t,x) := \{i,\ g_i(t,x) \leq \rho\}\,.$$

We suppose that there exist constants $\rho, \gamma > 0$ such that for all $x \in C(t)$ and all nonnegative reals λ_i

$$\sum_{i \in I_\rho(t,x)} \lambda_i |\nabla g_i(t,x)| \leq \gamma \left| \sum_{i \in I_\rho(t,x)} \lambda_i \nabla g_i(t,x) \right|, \qquad (R_\rho)$$

Proposition

Under the assumptions (1), (2) and (R_ρ), there exists $\eta > 0$ such that the set $C(t)$ is η-prox-regular for all $t \in I$. Moreover the set-valued map C is Lipschitz continuous with respect to the Hausdorff distance.

Numerical scheme

$$x^{n+1} = \mathrm{P}_{\tilde{C}(t^{n+1},x^{n})}(x^{n} + h\ f^{n})$$

with $\tilde{C}(t,x) = \left\{ y \in \mathbb{R}^d, \quad \forall i, g_i(t,x) + \nabla_x g_i(t,x) \cdot (y - x) \geq 0 \right\}.$

Previous assumptions $\Rightarrow$ x_h converges to x solution of (SP).

Outline

Granular media

Granular flows with inelastic shocks

$$\begin{cases} \ddot{\mathbf{q}} + \mathrm{N}(Q_0, \mathbf{q}) \ni f(t, \mathbf{q}) \\ \dot{\mathbf{q}}^+ = \mathrm{P}_{\mathcal{C}_{\mathbf{q}}}(\dot{\mathbf{q}}^-) \text{ (inelastic shock)} \\ \mathbf{q}(0) = \mathbf{q}_0 \\ \dot{\mathbf{q}}(0) = \mathbf{u}_0. \end{cases}$$

existence of a solution $\mathbf{q} \in W^{1,\infty}(I, \mathbb{R}^d)$ with $\dot{\mathbf{q}} \in BV(I, \mathbb{R}^d)$.

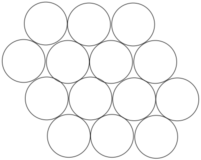
Improvements

Required assumptions :

Independence of $\mathbf{G}_{ij}(q)$

$$\mathbf{G}_{ij}(q) \cdot \mathbf{G}_{kl}(q) \leq 0.$$

Non-independent case :



L. PAOLI *Time-stepping approximation of rigid-body dynamics with perfect unilateral constraints. I-The inelastic impact case* Arch. Rational Mech. Anal. 198, no. 2, 457-503, 2010

Set defined by inequalities

With the previous notations ($C = \bigcap C_i$, g_i, ...) and the previous assumptions (1), (2) and (R_ρ), we obtain also the existence of a solution of

$$\begin{cases} \ddot{x} + \mathrm{N}(C(t), x) \ni f(t,x) \\ \dot{x}^+ = \mathrm{P}_{V(t,x)}(\dot{x}^-) \\ x(0) = x_0 \\ \dot{x}(0) = u_0. \end{cases}$$

where

$$V(t,x) = \left\{ z \in \mathbb{R}^d, \quad \forall i,\ \partial_t g_i(t,x) + \nabla_x g_i(t,x) \cdot z \geq 0 \right\}.$$

General set

If C is a Lipschitz set-valued map with η-prox-regular values and without "thin peaks”, we obtain the existence of a solution of

$$\begin{cases} \ddot{x}(t) + \mathrm{N}(C(t), x(t)) \ni f(t, x(t)) \\ \dot{x}(t^+) = \mathrm{P}_{W(t,x(t))}(\dot{x}(t^-)) \\ x(0) = x_0 \\ \dot{x}(0) = u_0 \end{cases}$$

with

$$W(t,x) = \left\{ v = \lim_{\epsilon \searrow 0} v_\epsilon, \ \text{ with } \ v_\epsilon \in \frac{C(t+\epsilon) - x}{\epsilon} \right\}.$$

Thanks for your attention !