On Evolution Equations Having Hypomonotonicities of Opposite Sign

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To take into account the nonsmoothness of the boundary of the convex set $C(t)$, the law of motion is formulated as

$$\begin{cases} \frac{du}{dt}(t) \in -N_{C(t)}(u(t)), a.e. t \in [0,T] \\ u(0) = u_0 \in C(0) \\ u(t) \in C(t) \quad \forall t \in [0,T], \end{cases} \tag{1}$$

where $N_{C(t)}(\cdot)$ denotes the normal cone operator of the convex set $C(t)$ in the sense of convex analysis in a Hilbert space H.

Existence and uniqueness of solutions of such systems and their classical variants :

▷ subjected to perturbation forces,

▷ non convex prox-regular sets C,

▷ state dependent set $C(t, x)$

▷ second-order sweeping processes ...

have been considered by many authors in the literature.

Introduction

On the other hand,
in [A. Bressan, A. Cellina, and G. Colombo ;'89]

$$\dot{x}(t) \in F(x(t)), \quad F(x) \subset \partial g(x), \quad x(0) = x_0, \qquad (2)$$

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• The lack of the minus sign in (2) yields that the distance among solutions increases and typically there is no uniqueness.

• Existence of solutions depends on arguments of convex analysis. This result has been generalized by many authors in different ways.

The dynamics

The purpose of the present talk is to study a Cauchy problem

$$\dot{x}(t) \in F(x(t)) - N_C(x(t)), \quad F(x) \subset \partial g(x), \quad x(0) = x_0 \in C, \tag{3}$$

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▷ the set valued mapping F **is not necessarily the whole subdifferential** of g , and we **take the plus sign**, instead of the classical minus.
▷ The system (3) can be considered as a non convex cyclical monotone differential inclusion under control term $u(t) \in -N_C(\cdot)$ which guarantees that the trajectory $x(t)$ always belongs to the desired set C for all $t \in [0, T]$.
We prove **(local) existence** of solutions.

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the authors proved the existence of solutions of problem (3),

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• the closed set C was supposed to be **compact and uniformly r-prox-regular**,

It is also worth mentioning that problem (3) in the more general form

$$\dot{x}(t) \in F(x(t)) - \partial V(x(t)),$$
$$F(x) \subset \partial g(x), \quad x(0) = x_0 \in domV, \tag{4}$$

- g is φ-convex of order two
- V has a φ-monotone subdifferential of order two (shortly $V \in MS(2)$)

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Notice that we can obtain (3) from (4), by taking $V = \delta_C$, the indicator function of the set C, i.e. $\delta_C(x) = 0$ for $X \in C$ and $\delta_C(x) = \infty$ for $x \in H \setminus C$.

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Indeed, as C is locally pox-reqular, then $V = \delta_C$ is pln (proposition 3.3 in [M.Mazade ad L.Thibault, '12) and so $MS(2)$.

Despite the similarity of (3) and (4), the problems are quite different, since in general with $V = \delta_C$ the level set $\{x \in H; V(x) \leq r\}$ is not compact, and **these were basic assumption** in previous works.

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In this talk, we give a new approach, in which **the compactness assumption is shifted** from the set C to the set-valued map F.

$x \longrightarrow N_C(x)$... weak smoothness

• While one would like to consider very general sets C there are limits to possible sets on which the existence of solutions (well-posedness) of sweeping processes

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• Namely, For a fixed closed subset C, the set-valued map $x \longrightarrow N_C(x)$ is not upper semicontunous which is needed for the proof of existence of solutions.

Uniformly $r-$ prox-regular set

• We say that C is uniformly r-prox-regular provided the inequality

$$\langle v, y - x\rangle \leq \frac{1}{2} r \|y - x\|^2$$

holds $\forall x, y \in C$, $\forall v$ the unit external normal to C.

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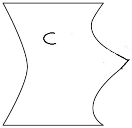
• an external tangent ball with radius smaller than $\frac{1}{2} r$ depending on the tangency point x, can be rolled around C with it's closure touching C only at x.



• In particular uniformly $r-$prox-regular sets can have **outside corners** and **outside cusps**, but not **inside corners**.

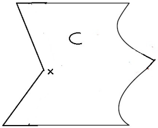
locally prox-regular set

Definition

For positive real numbers r and α, the closed set C is said to be (r, α) prox-regular at a point $\overline{x} \in C$ provided that for any $x \in C \cap B(\overline{x}, \alpha)$ and any $v \in N_C^P(x)$ such that $\|v\| \leq r$, one has

$$x = proj_C(x + v).$$

The set C is said to be r-uniformly prox-regular when $\alpha = +\infty$.



primal lower nice function

Let us now define primal lower nice function in a quantified way [Mazade and Thibault '12].

Definition

Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. The function f is said to be primal lower nice (pln, for short) on an open convex set $\mathcal{O}$ with $\mathcal{O} \cap domf \neq \emptyset$ if there exists some real number $c \geq 0$ such that for all $x \in \mathcal{O} \cap dom\partial^p f(x)$ and for all $v \in \partial^p f(x)$ on has

$$f(y) \geq f(x) + \langle v, y - x \rangle - c(1 + \|v\|)\|y - x\|^2, \qquad (5)$$

for each $y \in \mathcal{O}$.

The real $c \geq 0$ will be called a pln constant for f over $\mathcal{O}$ and we will say that f is c -pln on $\mathcal{O}$.

Auxiliary existence result

We have the following existence and uniqueness result [Mazade and Thibault '12].
Let C be an (r, α)-prox-regular set at the point $x_0 \in C$ and let any real number $\eta_0 \in]0, \alpha[$. Then for any $\overline{x} \in B(x_0, \alpha - \eta_0) \cap C$, any positive real number $\tau \leq T_0 - T$, and any mapping $h \in L^1([0, T], H)$ such that $\int_{T_0}^{T_0+\tau} \|h(s)\| ds < \eta_0/2$, the differential variational inequality

$$\dot{x}(t) \in -N_C(x(t)) + h(t), \quad x(0) = \overline{x}, a.e \quad t \in [T_0, T_0 + \tau], \tag{6}$$

has an absolutely continuous solution $x : [T_0, T_0 + \tau] \to B(\overline{x}, \eta_0) \cap C$. Moreover, $\|\dot{x}(t)\| \leq \|\dot{x}(t) - h(t)\| + \|h(t)\| \leq 2\|h(t)\| \quad a.e \quad t \in [T_0, T_0 + \tau]$.

necessarily condition of existence

With the previous assumptions we proved the following property of solution.

Lemma

The absolutely continuous solution $x : [T_0, T_0 + \tau] \to B(\overline{x}, \eta_0) \cap C$ *of*

$$\dot{x}(t) \in -N_C(x(t)) + h(t), \quad x(0) = \overline{x}, a.e \quad t \in [T_0, T_0 + \tau],$$

satisfies the following property

$$\langle \dot{x}(t), \dot{x}(t) \rangle = \langle h(t), \dot{x}(t) \rangle \quad a.e \quad t \in [T_0, T_0 + \tau].$$

Main result

we prove the existence of local solutions to the evolution problem

$$\dot{x}(t) \in F(x(t)) - N_C(x(t)), \quad x(0) = x_0 \in C, a.e \quad t \in [0, T], \tag{7}$$

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(HC_1) : the closed set C is (r, α)-prox-regular at the point $x_0 \in C$;

(HF_1) : $\mathcal{O} \subset H$ is an open convex set containing $\overline{B}(x_0, \eta_0)$ and $F : \mathcal{O} \to 2^H$ is an upper semicontinous set-valued mapping with nonempty weakly compact values ;

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(HF_2) : let $g : \mathcal{O} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinous function c-pln on $\mathcal{O}$ with $F(x) \subset \partial^C g(x)$, $\forall x \in \mathcal{O}$ such that g is locally bounded from above on $\mathcal{O}$.

By a solution of inclusion (7) we mean an absolutely continuous function $x(.) : [0, T] \to H, \quad x(0) = x_0 \in C$, such that the inclusion

$$\dot{x}(t) \in -N_C(x(t)) + f(t)$$

holds a.e. for some $f \in L^2([0, T], H)$ such that $f(t) \in F(x(t)) \quad a.e.$

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Theorem

Assume that H is the Hilbert space, (HC_1), (HF_1) and (HF_2) : $F(x) \subset \mathcal{K}$ for all $x \in C$ with $\mathcal{K}$ strongly compact in H hold. Then, there exists $\overline{T} > 0$ and an absolutely continuous function $x(.) : [0, \overline{T}] \rightsquigarrow B(x_0, \eta_0)$ a local solution to problem (7).

Sketch of the proof

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where α and η_0 are given by (HC_1).
Our purpose is to prove that there exists $x : [0, \overline{T}] \to B(x_0, \eta_0) \cap C$ a solution to the Cauchy problem (7)

Construction of approximates solutions

$x_k^n(.) : [t_k^n, t_{k+1}^n] \rightsquigarrow B(x_0, \eta_0) \cap C$ with $0 \leq k \leq n-1$, such that for each $k \in 0, \ldots, n-1$ (with $x_{-1}^n(t_0^n) = x_0$)

$$\dot{x}_k^n(t) \in -N_C(x_k^n(t)) + f_k^n(t), \quad x_k^n(t_k^n) = x_{k-1}^n(t_k^n), t \in [t_k^n, t_{k+1}^n],$$

$$f_k^n(t) \in F(x_{k-1}^n(t_k^n))$$

and

$$\|(\dot{x_k^n})(t)\| \leq 2M, a.e \quad t \in [t_k^n, t_{k+1}^n]$$

sequence of approximates solutions

Define $x_n(.), f_n(.) : [0, \bar{T}] \to H$ by

$$x_n(t) = \sum_{k=0}^{n-1} x_k^n(t)\chi_{[t_k^n, t_{k+1}^n]}(t), \quad f_n(t) = \sum_{k=0}^{n-1} f_k^n(t)\chi_{(t_k^n, t_{k+1}^n]}(t)$$

where χ_A is the characteristic function of the set A.
The mapping $x_n(.)$ is absolutely continuous on $[0, \bar{T}]$ with

$$x_n(t) \in B(x_0, \eta_0) \cap C; \quad \forall t \in [0, \bar{T}] \tag{8}$$

Further, putting

$$\theta_n(t) := t_k^n \quad if \quad t \in [t_k^n, t_{k+1}^n[\quad for \quad k \in 0, ..., n-1, \quad \theta_n(\bar{T}) = \tag{9}$$

we have

$$\dot{x}_n(t) \in -N_C(x_n(t)) + f_n(t), \quad x_n(0) = x_0, \quad a.e \quad t \in [0, \bar{T}] \tag{10}$$

Convergence of sequences

Let us define $Z_n(t) := \int_0^t f_n(s)ds$.
Then for all $t \in [0, \overline{T}]$ the set $\{Z_n(t), n \in N^*\}$ is contained in the strong compact set $\overline{T}\ \overline{co}(\{0\} \cup \mathcal{K})$ and so it is relatively strongly compact in H.

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▷ Then by Arzela Asccoli's theorem we get the relative compactness of the set $\{Z_n, n \in N^*\}$ with respect to the uniform convergence in $C([0, \overline{T}], H)$ and so we may assume that without loss of generality that $(Z_n)_n$ converges uniformly to some mapping Z.

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▷ then Z is absolutely continuous and $\dot{Z}(t) = f(t)$ for almost $t \in [0, \overline{T}]$

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Fix $m, n \in N^*$.
Put $\varepsilon_{m,n} := \|Z_m - Z_n\|_\infty \to 0$, and $w_n(t) := x_n(t) - Z_n(t)$, we obtain

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$$\frac{d}{dt}(\|w_m(t) - w_n(t)\|^2) \leq \frac{2M}{r}\|w_n(t) - w_m(t)\|^2 + \gamma\varepsilon_{m,n},$$

where γ is some positive constant independent of m, n and t.

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where γ is some positive constant independent of m, n and t.
As $\|w_m(0) - w_n(0)\| = 0$, the Gronwall inequality implies for almost t

$$\|w_m(t) - w_n(t)\|^2 \leq L\varepsilon_{m,n},$$

L is some positive constant independent of m, n and t. Hence $(w_n)_n$ converges uniformly to some mapping w and so $(x_n)_n$ **converges uniformly** to some mapping $x := w + Z$.

We prove that $\dot{x}(t) \in -N_C(x(t)) + f(t), \quad a.e \quad t \in [0, \bar{T}]$

By construction we have

$$-\dot{x}_n(t) + f_n(t) \in N_C(x_n(t)) \cap \overline{B}(0, M) = M\partial d_C(x_n(t)).$$

Since $(-\dot{x}_n + f_n)_n$ converges weakly in $L^2([0, \bar{T}], H)$ and $(x_n)_n$ converges strongly in $C([0, \bar{T}], H)$

By invoking a closure type Lemma

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By invoking a closure type Lemma

$$-\dot{x}(t) + f(t) \in M\partial^C d_C(x(t)), \quad a.e \quad t \in [0, \bar{T}]. \tag{11}$$

The last inclusion and $x(t) \in C$ ensure

$$\begin{cases} \dot{x}(t) \in -N_C(x(t)) + f(t), \quad a.e \quad t \in [0, \bar{T}] \\ x(t) \in B(x_0, \eta_0) \cap C; \quad \forall t \in [0, \bar{T}] \\ x(0) = x_0 \in C, \end{cases} \tag{12}$$

Convergence for the norm topology of L^2_H of the sequence $(\dot{x}_n(\cdot))$

- By exploiting that g is $c - pln$,
- Integrating

$\langle \dot{x}(t), \dot{x}(t) \rangle = \langle f(t), \dot{x}(t) \rangle$. and $\langle \dot{x}_n(t), \dot{x}_n(t) \rangle = \langle f_n(t), \dot{x}_n(t) \rangle$.
we obtain

$$\limsup_n \int_0^{\overline{T}} \|\dot{x}_n(t)\|^2 dt \leq \int_0^{\overline{T}} \|\dot{x}(t)\|^2 dt.$$

By the weak lower semicontinuity of the norm, we deduce that

$$\lim_n \int_0^{\overline{T}} \|\dot{x}_n(t)\|^2 dt = \int_0^{\overline{T}} \|\dot{x}(t)\|^2 dt,$$

which implies that $(\dot{x}_n)_n$ converges to $\dot{x}$ in the **strong topology** $L^2([0, \bar{T}], H)$.
Therefore, there exists a subsequence, still denoted by $(\dot{x}_n)_n$ which converges pointwise a.e. to $\dot{x}$.

Define the set-valued mapping $G(x) := -M\partial d_C(x)$, $x \in H$. Fix $t \in [0, \bar{T}] \setminus \mathcal{N}$, we have ($d^*(A, B) := \sup\{d(a, B) : a \in A\}$ for $A, B \subset H$)

$\dot{x}(t) \in -N_C(x(t)) + F(x(t)), \quad a.e \quad t \in [0, \bar{T}].$

Define the set-valued mapping $G(x) := -M\partial d_C(x)$, $x \in H$. Fix $t \in [0, \bar{T}] \setminus \mathcal{N}$, we have $(d^*(A, B) := \sup\{d(a, B) : a \in A\}$ for $A, B \subset H)$

$$d(\dot{x}_n(t), G(x(t))+F(x(t)) \leq d^*(G(x_n(t))+F(x_n(\theta_n(t))), G(x(t))+$$

$$\leq d^*(G(x_n(t)), G(x(t))) + d^*(F(x_n(\theta_n(t))), F(x(t))).$$

Since $\dot{x}_n(t) \to \dot{x}(t)$, $x_n(t) \to x(t)$ and $\dot{x}_n(\theta_n(t)) \to \dot{x}(t)$ strongly in H, the upper semicontinuity of F and G give

$$d(\dot{x}(t), G(x(t)) + F(x(t)) = 0.$$

As $x(t) \in C$ we finally get

$$\dot{x}(t) \in G(x(t)) + F(x(t)) = -M\partial^p d_C(x(t)) + F(x(t))$$

$$\subset -N_C(x(t)) + F(x(t)).$$

This completes the proof of Theorem. ∎

Comments and open problem

▷ In this talk, the compactness assumption is shifted from the set C to the perturbation F. More precisely, we cannot use Arzela-Ascolis theorem .
However we prove that (x_n) is a Cauchy sequence in the space $C([0,T],H)$. These arguments, used with convex-valued mapping F and uniformly prox-regular set C (see [Bounkhel and Thibault '05]), still hold for (3).
▷ The property

$$\langle \dot{x}(t), \dot{x}(t)\rangle = \langle f(t), \dot{x}(t)\rangle \quad a.e \quad t$$

guarantees the convergence for the norm topology of L^2_H of the sequence $(\dot{x}_n(\cdot))$.

Comments

▷ We can extent the existence result to the non constant set-valued map $C(t) := C + a(t)$ where C is a fixed closed locally prox-regular subset x_0, and $a(t)$ an absolutely continuous mapping from $[0, T]$ into H, with $a(0) = 0$. Also the previous property becomes

$$\langle h(t) - \dot{x}(t), \dot{x}(t) - \dot{a}(t) \rangle = 0 \quad a.e \quad t$$

open problem

Further, it would be interesting to address the validity of the preceding result to the general time-dependent when $C(t)$

$$\dot{x}(t) \in F(x(t)) - N_{C(t)}(x(t)),$$

$$\mathrm{F(x)} \subset \partial g(x), \quad x(0) = x_0 \in C(0),$$

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