Growth Models for Tree Stems and Vines

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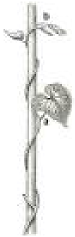
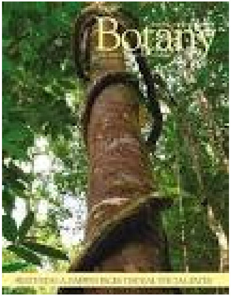
Stabilizing stem growth

what kind of stabilizing feedback is used here?

Growth in the presence of obstacles



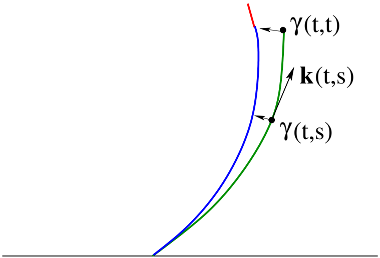


Are the growth equations still well posed, when an obstacle is present?

What additional feedback produces curling around other branches?

A model of stem growth (F. Ancona, A.B., O. Glass)

- New cells are born at the tip of the stem
- Their length grows in time, at an exponentially decreasing rate



$\gamma(t, s) =$ position at time t of the cell born at time s

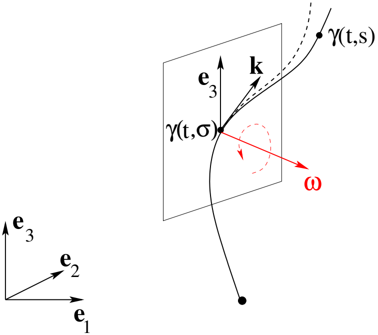
Unit tangent vector to the stem: $\mathbf{k}(t, s) = \partial_s \gamma(t, s)$

Stabilizing growth in the vertical direction

stem not vertical $\Longrightarrow$ local change in curvature

$e^{-\beta(t-s)}$ = stiffness factor, $\omega = \mathbf{k}(t,\sigma) \times \mathbf{e}_3$ = angular velocity

$$\partial_t \gamma(t,s) \;=\; \int_0^s e^{-\beta(t-\sigma)} \big(\mathbf{k}(t,\sigma) \times \mathbf{e}_3\big) \times \big(\gamma(t,s) - \gamma(t,\sigma)\big)\, d\sigma$$

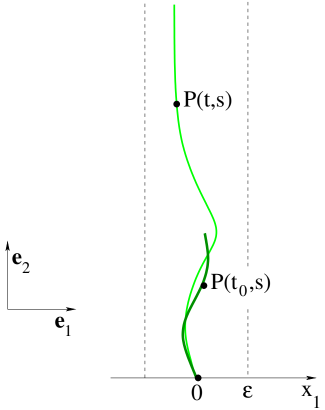


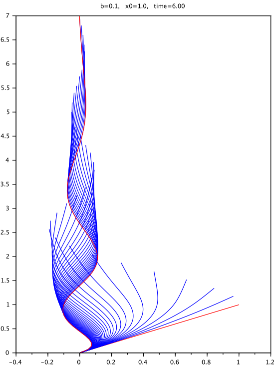
$$\begin{cases} \gamma(t_0, s) \;=\; \overline{\gamma}(s) & s \in [0, t_0] \\ \gamma_{ss}(t,s)\Big|_{s=t} \;=\; 0 & \end{cases}$$

We say that the growth equation is **stable in the vertical direction** if for any initial time $t_0 > 0$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that

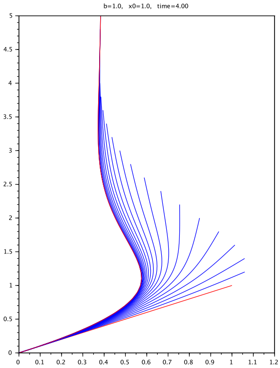
$$\big|\mathbf{e}_1 \cdot \partial_s \gamma(t_0, s)\big| \;\leq\; \delta \qquad \text{for all } s \in [0, t_0] \qquad\qquad \text{implies}$$

$$\Big|\mathbf{e}_1 \cdot \gamma(t, s)\Big| \;\leq\; \varepsilon \qquad \Big|\mathbf{e}_1 \cdot \partial_s \gamma(t, s)\Big| \;\leq\; \varepsilon\,, \qquad \text{for all } t > t_0, \quad s \in [0, t]$$

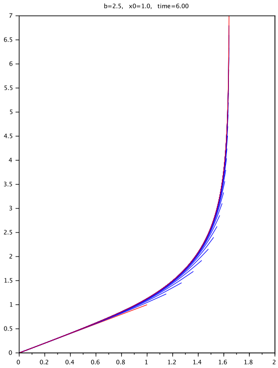




$\beta = 0.1$



$\beta = 1.0$



$\beta = 2.5$

- stability is always achieved
- increasing the stiffness reduces oscillations

Analytical results (F. Ancona, A.B., O. Glass, 2017)

$$\beta = \text{stiffening constant}$$

- If $\beta^4 - \beta^3 \geq 4$, then the growth is stable in the vertical direction (non-oscillatory regime: $\beta \geq \beta_0 \approx 1.7485$)
- If $\beta > \beta^* = (48 + \sqrt{9504})/160$, then growth is still stable in the vertical direction (oscillatory regime, $\beta^* \approx 0.9093$)
- Stability apparently holds for all $\beta > 0$ (??)

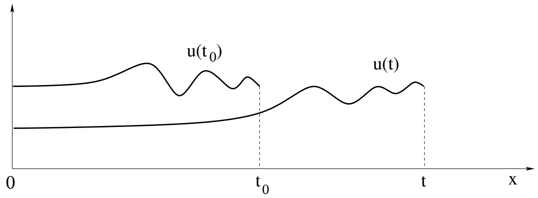
A linearized problem

Key step: prove stability for the **linearized system**

$$u_t + u_x = -\int_x^{\infty} e^{-\beta y} u(y)\, dy \qquad \text{for} \quad x \in [0, +\infty[$$

with Neumann boundary condition at $x = 0$

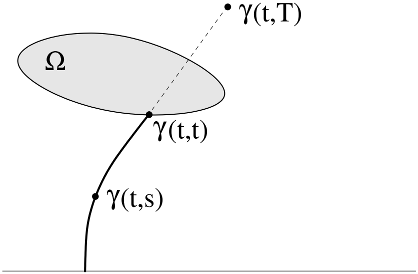
$$u_x(t, 0) = 0$$



$$u(t, x) \approx \mathbf{e}_1 \cdot \gamma_s(t,\ t - x)$$

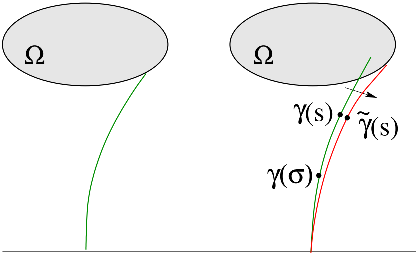
Growth with obstacles

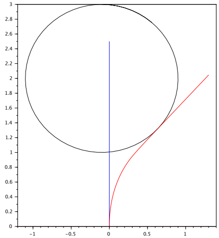
- Obstacle: $\Omega \subset \mathbb{R}^3$ open set, with smooth boundary
- At time t the stem $\gamma(t, \cdot)$ is a curve of length t remaining outside the obstacle



- Basic space: $\gamma(t, \cdot) \in H^2([0, T]\,;\,\mathbb{R}^3)$
- For $s \in]t, T]$ define $\gamma(t, s) = \gamma(t, t) + \gamma_s(t, t)(s - t)$
- Time-dependent constraint: $\gamma(t, s) \notin \Omega$ for $s \in [0, t]$

A *push-out* operator





$\omega(\sigma)$ = additional bending of the stem caused the obstacle, at the point $\gamma(\sigma)$

$$\widetilde{\gamma}(s) - \gamma(s) \;=\; \int_0^s \omega(\sigma) \times (\gamma(s) - \gamma(\sigma))d\sigma \qquad\qquad s \in [0, t]$$

Among all infinitesimal deformations that push the stem outside the obstacle,

minimize the elastic energy: $\mathcal{E} \;=\; \dfrac{1}{2}\displaystyle\int_0^t e^{\beta(t-\sigma)}|\omega(\sigma)|^2\, d\sigma$

The evolution equation with constraints

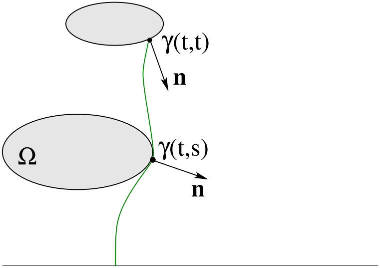
$\Psi(\sigma) = \Psi\Big(t, \sigma, \gamma(t,\sigma), \gamma_s(t,\sigma)\Big)$ = upward bending, as response to gravity

without obstacle: $$\gamma_t(t,s) \;=\; \int_0^s \Psi(\sigma) \times \Big(\gamma(t,s) - \gamma(t,\sigma)\Big)\, d\sigma$$

with obstacle: $$\gamma_t(t,s) \;=\; \int_0^s \Big(\Psi(\sigma) + \bar\omega(t,\sigma)\Big) \times \Big(\gamma(t,s) - \gamma(t,\sigma)\Big)\, d\sigma$$

$$\bar\omega(\cdot) \;=\; \underset{\omega(\cdot)}{\operatorname{argmin}} \int_0^t e^{\beta(t-\sigma)} |\omega(\sigma)|^2 \, d\sigma$$

The instantaneous minimization problem



$$\bar{\omega}(\cdot) \;=\; \underset{\omega(\cdot)}{\operatorname{argmin}} \, \frac{1}{2}\int_0^t e^{\beta(t-\sigma)}|\omega(\sigma)|^2\,d\sigma$$

subject to the unilateral constraints at points of contact:

$$\langle \gamma_t(t,s)\,,\; \mathbf{n}(t,s)\rangle \;\geq\; 0 \qquad \text{whenever} \;\; \gamma(t,s)\in\partial\Omega,$$

If the tip of the stem touches the obstacle, one also needs

$$\langle \gamma_s(t,t)+\gamma_t(t,t)\,,\; \mathbf{n}(t,t)\rangle \;\geq\; 0$$

Necessary conditions for optimality

The solution to the constrained minimization problem

$$\bar{\omega}(\cdot) \;=\; \operatorname*{argmin}_{\omega \in \mathcal{A}} \int_0^t e^{\beta(t-\sigma)} |\omega(\sigma)|^2 \, d\sigma$$

admits the representation:

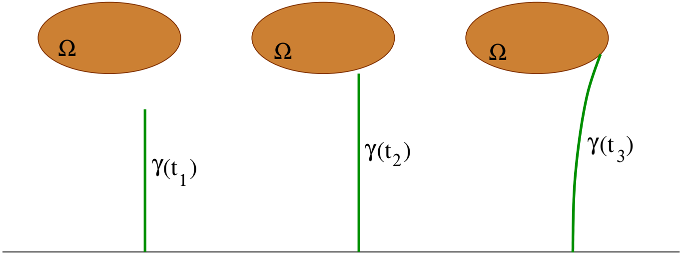
$$\bar{\omega}(s) \;=\; -\int_s^t \left(\int_{[\sigma,t]} e^{-\beta(t-s)} \, \mathbf{n}(\gamma(t,s'))d\mu(s') \right) \times \gamma_s(t,\sigma)d\sigma$$

where μ is a positive measure, supported on the contact set

$$\chi(t) \doteq \Big\{ s' \in [0,t] \, ; \;\; \gamma(t,s') \in \partial\Omega \Big\}$$

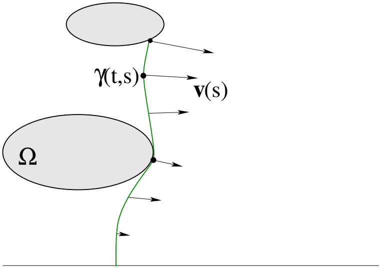
Discontinuous evolution problems

Growth with obstacles yields an evolution equation with discontinuous right hand side



Can be reformulated as a differential inclusion with u.s.c. right hand side

A cone of admissible reactions



$$\chi(t) \doteq \{s' \in [0,t]\,;\ \gamma(t,s') \in \partial\Omega\}$$

$=$ contact set

Cone of admissible velocities produced by the obstacle reaction:

$$\Lambda(\gamma(t)) \doteq \Big\{ \mathbf{v} : [0,t] \mapsto \mathbb{R}^3\,; \qquad \mathbf{v}(s) = \int \mathcal{K}(s,s')\, d\mu(s')$$

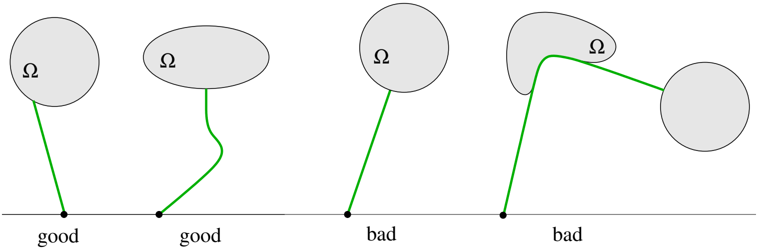
$$\text{for some positive measure } \mu \text{ supported on } \chi(t) \Big\}$$

differential inclusion: $$\gamma_t(t,s) \in \int_0^s \Psi(\sigma) \times \Big(\gamma(t,s) - \gamma(t,\sigma)\Big)\, d\sigma + \Lambda(\gamma(t))$$

Well-posedness of the stem growth model with obstacle

Theorem (A.B. - M.Palladino, 2016-17)

Solutions exist and are unique except if a (highly non-generic) breakdown configuration occurs.



(B) *The tip of the stem touches the obstacle perpendicularly, namely*

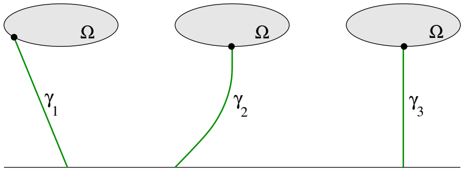
$$\overline{\gamma}(t_0) \in \partial\Omega\,, \qquad \overline{\gamma}_s(t_0) = -\mathbf{n}(\overline{\gamma}(t_0)). \tag{1}$$

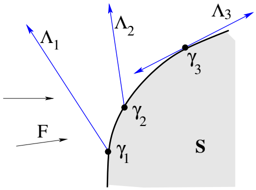
Moreover,

$$\overline{\gamma}_{ss}(s) = 0 \qquad \textit{for all } s \in\ ]0,t[\ \textit{ such that } \overline{\gamma}(s) \notin \partial\Omega\,. \tag{2}$$

Geometric interpretation

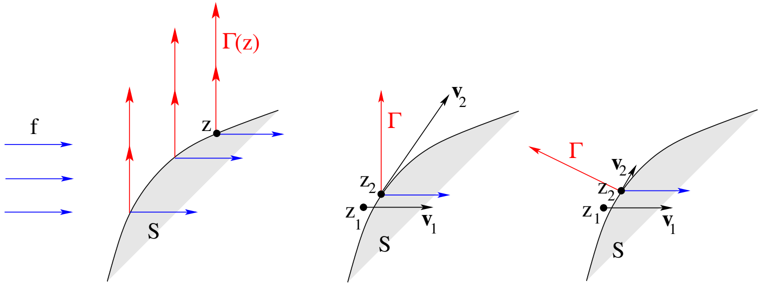
$$\frac{d}{dt}\gamma(t) \;\in\; F(\gamma(t)) + \Lambda(\gamma(t)) \qquad\qquad \gamma \notin \mathbf{S}$$





Well posedness of evolution equations with constraints

$$\frac{d}{dt}\mathbf{z}(t) \in f(\mathbf{z}(t)) + \Gamma(\mathbf{z}(t)) \qquad\qquad \mathbf{z} \notin \mathbf{S}$$

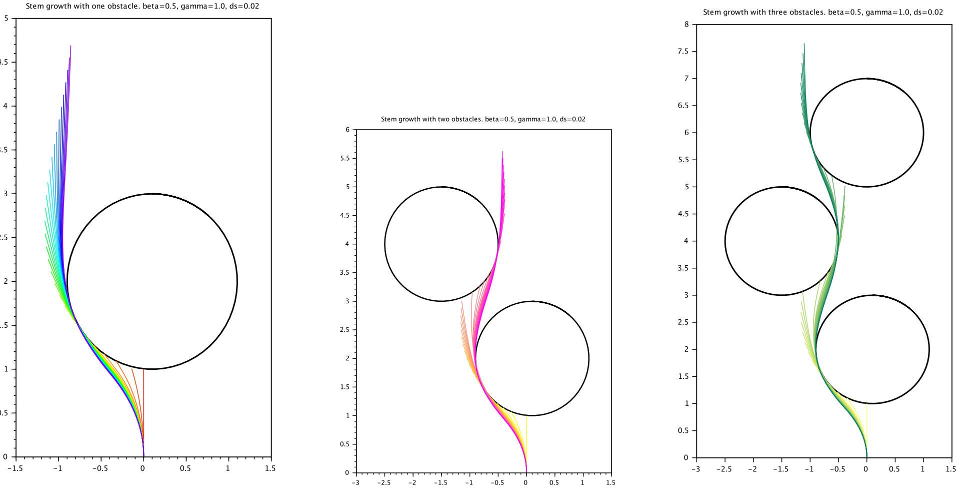


If f is Lipschitz and $\Gamma(z) = N_{\mathbf{S}}(z) =$ outer normal cone to $\mathbf{S}$ at a boundary point $\mathbf{z}$, then

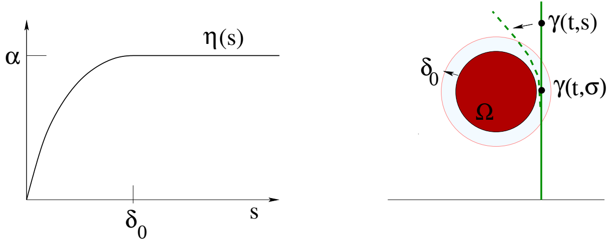
$$\frac{d}{dt}\big\|\mathbf{z}_1(t) - \mathbf{z}_2(t)\big\| \leq C\,\big\|\mathbf{z}_1(t) - \mathbf{z}_2(t)\big\| \tag{1}$$

Main idea: introduce an equivalent "Riemann metric" so that the cones $\Gamma(\mathbf{z})$ become perpendicular to the boundary of $\mathbf{S}$

Numerical simulations (Wen Shen, 2016)



- The stem bends toward the obstacle, at points which are sufficiently close (i.e., at a distance $< \delta_0$ from the obstacle)

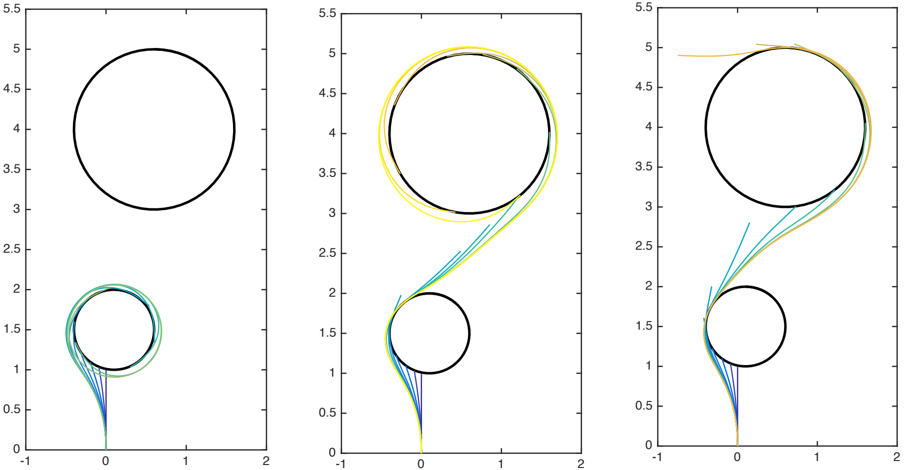


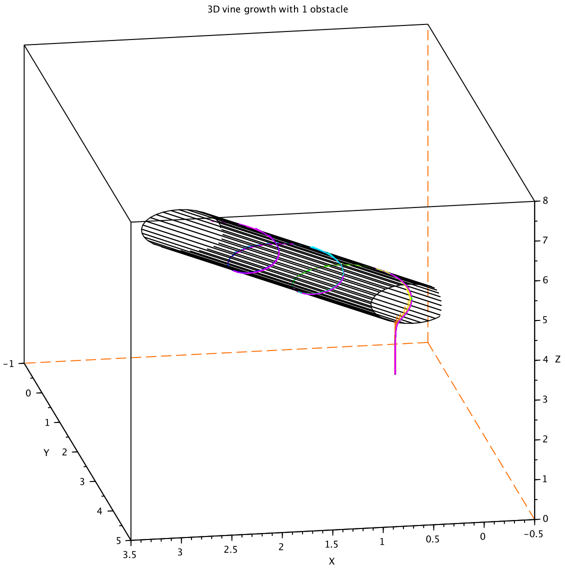
$$\psi(x) \doteq \eta\big(d(x,\Omega)\big)$$

In the case of a vine that clings to a branch of another tree, the evolution equation contains an additional term ($\Longrightarrow$ bending toward the obstacle)

$$\gamma_t(t,s) = \int_0^s e^{-\beta(t-\sigma)}\Big(\nabla\psi(\gamma(t,\sigma))\times\gamma_s(t,\sigma)\Big)\times\big(\gamma(t,s)-\gamma(t,\sigma)\big)\,d\sigma \;+\; \cdots$$

Numerical simulations (Wen Shen)




3D vine growth with 1 obstacle
X
Y
Z
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
−1
0
1
2
3
4
5
0
1
2
3
4
5
6
7
8

References

[1] F. Ancona, A. Bressan, O. Glass, and W. Shen, Feedback stabilization of stem growth, *J. Dynam. Diff. Equat.*, submitted.

[2] A. Bressan, M. Palladino, and W. Shen, Growth models for tree stems and vines, *J. Differential Equations*, to appear.

[3] A. Bressan and M. Palladino, Well-posedness of a model for the growth of vines in the presence of obstacles, *Discr. Cont. Dyn. Syst.*, to appear.

- MATLAB source codes: http://math.psu.edu/shen_w/STEM-VINE-SIM/