A Mayer Problem for A Controlled Sweeping Process

Chems Eddine Arroud and Giovanni Colombo

University Of Jijel Algeria, Università di Padova

20 septembre 2017

Preliminaries

The first differential inclusions problems have been studied in the early 70s by H. Brezis

$$-\dot{x}(t) \in \partial\varphi(x(t)) \quad t \in [0, T]$$

thanks to the theory of maximal monotone operators.

Preliminaries

The first differential inclusions problems have been studied in the early 70s by H. Brezis

$$-\dot{x}(t) \in \partial\varphi(x(t)) \quad t \in [0, T]$$

thanks to the theory of maximal monotone operators.
"J.J. Moreau ", Evolution problems associated with a moving convex set in a Hilbert space

$$-\dot{x}(t) \in \partial I_{C(t)}(x(t)), \quad x(0) \in C(0)$$

Preliminaries

The first differential inclusions problems have been studied in the early 70s by H. Brezis

$$-\dot{x}(t) \in \partial\varphi(x(t)) \quad t \in [0, T]$$

thanks to the theory of maximal monotone operators.
"J.J. Moreau ", Evolution problems associated with a moving convex set in a Hilbert space

$$-\dot{x}(t) \in \partial I_{C(t)}(x(t)), \quad x(0) \in C(0)$$

where $\partial I_{C(t)}$ is the subdifferential of the indicator function of a closed convex C (normal cone) :

$$-\dot{x}(t) \in N_{C(t)}(x(t)), \quad x(0) \in C(0)$$

Formally, the sweeping process is the differential inclusion with initial condition

$$-\dot{x}(t) \in N_{C(t)}(x(t)), \quad x(0) = x_0 \in C(0)$$

Here $N_C(x)$ denotes the normal cone to C at x in C. In particular, Formally, the sweeping process is the differential inclusion with initial condition

$$-\dot{x}(t) \in N_{C(t)}(x(t)), \quad x(0) = x_0 \in C(0)$$

Here $N_C(x)$ denotes the normal cone to C at x in C. In particular,

$$N_C(x) = \{0\} \text{ if } x \in C$$
$$N_C(x) = \emptyset \text{ if } x \notin C$$

The perturbed sweeping process :

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + f(x(t)), \quad x(0) = x_0 \in C(0)$$

The perturbed sweeping process :

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + f(x(t)), \quad x(0) = x_0 \in C(0)$$

Given a dynamics, it is impossible resisting to the temptation of putting some control

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + f(x(t), u), \quad u \in U$$

Prox-regular set

We say that C is ρ-prox-regular provided the inequality

$$\langle \zeta, y - x \rangle \leq \frac{\|y-x\|^2}{2\rho}$$

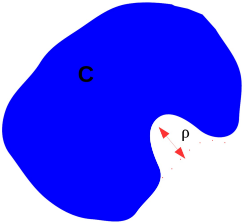
holds for all $x, y \in C$, for every ζ the unit external normal to C

Prox-regular set

We say that C is ρ-prox-regular provided the inequality

$$\langle \zeta, y - x \rangle \leq \frac{\|y-x\|^2}{2\rho}$$

holds for all $x, y \in C$, for every ζ the unit external normal to C



The dynamics

Let the problem P

$$\text{Minimize } h(x(T))$$

The dynamics

Let the problem P

$$\text{Minimize } h(x(T))$$

Subject to

$$\begin{cases} \dot{x}(t) \in -N_{C(t)}(x(t)) + f(x(t), u(t)), \\ x(0) = x_0 \in C(0) , \end{cases} \tag{1}$$

with respect to $u : [0, T] \rightsquigarrow U$, u is measurable.

Assumptions

H_1 : $C : [0, \infty) \rightsquigarrow R^n$ is a set-valued map with the following properties :

Assumptions

H_1 : $C : [0, \infty) \rightsquigarrow R^n$ is a set-valued map with the following properties :

- $H_{1.1}$: for all $t \in [0, T]$, $C(t)$ is nonempty and compact and there exists $r > 0$ such that $C(t)$ is uniformly r-prox regular.
- $H_{1.2}$: C is γ_- Lipschitz and has C^3 boundary.

Assumptions

H_1 : $C : [0, \infty) \rightsquigarrow R^n$ is a set-valued map with the following properties :

- $H_{1.1}$: for all $t \in [0, T]$, $C(t)$ is nonempty and compact and there exists $r > 0$ such that $C(t)$ is uniformly r-prox regular.
- $H_{1.2}$: C is γ_- Lipschitz and has C^3 boundary.
- $H_{1.3}$: $C(t) = \{x : g(t, x) \leq 0\}$ with $g(., x)$ lipschitz and of class $C^{2,1}$.

Assumptions

H_1 : $C : [0, \infty) \rightsquigarrow R^n$ is a set-valued map with the following properties :

- $H_{1.1}$: for all $t \in [0, T]$, $C(t)$ is nonempty and compact and there exists $r > 0$ such that $C(t)$ is uniformly r-prox regular.
- $H_{1.2}$: C is γ_- Lipschitz and has C^3 boundary.
- $H_{1.3}$: $C(t) = \{x : g(t,x) \leq 0\}$ with $g(., x)$ lipschitz and of class $C^{2,1}$.

H_2 : $U \in R^n$ is compact and convex.

Assumptions

H_1 : $C : [0, \infty) \rightsquigarrow R^n$ is a set-valued map with the following properties :

- $H_{1.1}$: for all $t \in [0, T]$, $C(t)$ is nonempty and compact and there exists $r > 0$ such that $C(t)$ is uniformly r-prox regular.
- $H_{1.2}$: C is γ_- Lipschitz and has C^3 boundary.
- $H_{1.3}$: $C(t) = \{x : g(t, x) \leq 0\}$ with $g(., x)$ lipschitz and of class $C^{2,1}$.

H_2 : $U \in R^n$ is compact and convex.

H_3 : $f : R^n \times U \rightsquigarrow R^n$ such that there exist $\beta \geq 0$ with

- $H_{3.1}$: $|f(x, u)| \leq \beta$ for all (x, u);
- $H_{3.2}$: $f(x, u)$ is of class C^1 for all x and u and $f(., .)$ is Lipschitz with lipschitz constant k;
- $H_{3.3}$: $f(x, U)$ is convex for all $x \in R^n$;

Assumptions

H_1 : $C : [0, \infty) \rightsquigarrow R^n$ is a set-valued map with the following properties :

- $H_{1.1}$: for all $t \in [0, T]$, $C(t)$ is nonempty and compact and there exists $r > 0$ such that $C(t)$ is uniformly r-prox regular.
- $H_{1.2}$: C is γ_- Lipschitz and has C^3 boundary.
- $H_{1.3}$: $C(t) = \{x : g(t, x) \leq 0\}$ with $g(., x)$ lipschitz and of class $C^{2,1}$.

H_2 : $U \in R^n$ is compact and convex.

H_3 : $f : R^n \times U \rightsquigarrow R^n$ such that there exist $\beta \geq 0$ with

- $H_{3.1}$: $|f(x, u)| \leq \beta$ for all (x, u) ;
- $H_{3.2}$: $f(x, u)$ is of class C^1 for all x and u and $f(., .)$ is Lipschitz with lipschitz constant k ;
- $H_{3.3}$: $f(x, U)$ is convex for all $x \in R^n$;

H_4 : $h : R \rightsquigarrow R$ is of class C^1

Main result

Theorem

Let (x_*, u_*) be a global minimizer satisfying the outward (or inward) pointing condition. Then there exist a BV adjoint vector $p : [0, T] \to R^n$,a finite signed Radon measure μ on $[0, T]$, and measurable vectors $\xi, \eta : [0, T] \to R^n$, with $\xi(t) \geq 0$ for μ-a.e. t and $0 \leq \eta(t) \leq \beta + \gamma$ for a.e. t, satisfying :

Main result

Theorem

Let (x_*, u_*) be a global minimizer satisfying the outward (or inward) pointing condition. Then there exist a BV adjoint vector $p : [0, T] \to R^n$,a finite signed Radon measure μ on $[0, T]$, and measurable vectors $\xi, \eta : [0, T] \to R^n$, with $\xi(t) \geq 0$ for μ-a.e. t and $0 \leq \eta(t) \leq \beta + \gamma$ for a.e. t, satisfying :

- (adjoint equation)

Main result

Theorem

Let (x_*, u_*) be a global minimizer satisfying the outward (or inward) pointing condition. Then there exist a BV adjoint vector $p : [0, T] \to R^n$,a finite signed Radon measure μ on $[0, T]$, and measurable vectors $\xi, \eta : [0, T] \to R^n$, with $\xi(t) \geq 0$ for μ-a.e. t and $0 \leq \eta(t) \leq \beta + \gamma$ for a.e. t, satisfying :

- (adjoint equation)

for all continuous functions $\varphi : [0, T] \to R^n$

$$
\begin{aligned}
-\int_{[0,T]} \langle \varphi(t), dp \rangle = & -\int_{[0,T]} \langle \varphi(t), \nabla_x d(x_*(t), C(t)) \rangle \xi(t)\, d\mu \\
& - \int_{[0,T]} \langle \varphi(t), \eta(t) \nabla_x^2 d(x_*(t), C(t)) p(t) \rangle\, dt \\
& + \int_{[0,T]} \langle \varphi(t), \nabla_x f(x_*(t), u_*(t)) p(t) \rangle\, dt,
\end{aligned} \tag{2}
$$

Theorem

- (transversality condition) $$-p(T) = \nabla h(x_*(T)),$$

Theorem

- (transversality condition) $\qquad -p(T) = \nabla h(x_*(T)),$
- (maximality condition)

$$\langle p(t), \nabla_u f(x_*(t), u_*(t))u_*(t)\rangle = \max_{u\in U}\langle p(t), \nabla_u f(x_*(t), u_*(t))u\rangle \quad \text{for a.e. } t \in [0, T]. \tag{3}$$

.

The Cauchy problem (P) has one and only one solution by

- M.Sene, L.Thibault, Regularization of dynamical systems associated with prox-regular moving sets,

JNCA,Vol 15, Number 4 or 5

The Cauchy problem (P) has one and only one solution by

- M.Sene, L.Thibault, Regularization of dynamical systems associated with prox-regular moving sets, JNCA,Vol 15, Number 4 or 5

and also that for every fixed control $u_* \in U$ (given a minimizer u_*) the sequence x^*_ε of solutions to

$$\begin{cases} \dot{x}(t) = -\frac{1}{\varepsilon}(x(t) - proj_{C(t)}(x(t))) + f(x(t), u_*(t)), \\ x(0) = x_0 \in C(0) , \end{cases} \tag{4}$$

The Cauchy problem (P) has one and only one solution by

- M.Sene, L.Thibault, Regularization of dynamical systems associated with prox-regular moving sets,

JNCA,Vol 15, Number 4 or 5

and also that for every fixed control $u_* \in U$ (given a minimizer u_*) the sequence x^*_ε of solutions to

$$\begin{cases} \dot{x}(t) = -\frac{1}{\varepsilon}(x(t) - proj_{C(t)}(x(t))) + f(x(t), u_*(t)), \\ x(0) = x_0 \in C(0) , \end{cases} \tag{4}$$

converge weakly in $W^{1,2}([0, T], R^n)$ to the solution of

$$\begin{cases} \dot{x}(t) \in -N_{C(t)}(x(t)) + f(x(t), u_*(t))), \\ x(0) = x_0 \in C(0) , \end{cases} \tag{5}$$

Regularised Minimization Problem

Regularised Minimization Problem

Let the problem P_ε

$$\text{Minimize } J(x,u;u_*) := h(x(T)) + \frac{1}{2}\int_0^T \|u(t)-u_*(t)\|^2 dt$$

Regularised Minimization Problem

Let the problem P_ε

$$\text{Minimize } J(x,u;u_*) := h(x(T)) + \frac{1}{2}\int_0^T \|u(t)-u_*(t)\|^2 dt$$

Subject to

$$\dot{x}(t) = -\frac{1}{\varepsilon}(x(t) - proj_{C(t)}(x(t))) + f(x(t),u(t)), \qquad (6)$$

$$x(0) = x_0$$

over $u : [0,T] \rightsquigarrow U$, u is measurable.

Regularised Minimization Problem

Let the problem P_ε

$$\text{Minimize } J(x,u;u_*) := h(x(T)) + \frac{1}{2}\int_0^T \|u(t) - u_*(t)\|^2 dt$$

Subject to

$$\dot{x}(t) = -\frac{1}{\varepsilon}(x(t) - proj_{C(t)}(x(t))) + f(x(t), u(t)), \qquad (6)$$

$$x(0) = x_0$$

over $u : [0, T] \rightsquigarrow U$, u is measurable.

• The corresponding solutions x_ε of (6) are uniformly Lipschitz, with Lipschitz constant $\gamma + 2\beta$.

Regularised Minimization Problem

Let the problem P_ε

$$\text{Minimize } J(x, u; u_*) := h(x(T)) + \frac{1}{2}\int_0^T \|u(t) - u_*(t)\|^2 dt$$

Subject to

$$\dot{x}(t) = -\frac{1}{\varepsilon}(x(t) - proj_{C(t)}(x(t))) + f(x(t), u(t)), \quad (6)$$

$$x(0) = x_0$$

over $u : [0, T] \rightsquigarrow U$, u is measurable.

• The corresponding solutions x_ε of (6) are uniformly Lipschitz, with Lipschitz constant $\gamma + 2\beta$.

• Let $(x_\varepsilon, u_\varepsilon)$ be solution of the problem (P_ε). Then there exists a sequence $\varepsilon_n \downarrow 0$ such that

$$x_{\varepsilon_n} \to x_* \text{ weakly in } W^{1,2}([0, T]; R^n)$$

$$u_{\varepsilon_n} \to u_* \text{ strongly in } L^2([0, T]; R^m).$$

.

Adjoint equation

The adjoint equation for P_ε

$$\begin{cases} -\dot{p}_\varepsilon(t) & = p_\varepsilon(t)\left(\frac{-1}{2\varepsilon}\nabla_x^2 d^2(x_\varepsilon(t), C(t)) + \nabla_x f(x_\varepsilon(t), u_\varepsilon(t))\right) \\ -p_\varepsilon(T) & = \nabla h(x_\varepsilon(T)), \end{cases} \tag{7}$$

Adjoint equation

The adjoint equation for P_ε

$$\begin{cases} -\dot{p}_\varepsilon(t) & = p_\varepsilon(t)\left(\frac{-1}{2\varepsilon}\nabla_x^2 d^2(x_\varepsilon(t), C(t)) + \nabla_x f(x_\varepsilon(t), u_\varepsilon(t))\right) \\ -p_\varepsilon(T) & = \nabla h(x_\varepsilon(T)), \end{cases} \tag{7}$$

$$\frac{-1}{2\varepsilon}\nabla_x^2 d^2(x_\varepsilon(t), C(t)) = \frac{-1}{\varepsilon}\Big(d(x_\varepsilon(t), C(t))\nabla_x^2 d(x_\varepsilon(t), C(t)) + \nabla_x d(x_\varepsilon(t), C(t)) \otimes \nabla_x d(x_\varepsilon(t), C(t))\Big) \tag{8}$$

Monotonicity of The Distance

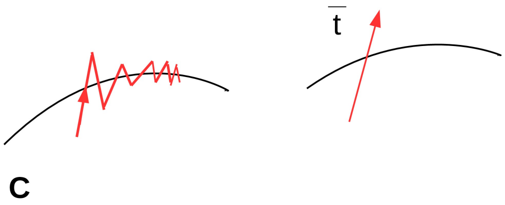
$$\nabla_x d(x_\varepsilon(t), C(t)) \otimes \nabla_x d(x_\varepsilon(t), C(t))$$

Monotonicity of The Distance

$$\nabla_x d(x_\varepsilon(t), C(t)) \otimes \nabla_x d(x_\varepsilon(t), C(t))$$
$$I_\partial := \{t \in [0, T] : x_*(t) \in \partial C(t)\}$$

Monotonicity of The Distance

$$\nabla_x d(x_\varepsilon(t), C(t)) \otimes \nabla_x d(x_\varepsilon(t), C(t))$$
$$I_\partial := \{t \in [0,T] : x_*(t) \in \partial C(t)\}$$



So in this work we requiring a **strong*outward pointing condition*** on f in order to treat the discontinuity of second derivatives of the squared distance function at the boundary of $C(t)$.

Problem without the strong condition

We consider the problem of minimizing the cost $h(x(T))$ at the endpoint of a trajectory x subject to the finite dimensional dynamics

$$\dot{x} \in -N_C(x) + f(x, u), \quad x(0) = x_0,$$

Problem without the strong condition

We consider the problem of minimizing the cost $h(x(T))$ at the endpoint of a trajectory x subject to the finite dimensional dynamics

$$\dot{x} \in -N_C(x) + f(x,u), \quad x(0) = x_0,$$

The regularized problem

$$\dot{x}(t) = \frac{-1}{\varepsilon} \nabla \Psi(x(t)) + f(x(t), u(t)),\ x(0) = x_0, \tag{9}$$

$$\Psi(x) = \frac{1}{3} \psi^3(x)\, 1_{(0,+\infty)}(\psi(x)).$$

The adjoint equation

$$-\dot{p}_\varepsilon(t) = \Big(\frac{-1}{\varepsilon}\nabla^2\Psi(x_\varepsilon(t)) + \nabla_x f(x_\varepsilon(t), u_\varepsilon(t))\Big)p_\varepsilon(t)$$

The adjoint equation

$$-\dot{p}_\varepsilon(t) = \Big(\frac{-1}{\varepsilon}\nabla^2\Psi(x_\varepsilon(t)) + \nabla_x f(x_\varepsilon(t), u_\varepsilon(t))\Big)p_\varepsilon(t)$$

$$\begin{aligned}
-\int_{[0,T]}\langle\varphi(t), dp(t)\rangle = &-\int_{[0,T]}\langle\varphi(t), n_*(t)\rangle\xi(t)\,d\nu(t)\\
&-\int_{[0,T]}\langle\varphi(t), \nabla_x^2 d(x_*(t))p(t)\rangle\eta(t)\,dt\\
&+\int_{[0,T]}\langle\varphi(t), \nabla_x f(x_*(t), u_*(t))p(t)\rangle\,dt,
\end{aligned}$$

Transversality condition

$$-p_\varepsilon(T) = \nabla h(x_\varepsilon(T))$$

Transversality condition

$$-p_\varepsilon(T) = \nabla h(x_\varepsilon(T))$$

$$-p(T) = \nabla h(x_*(T))$$

Transversality condition

$$-p_\varepsilon(T) = \nabla h(x_\varepsilon(T))$$

$$-p(T) = \nabla h(x_*(T))$$

Maximality condition

$$\langle p_\varepsilon(t), \nabla_u f(x_\varepsilon(t), u_\varepsilon(t))u_\varepsilon(t)\rangle - \langle u_\varepsilon(t) - u_*(t), u_\varepsilon(t)\rangle = \max_{u\in U}\{\langle p_\varepsilon(t), \nabla_u f(x_\varepsilon(t), u_\varepsilon(t))u\rangle - \langle u_\varepsilon(t) - u_*(t), u\rangle\}$$

Transversality condition

$$-p_\varepsilon(T) = \nabla h(x_\varepsilon(T))$$

$$-p(T) = \nabla h(x_*(T))$$

Maximality condition

$$\langle p_\varepsilon(t), \nabla_u f(x_\varepsilon(t), u_\varepsilon(t))u_\varepsilon(t)\rangle - \langle u_\varepsilon(t) - u_*(t), u_\varepsilon(t)\rangle = \max_{u\in U}\{\langle p_\varepsilon(t), \nabla_u f(x_\varepsilon(t), u_\varepsilon(t))u\rangle - \langle u_\varepsilon(t) - u_*(t), u\rangle\}$$

$$\langle p(t), \nabla_u f(x_*(t), u_*(t))u_*(t)\rangle = \max_{u\in U}\langle p(t), \nabla_u f(x_*(t), u_*(t))u\rangle \quad \text{for a.e. } t \in [0, T].$$

Difference between the two results

$$\dot{x} \in -N_{C(t)}(x) + f(x,u), \quad x(0) = x_0,$$

Difference between the two results

$$\dot{x} \in -N_{C(t)}(x) + f(x,u), \quad x(0) = x_0,$$

$C(t)$ **is moving , we require the outward pointing condition.**

Difference between the two results

$$\dot{x} \in -N_{C(t)}(x) + f(x,u), \quad x(0) = x_0,$$

$C(t)$ **is moving , we require the outward pointing condition.**

$$\dot{x} \in -N_{C}(x) + f(x,u), \quad x(0) = x_0,$$

Difference between the two results

$$\dot{x} \in -N_{C(t)}(x) + f(x,u), \quad x(0) = x_0,$$

$C(t)$ **is moving , we require the outward pointing condition.**

$$\dot{x} \in -N_{C}(x) + f(x,u), \quad x(0) = x_0,$$

$C(t)$ **is constant, but we do not require the outward pointing condition.**

Example

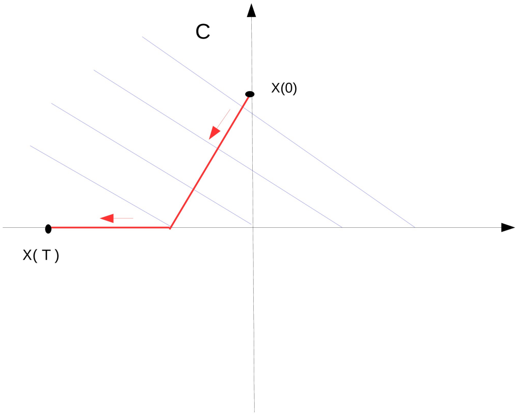
Example

The state space is $R^2 \ni (x, y)$, the constraint $C(t)$ is constant and equals $C := \{(x, y) : y \geq 0\}$, the upper half plane.
We wish to minimize $x(1) + y(1)$ subject to

$$\begin{cases} (\dot{x}(t), \dot{y}(t)) & \in -N_C(x(t), y(t)) + (u_1(t), u_2(t)) \\ (x(0), y(0)) & = (0, y_0), \qquad y_0 \geq 0, \end{cases} \tag{10}$$

where the controls (u_1, u_2) belong to $[-1, 1] \times [-1, \frac{-1}{2}] =: U$.

Example



Example

$$\dot{p}^x = 0,\ \dot{p}^y = 0 \text{ a.e. on } [0, T],\ p^x(1) = p^y(1) = -1$$

Example

$$\dot{p}^x = 0,\ \dot{p}^y = 0 \text{ a.e. on } [0, T],\ p^x(1) = p^y(1) = -1$$

p^x is continous at $t = 1$ and $p^y(1-) + 1 = 1$, namely $p^y(1-) = 0$.

Example

Example

The adjoint vector (p^x, p^y) is :

$$\begin{aligned} p^x(t) &= -1 \quad \text{for all } t \in [0,1] \\ p^y(t) &= 0 \quad \text{for all } t \in [0,1[\\ p^y(1) &= -1. \end{aligned}$$

The maximum condition

$$\langle(-1,-1),(u_*^x,u_*^y)\rangle = \max_{|u_1|\le 1, 1\le u_2 \le \frac{-1}{2}} \langle(-1,-1),(u_1,u_2)\rangle \quad t=1$$

$$\langle(-1,0),(u_*^x,u_*^y)\rangle = \max_{|u_1|\le 1, 1\le u_2 \le \frac{-1}{2}} \langle(-1,0),(u_1,u_2)\rangle \quad 0\le t<1$$

which gives $u_*^x = -1$ and $u_*^y(1) = -1$, while no information is available for $u_*^y(t)$ if $0 \le t < 1$ (**the optimal solution belongs to** $intC$). If we assume that u_*^y is constant, then the expected optimal control $u_*^y = -1$ is found.

References

Ch. Arroud and G. Colombo.A Maximum Principle for the controlled Sweeping Process. J of SVVA, 10.1007/s11228-017-0400-4 (2017).

S. Adly, F. Nacry, and L. Tibault, *Preservation of prox-regularity of sets with application to constrained optimization* SIAM J. Optim. 26 (2016), 448-473.

L. Ambrosio, Halil Mete Soner, *Level set approach to mean curvature flow in arbitrary codimension* J. Differential Geom. 43 (1996), 693-737.

A. Bressan and B. Piccoli, *Introduction to the Mathematical Theory of Control* (2007).

G. Colombo, R. Henrion, N.D. Hoang and B.S. Mordukhovich, *Optimal control of the sweeping process.* Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms 19 (2012), No. 1-2, 117-159.

G. Colombo, R. Henrion, N.D. Hoang and B.S. Mordukhovich, *Discrete approximations of a controlled sweeping process.* Set-Valued and Variational Analysis 23 (2015), 69-86.

M. Sene, L. Thibault, *Regularization of dynamical systems associated with prox-regular moving sets*, Journal of Nonlinear and Convex Analysis 15 (2014), 647-663.

O.S. Serea *Optimality conditions for reflecting boundary control problems* Nonlinear differ. Equ. Appl.20 (2013), 1225-1242.

H.J. Sussmann, *A Pontryagin Maximum Principle for systems of flows*, in Recent Advances in Learning and Control,.

R.B. Vinter, *Optimal Control*, Birkhäuser, Boston (2000)