Optimal piezoelectric energy harvesting strategy

Joint work with B. Kaltenbacher

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Žitná 25, Praha 1

Padova
September 26, 2017

Plan of the talk

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- Experimental observations

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- Problems in constitutive modeling, Principles of Thermodynamics

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- Optimal energy harvesting process - necessary optimality conditions

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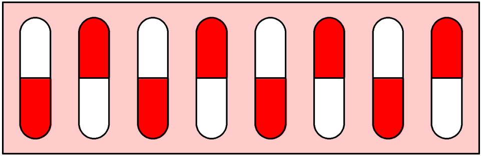
Magnetostrictive and piezoelectric materials

Magnetostrictive and piezoelectric materials exhibit mechanical deformation under the influence of electric or magnetic field and, vice versa, produce electric or magnetic field under mechanical loading.



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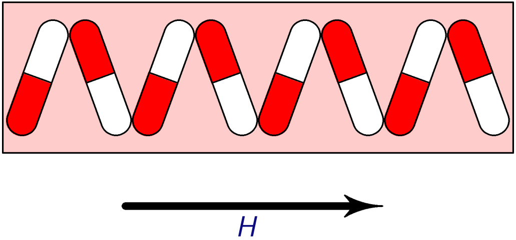

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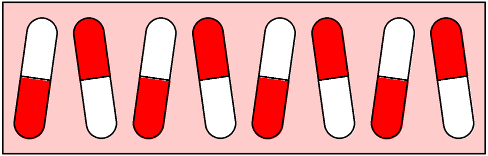

Hysteresis!

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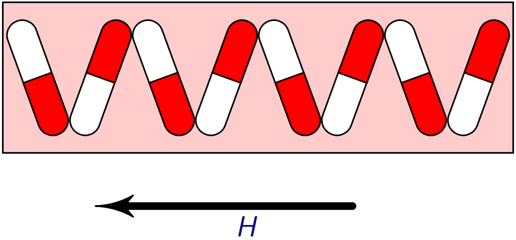

Applications: Actuators, sensors, harvesters, active or passive damping

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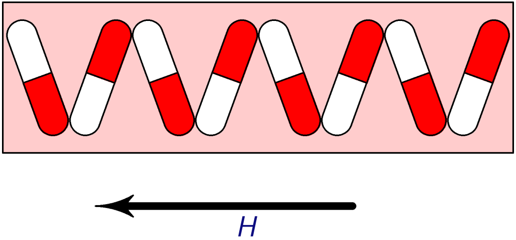
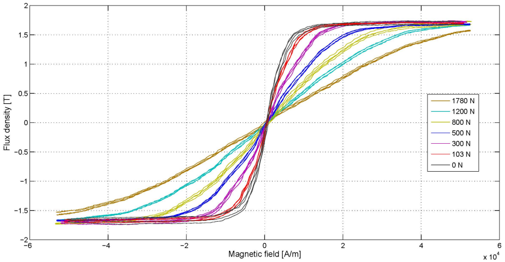


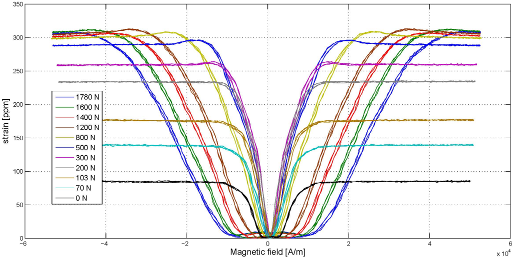
Applications: Actuators, sensors, harvesters, active or passive damping

A 2 input (e.g., strain ε and electric field E) – 2 output (dielectric displacement D and stress σ) model is necessary for describing these phenomena.

Magnetic and magnetoelastic curves of Galfenol at various preloads

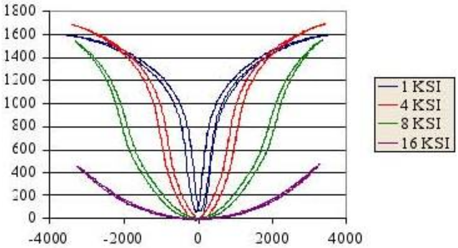
Measured by Daniele Davino, Università del Sannio, Benevento





Terfenol D, commercial presentation by Etrema Products Inc.

Strain (ppm)



Applied field (Oe $\approx$ 80 A/m)

Problems in constitutive modeling

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A constitutive relation $(D, \sigma) = \mathcal{F}[E, \varepsilon]$ is compatible with the First and the Second Principle of Thermodynamics only if there exists a free energy operator $W = \mathcal{W}[E, \varepsilon]$ such that for all isothermal processes we have

$$\dot{D}E + \dot{\varepsilon}\sigma - \dot{W} = \Delta \geq 0\,,$$

where Δ is the dissipation rate.

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Scalar counterparts of this energy balance are known, e.g., for the Preisach model for ferromagnetism: If $m = \mathcal{P}[h]$ is the constitutive relation between the magnetic field h and the magnetization m with a Preisach operator $\mathcal{P}$ and with the associated Preisach free energy operator $W = \mathcal{W}[h]$, then the inequality

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!!! Hysteresis losses can influence the harvester efficiency.

Preisach operator

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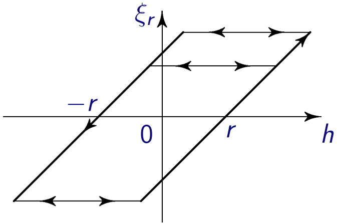
Let $\mathfrak{p}_r$ be the mapping which with a parameter $r > 0$ and with a function $h \in W^{1,1}(0,T)$ associates the solution $\xi_r \in W^{1,1}(0,T)$ of the constrained rate independent equation

$$|h(t) - \xi_r(t)| \le r\,,$$

$$\dot{\xi}_r(t)(h(t) - \xi_r(t)) = r|\dot{\xi}_r(t)|\,,$$
$$\xi_r(0) = \min\{h(0) + r, \max\{0, h(0) - r\}\}.$$



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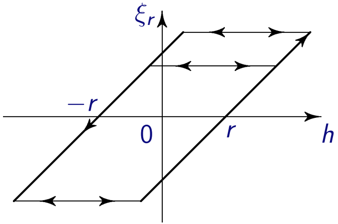

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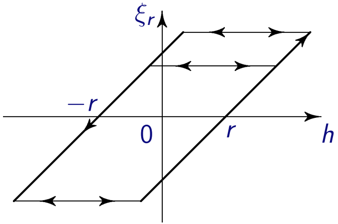

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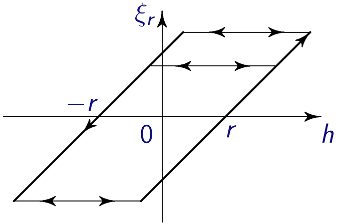

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Preisach operator and Preisach free energy

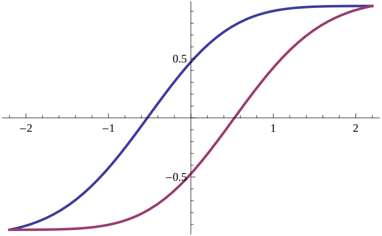
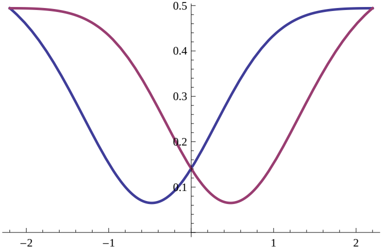


Fig. 1: The Preisach constitutive relation $m = \mathcal{P}[h]$.



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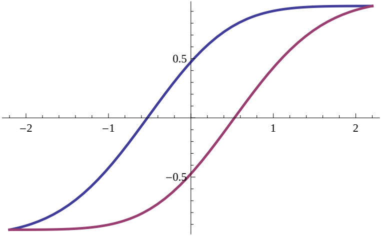

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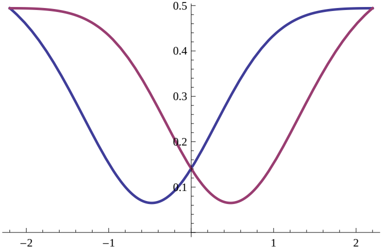

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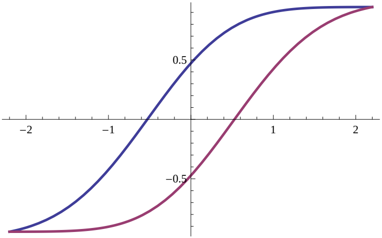

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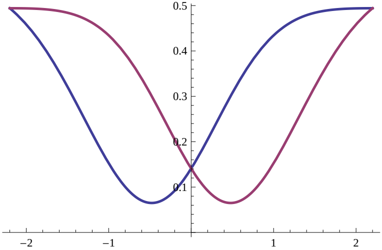

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Conjecture: The Preisach free energy operator describes the electro-mechanical or magneto-mechanical interaction.

Piezoelectricity

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In order to model the self-similar behavior in the input- (strain ε and the electric field E) output (dielectric displacement D and stress σ) hysteresis diagram, the simplest choice is

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D &= \omega\varepsilon + \kappa E + P, \\
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As a correction of the model, a modification including a mean field feedback correction term has recently been proposed in the form

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where $\rho, \nu, K, \omega, \kappa, \alpha$ are physical constants.

Piezoelectric harvester: Case without inductance

The dynamics of a piezoelectric harvester subject to an impressed time-dependent mechanical force $\sigma_{imp}(t)$ can be described by the system

$$\begin{aligned} \rho\ddot{\varepsilon}+\nu\dot{\varepsilon}+K\varepsilon - \omega E + f'(\varepsilon)\mathcal{W}[u] + \tfrac{1}{2}a'(\varepsilon)\mathcal{P}^2[u] &= \sigma_{imp}, \\ \frac{\mathrm{d}}{\mathrm{d}t}(\omega\varepsilon + \kappa E + \mathcal{P}[u]) + \alpha E &= 0, \end{aligned}$$

where $\rho, \nu, K, \omega, \kappa, \alpha$ are physical constants.

We have $D = \omega\varepsilon + \kappa E + \mathcal{P}[u]$ and $E = f(\varepsilon)u + a(\varepsilon)\mathcal{P}[u]$, hence,

$$u + \frac{1+\kappa a(\varepsilon)}{\kappa f(\varepsilon)}\mathcal{P}[u] = \frac{D-\omega\varepsilon}{\kappa f(\varepsilon)}. \tag{1}$$

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Theorem (K+K 2016). *The operator $\mathcal{R}$ which with $D, \varepsilon \in C[0,T]$ associates the solution $u = \mathcal{R}[D,\varepsilon] \in C[0,T]$ of equation* (**??**) *is Lipschitz continuous.*

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Our equations thus can be reduced to a simple ODE system with a locally Lipschitz continuous right-hand side, for which all results about local existence, uniqueness, and continuous data dependence are available.

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$$\begin{aligned} \rho\ddot{\varepsilon}+\nu\dot{\varepsilon}+K\varepsilon - \omega E + f'(\varepsilon)\mathcal{W}[u] + \tfrac{1}{2}a'(\varepsilon)\mathcal{P}^2[u] &= \sigma_{imp}, \\ \frac{\mathrm{d}}{\mathrm{d}t}(\omega\varepsilon + \kappa E + \mathcal{P}[u]) + \alpha E + \beta\Phi &= 0, \\ \dot{\Phi} - E &= 0. \end{aligned}$$

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Passing to the new variable $D = \omega\varepsilon + \kappa E + \mathcal{P}[u]$ and substituting $u = \mathcal{R}[D, \varepsilon]$, we obtain as before an ODE system for the unknowns D, ε, Φ with a locally Lipschitz continuous right-hand side.

Energy balance

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we multiply the first equation by $\dot{\varepsilon}$, the second equation by E, the third equation by $\beta\Phi$, and sum them up to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho}{2}\dot{\varepsilon}^2 + \frac{c}{2}\varepsilon^2 + \frac{\kappa}{2}E^2 + \frac{\beta}{2}\Phi^2 + f(\varepsilon)\mathcal{W}[u] + \frac{1}{2}b(\varepsilon)\mathcal{P}^2[u]\right)$$
$$+\nu\dot{\varepsilon}^2 + \alpha E^2 + f(\varepsilon)\left(u\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}[u] - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}[u]\right) = \dot{\varepsilon}\,\sigma_{imp}.$$

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\rho\ddot{\varepsilon}+\nu\dot{\varepsilon}+K\varepsilon-\omega E+f'(\varepsilon)\mathcal{W}[u]+\tfrac{1}{2}a'(\varepsilon)\mathcal{P}^2[u] &= \sigma_{imp},\\
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&\quad+\nu\dot{\varepsilon}^2+\alpha E^2+f(\varepsilon)\left(u\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}[u]-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}[u]\right)=\dot{\varepsilon}\,\sigma_{imp}.
\end{aligned}
$$

The solution thus remains bounded in the whole existence range. This implies in turn that the solution exists globally and depends continuously on the data and on the physical parameters.

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$$u + \lambda B(\varepsilon)\mathfrak{p}_r[u] \quad = \quad A(D,\varepsilon).$$

The inversion formula is explicit in terms of the play operator with moving threshold:

$$\xi = \mathfrak{p}_r[u] = \mathfrak{p}_{R(\varepsilon)}\left[\frac{A(D,\varepsilon)}{1+\lambda B(\varepsilon)}\right], \quad R(\varepsilon) = \frac{r}{1+\lambda B(\varepsilon)}.$$

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All operators in the balance equation thus admit a representation in terms of ξ

$$\begin{aligned} \mathcal{R}[D,\varepsilon] &= A(D,\varepsilon) - \lambda B(\varepsilon)\xi, \\ (\mathcal{P}\circ\mathcal{R})[D,\varepsilon] &= \lambda\xi, \\ (\mathcal{W}\circ\mathcal{R})[D,\varepsilon] &= \frac{\lambda}{2}\xi^2. \end{aligned}$$

Special case: The play operator II

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The system of balance equations has the form

$$\dot{y}(t) = F(t, y(t), \xi(t); \theta),$$

where $y = (\varepsilon, \dot{\varepsilon}, D, \Phi)$ is the unknown vector function, and $\theta \in \Theta$ is the constant vector of physical parameters.

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where $y = (\varepsilon, \dot{\varepsilon}, D, \Phi)$ is the unknown vector function, and $\theta \in \Theta$ is the constant vector of physical parameters. The moving play operator $\xi = \mathfrak{p}_{R(\varepsilon)}\left[\frac{A(D,\varepsilon)}{1+\lambda B(\varepsilon)}\right]$ admits a representation in terms of differential inclusion

$$\dot{\xi}(t) \in \partial I_{[-1,1]}(a(t)), \quad a = \frac{1}{r}(A(D,\varepsilon) - (1 + \lambda B(\varepsilon))\xi),$$

where $I_{[-1,1]}$ is the indicator function of the interval $[-1, 1]$ and $\partial I_{[-1,1]}$ is its subdifferential.

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$$\dot{a}(t) + \partial I_{[-1,1]}(a(t)) \ni g(t, y(t), a(t); \theta).$$

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The system of balance equations has the form

$$\dot{y}(t) = \hat{F}(t, y(t), a(t); \theta),$$

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The next goal is to maximize the harvested energy

$$\int_0^T J(t, y(t), a(t); \theta)(t)\,\mathrm{d}t \quad \longrightarrow \quad \min$$

with respect to the physical parameter vector $\theta \in \Theta$ if $y(0), a(0)$ are given.

Approximation

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Complement the cost functional with the term $|\theta - \theta_*|^2$, where θ_* is a value where the minimum is achieved, and replace the system

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with

$$\begin{aligned} \dot{y}_\gamma(t) &= \hat{F}(t, y_\gamma(t), a_\gamma(t); \theta_\gamma) \\ \dot{a}_\gamma(t) + \frac{1}{\gamma}\Psi'(a_\gamma(t)) &= g(t, y_\gamma(t), a_\gamma(t); \theta_\gamma), \end{aligned}$$

for $\gamma > 0$, where

$$\Psi(a) = \frac{1}{6}((a^2 - 1)^+)^3;$$

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$$\Psi(a) = \frac{1}{6}((a^2 - 1)^+)^3;$$

For $\gamma \to 0$, (y_γ, a_γ) converge strongly to solutions (y_*, a_*) in $W^{1,2}(0, T)$ of the system

$$\dot{y}_*(t) = \hat{F}(t, y_*(t), a_*(t); \theta_*)$$
$$\dot{a}_*(t) + \partial I_{[-1,1]}(a_*(t)) \ni g(t, y_*(t), a_*(t); \theta_*).$$

Necessary optimality conditions

Theorem. *Let $\hat{F}, g, J$ be continuously differentiable, and let (y_*, a_*, θ_*) be a local maximizer of the problem. Then there exist adjoint states* $p_* \in W^{1,2}(0,T;\mathbb{R}^n)$, $q_* \in BV(0,T)$ *such that*

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-\dot{p}_*(t) &= \partial_y \hat{F}(t, y_*(t), a_*(t); \theta_*) \cdot p_*(t) + \partial_y g(t, y_*(t), a_*(t); \theta_*)\, q_*(t) \\
&\quad - \partial_y J(t, y_*(t), a_*(t); \theta_*) \quad \text{for} \quad t \in (0,T), \\
&\quad\ p_*(T) = 0, \\
&q_*(t)\, g(t, y_*(t), a_*(t); \theta_*) = 0 \quad \text{for a. e. } t \in \{s \in (0,T) : |a_*(s)| = 1\}, \\
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0 \in \int_0^T \Big(&\partial_\theta J(t, y_*(t), a_*(t); \theta_*) - \partial_\theta \hat{F}(t, y_*(t), a_*(t); \theta_*) \cdot p_*(t) \\
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- Distinguish the cases that $a_*(t)$ is on the boundary or in the interior of the admissible interval $[-1,1]$.

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- Real time control of piezoelectric or magnetostrictive actuators and sensors, where algorithms for fast and accurate inversion of hysteresis operators are of central importance;
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- Can be coupled with the full system of balance PDEs describing, e.g., vibrations of piezoelectric beams.

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