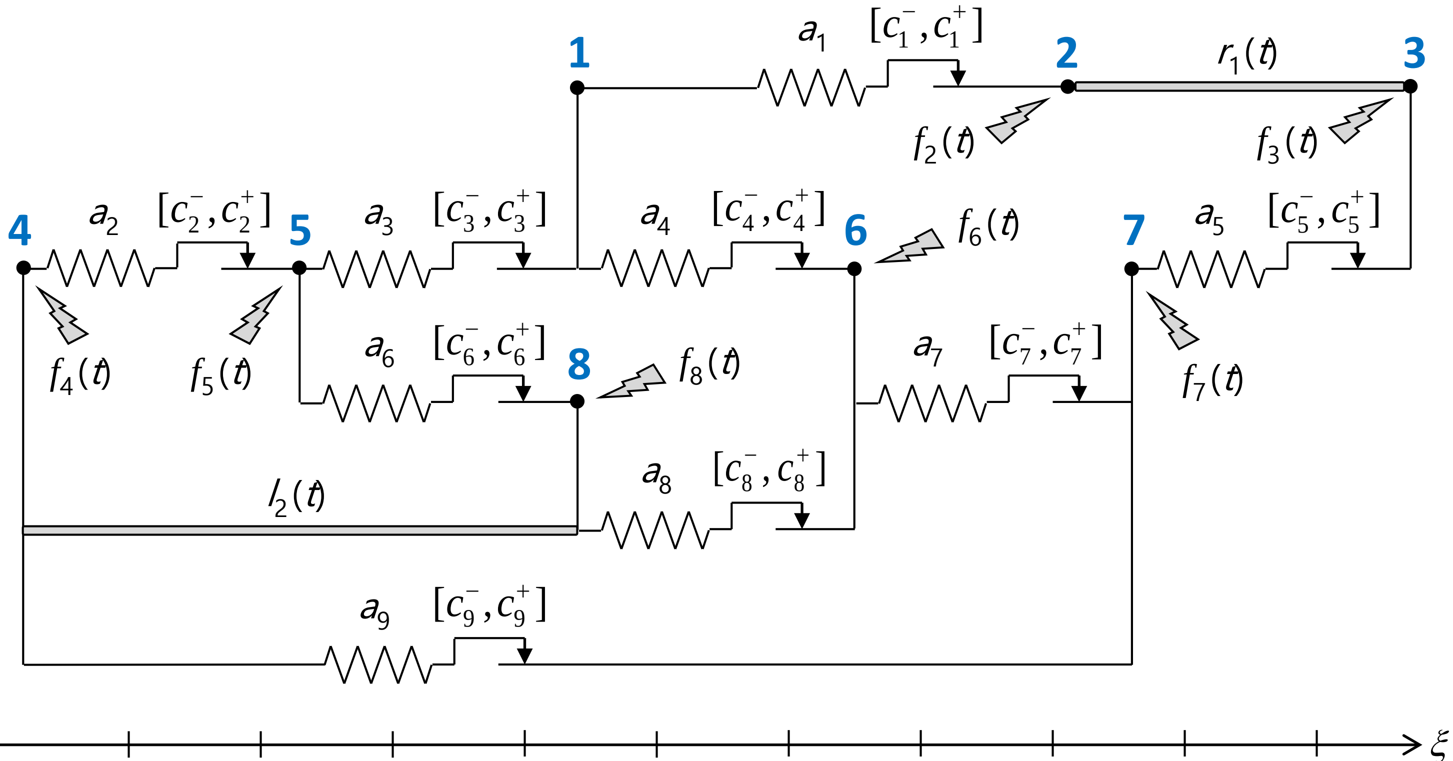
Stabilization of quasistatic evolution of elastoplastic systems subject to periodic loading

Oleg Makarenkov

Department of Mathematical Sciences
University of Texas at Dallas

in cooperation with Ivan Gudoshnikov

A parallel network of elastoplastic springs



$(i_1, j_1) = (1,2)$

$(i_2, j_2) = (4,5)$

$(i_3, j_3) = (5,1)$

$(i_4, j_4) = (1,6)$

$(i_5, j_5) = (7,3)$

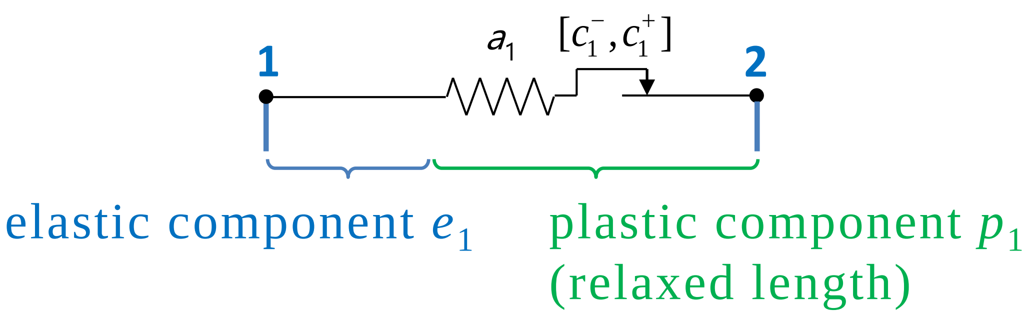
$(i_6, j_6) = (5,8)$

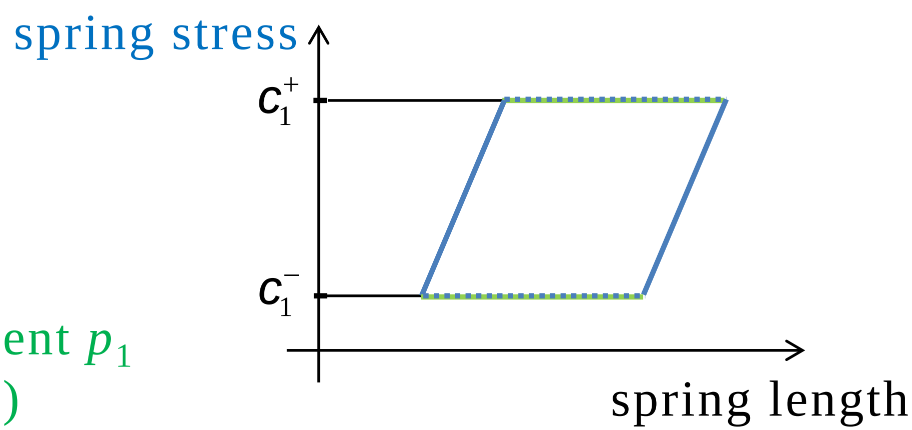
$(i_7, j_7) = (6,7)$

$(i_8, j_8) = (8,6)$

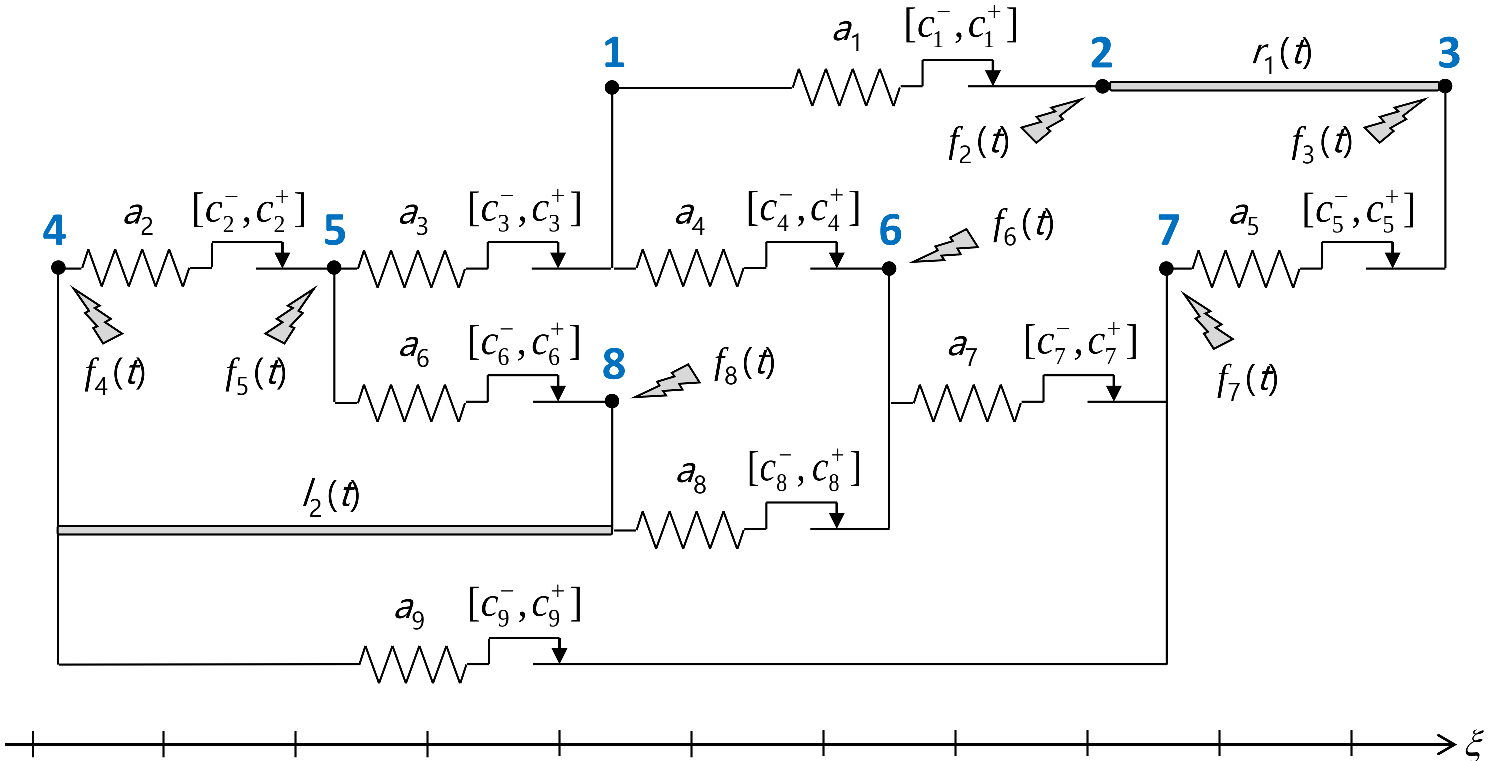
$(i_9, j_9) = (4,7)$

Elastoplastic spring:





A parallel network of elastoplastic springs



$(i_1, j_1) = (1,2)$

$(i_2, j_2) = (4,5)$

$(i_3, j_3) = (5,1)$

$(i_4, j_4) = (1,6)$

$(i_5, j_5) = (7,3)$

$(i_6, j_6) = (5,8)$

$(i_7, j_7) = (6,7)$

$(i_8, j_8) = (8,6)$

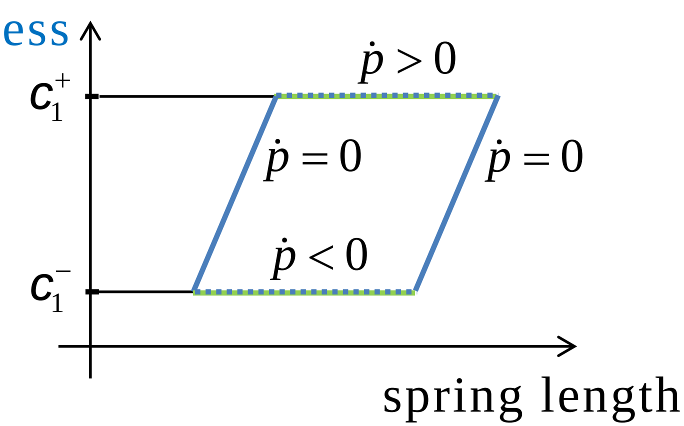
$(i_9, j_9) = (4,7)$

Elastic deformation: $s = Ae$

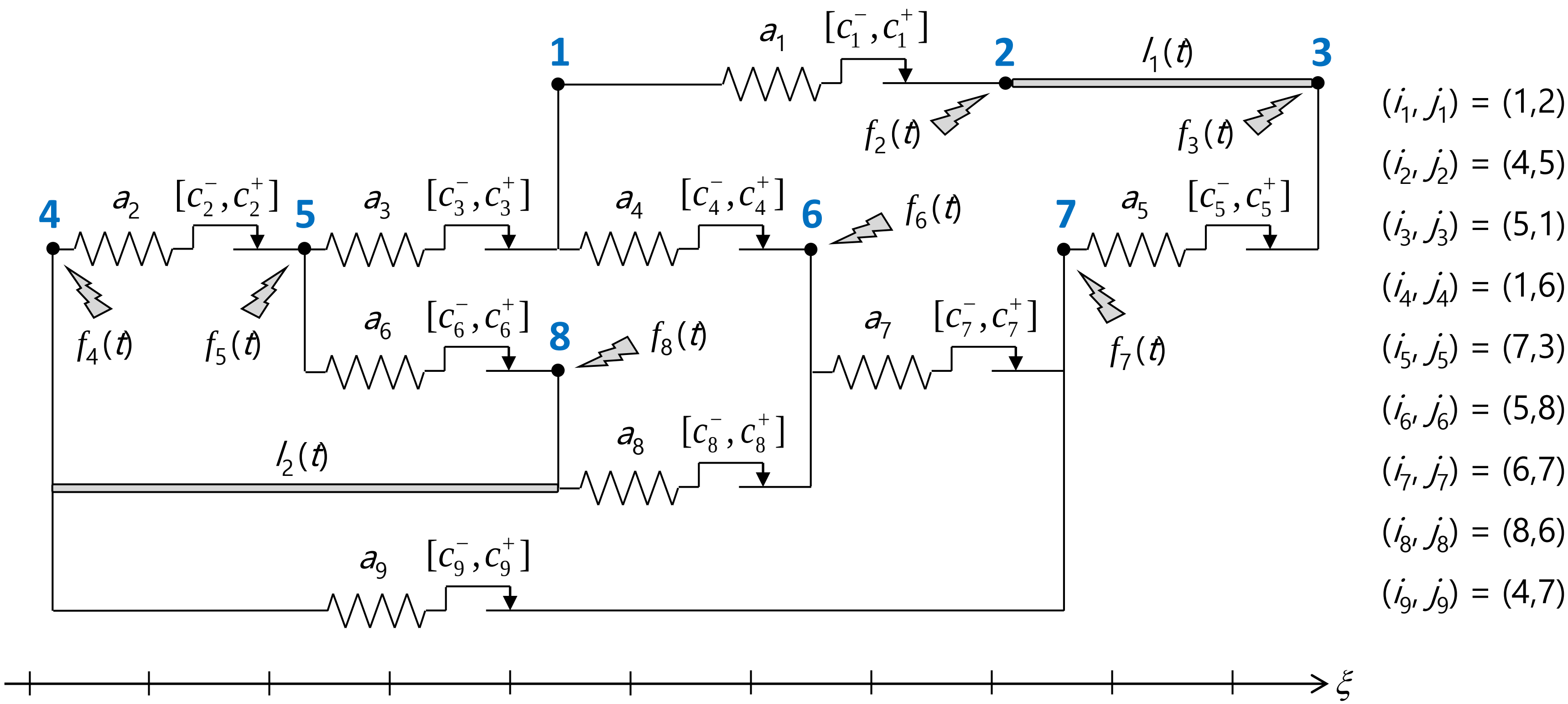
Plastic deformation: $\dot{p} \in N_C(s)$

$$C = [c_1^-, c_1^+] \times \ldots \\ \ldots \times [c_m^-, c_m^+]$$

$$N_{[c_1^-, c_1^+]}(s) = \begin{cases} [0, \infty), & \text{if } s = c_1^+, \\ \{0\}, & \text{if } s \in (c_1^-, c_1^+), \\ (-\infty, 0], & \text{if } s = c_1^-. \end{cases}$$



Initial system of variational inequalities



Elastic deformation: $s = Ae,$

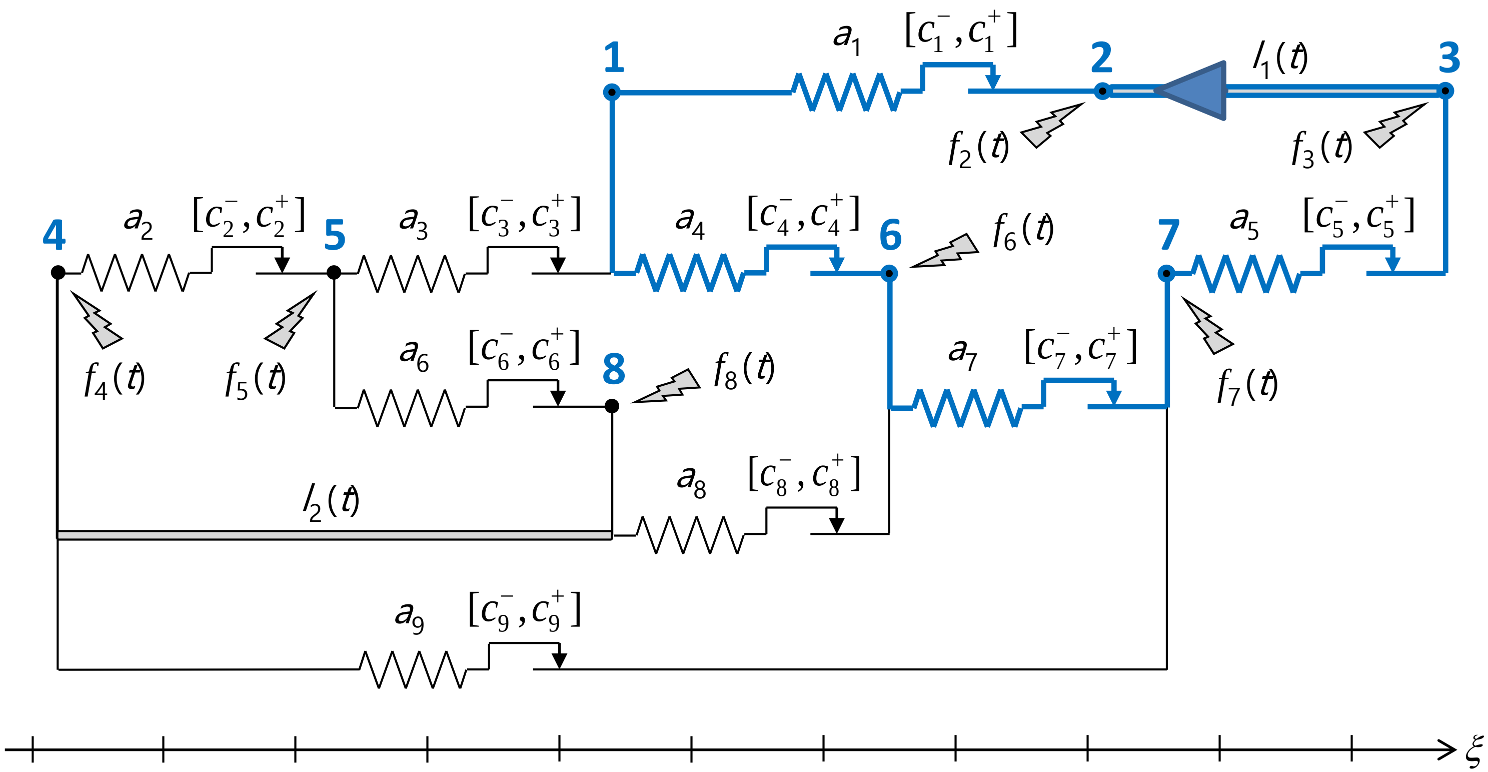
Plastic deformation: $\dot{p} \in N_C(s),$

Geometric constraint: $e + p \in D\mathfrak{R}^n,$

Enforced constraint: $R^T(e+p) = l(t),$

Static balance: $s^1 + \ldots + s^m + r^1 + \ldots + r^q + f(t) = 0.$

Tension/compression law



$(i_1, j_1) = (1,2)$

$(i_2, j_2) = (4,5)$

$(i_3, j_3) = (5,1)$

$(i_4, j_4) = (1,6)$

$(i_5, j_5) = (7,3)$

$(i_6, j_6) = (5,8)$

$(i_7, j_7) = (6,7)$

$(i_8, j_8) = (8,6)$

$(i_9, j_9) = (4,7)$

Elastic deformation: $s = Ae,$

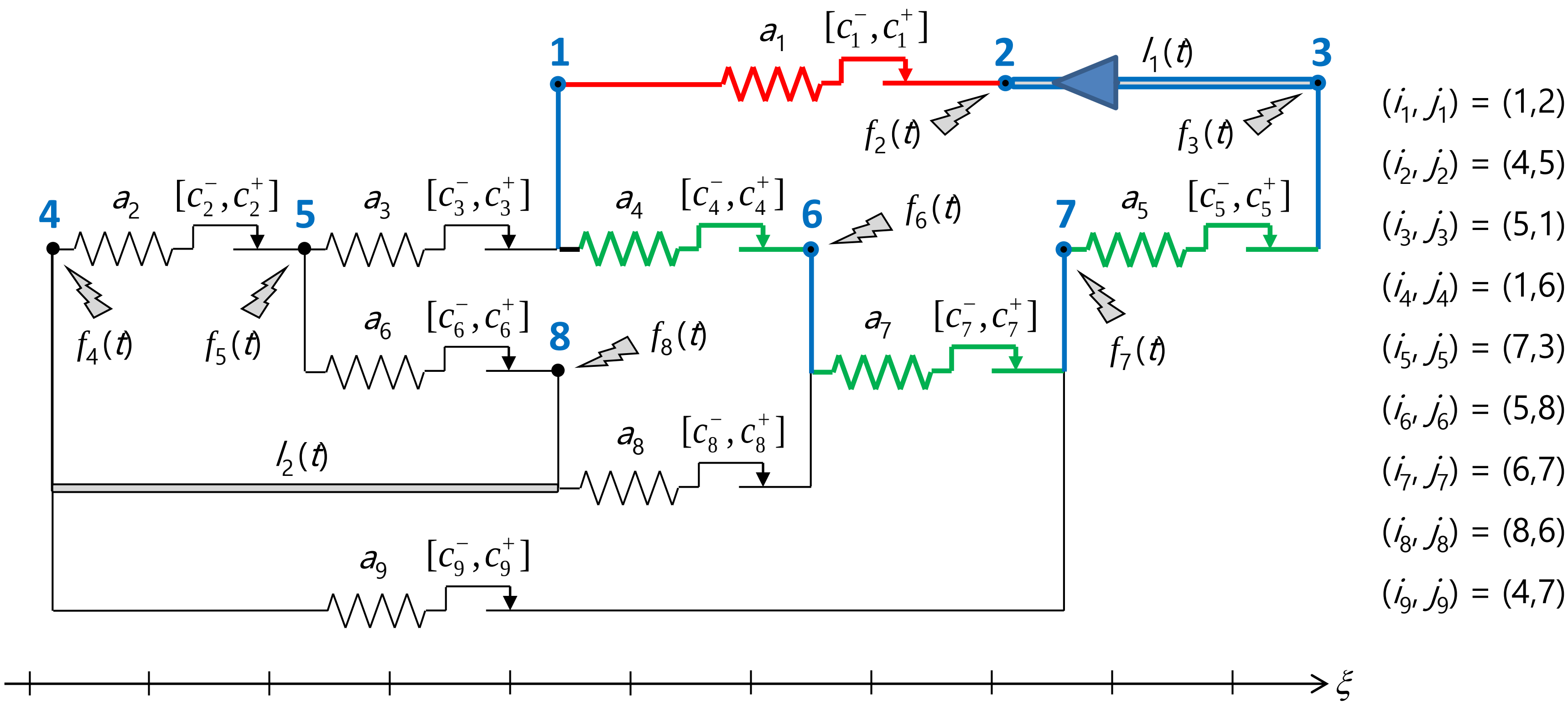
Plastic deformation: $\dot{p} \in N_C(s),$

Geometric constraint: $e + p \in D\Re^n,$

Enforced constraint: $R^T(e+p) = l(t),$

Static balance: $s^1 + \ldots + s^m + r^1 + \ldots + r^q + f(t) = 0.$

Tension/compression law



Elastic deformation: $s = Ae,$

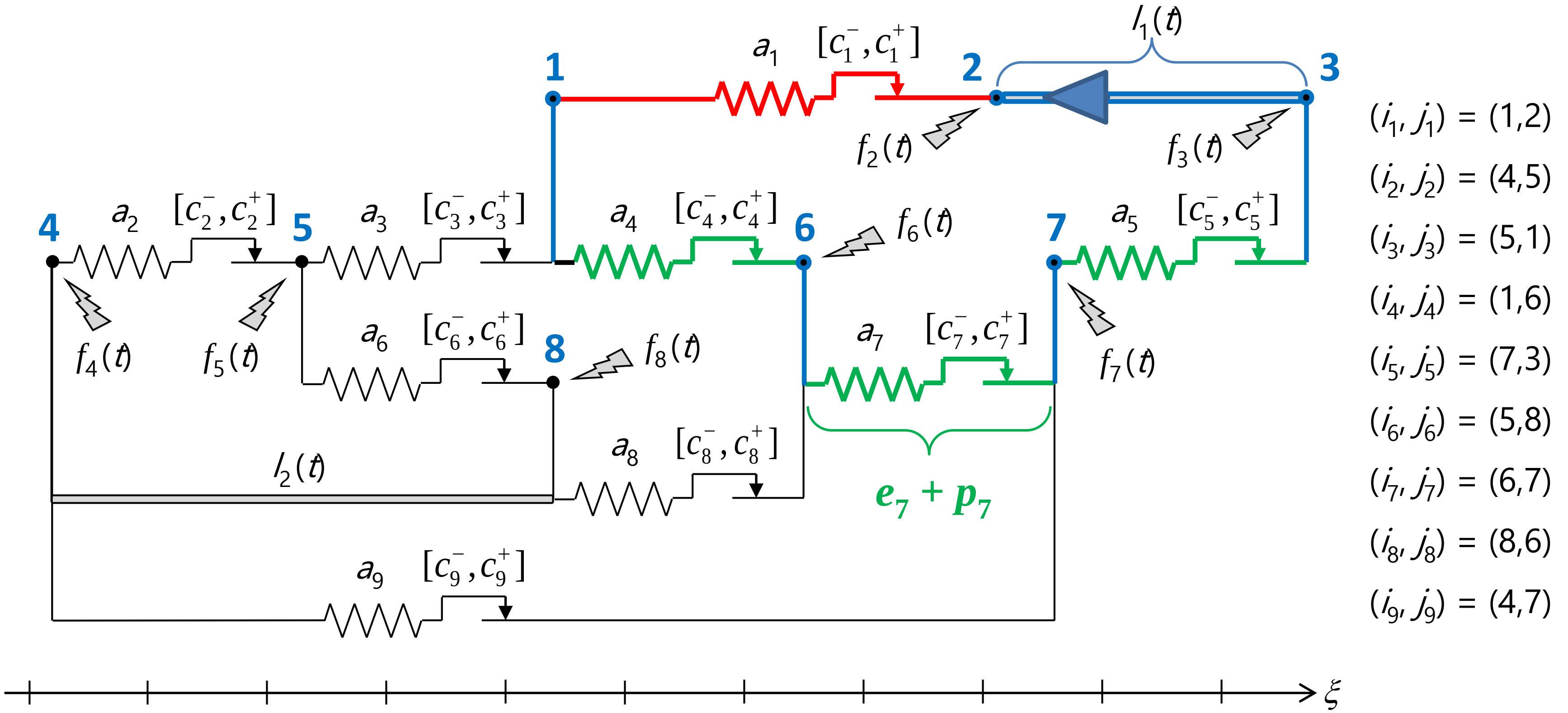
Plastic deformation: $\dot{p} \in N_C(s),$

Geometric constraint: $e + p \in D\Re^n,$

Enforced constraint: $R^T(e+p) = l(t),$

Static balance: $s^1 + \ldots + s^m + r^1 + \ldots + r^q + f(t) = 0.$

Tension/compression law



Elastic deformation: $s = Ae,$

Plastic deformation: $\dot{p} \in N_C(s),$

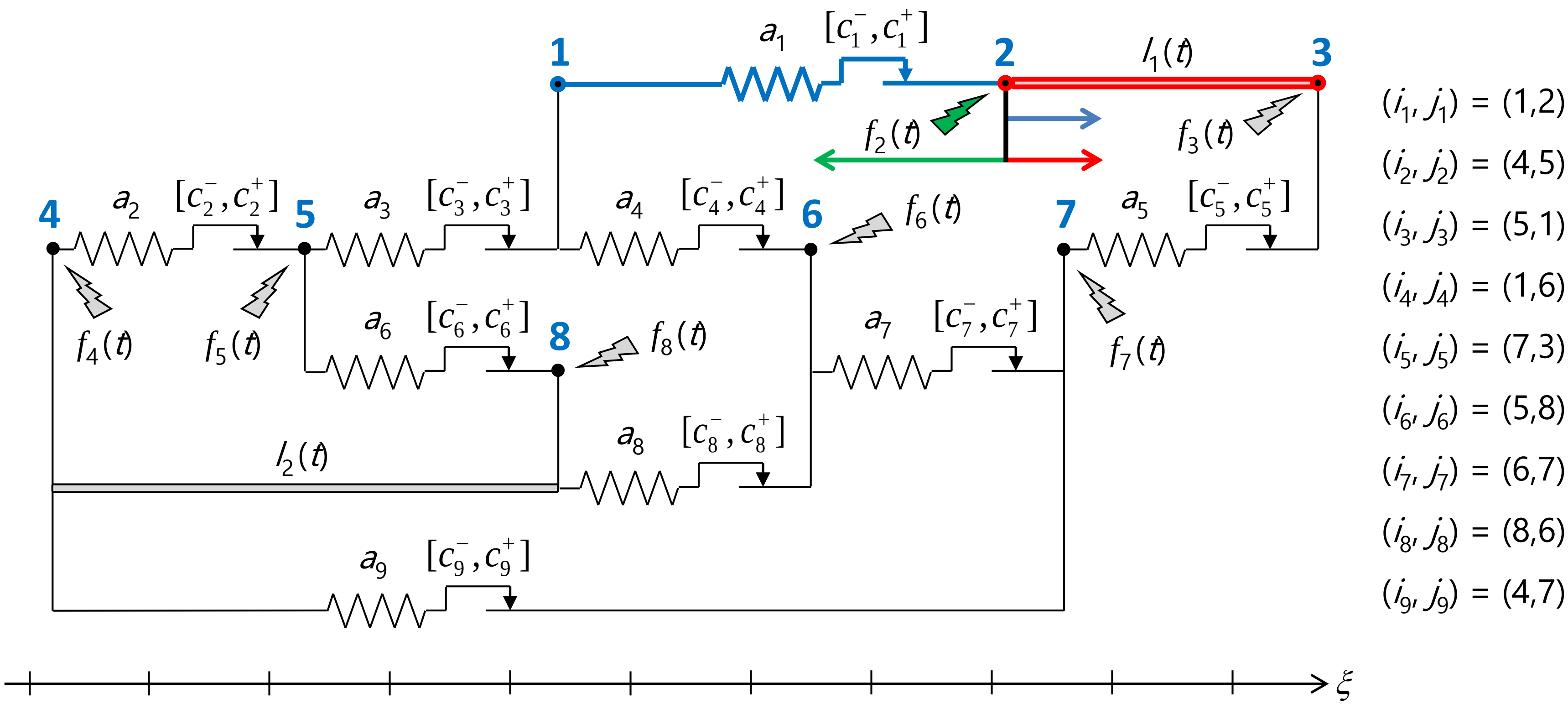
Geometric constraint: $e + p \in D\mathfrak{R}^n,$

Enforced constraint: $R^T(e+p) = l(t),$

Static balance: $s^1 + \ldots + s^m + r^1 + \ldots + r^q + f(t) = 0.$

For enforced constraint **1**:
$(e_4+p_4)+(e_7+p_7)+(e_5+p_5)-(e_1+p_1)=l_1(t)$

Static balance law



Elastic deformation : $s = Ae,$

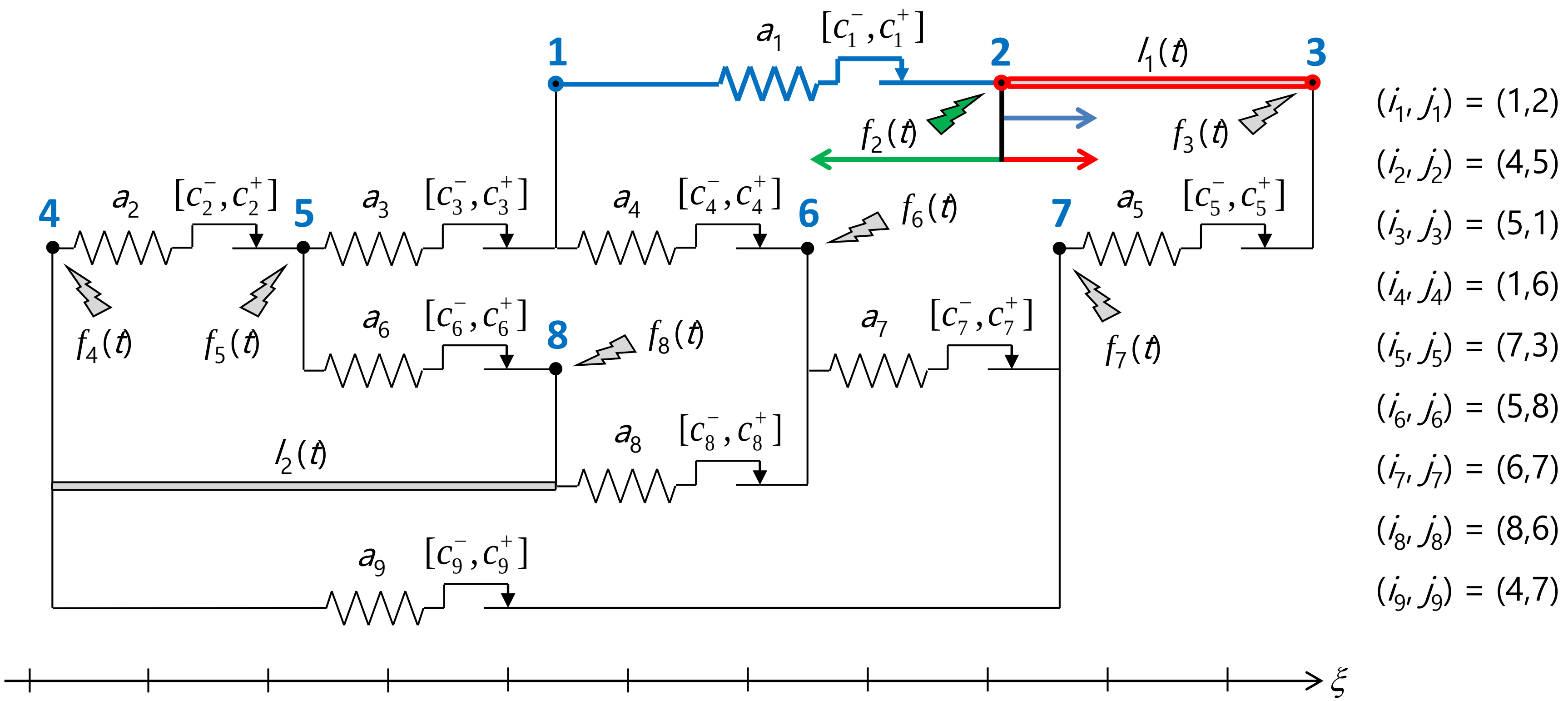
Plastic deformation : $\dot{p} \in N_C(s),$

Geometric constraint : $e + p \in D\Re^n,$

Enforced constraint : $R^T(e + p) = l(t),$

Static balance : $s^1 + \ldots + s^m + r^1 + \ldots + r^q + f(t) = 0.$

Static balance law



Elastic deformation: $s = Ae,$

Plastic deformation: $\dot{p} \in N_C(s),$

Geometric constraint: $e + p \in D\Re^n,$

Enforced constraint: $R^T(e+p) = l(t),$

Static balance: $s^1 + \ldots + s^m + r^1 + \ldots + r^q + f(t) = 0.$

For node **2**:
$-s_1 + r_1 + f_2(t) = 0$

Moreau sweeping process

Elastic deformation: $s = Ae,$

Plastic deformation: $\dot{p} \in N_C(s),$

Geometric constraint: $e + p \in D(\Re^n),$

Enforced constraint: $R^T(e+p) = l(t),$

⟺

$e + p \in U + g(t)$

$U = \{x \in D(\Re^n) : R^T x = 0\}$

$g(t) = \left(D\bar{\xi}l(t)\right)\Big|_V$

$V = A^{-1}U^{\perp}$

Graph theory: $s^1 + \ldots + s^m = -D^T s$ $\quad r^1 + \ldots + r^q = -D^T Rr$

Static balance: $s^1 + \ldots + s^m + r^1 + \ldots + r^q + f(t) = 0$

Algebra: $\mathrm{Ker} D^T = \left(DR^n\right)^{\perp} \subset U^{\perp}$

⟺

$e + h(t) \in V$

$f(t) = -D^T \bar{h}(t)$

$h(t) = \left(A^{-1}\bar{h}(t)\right)\Big|_U$

$\dot{p} \in N_C(Ae)$

$e + p - g(t) + h(t) \in U$

$e + h(t) - g(t) \in V$

$y = e + h(t) - g(t)$

⟷

$z = e + p + h(t) - g(t)$

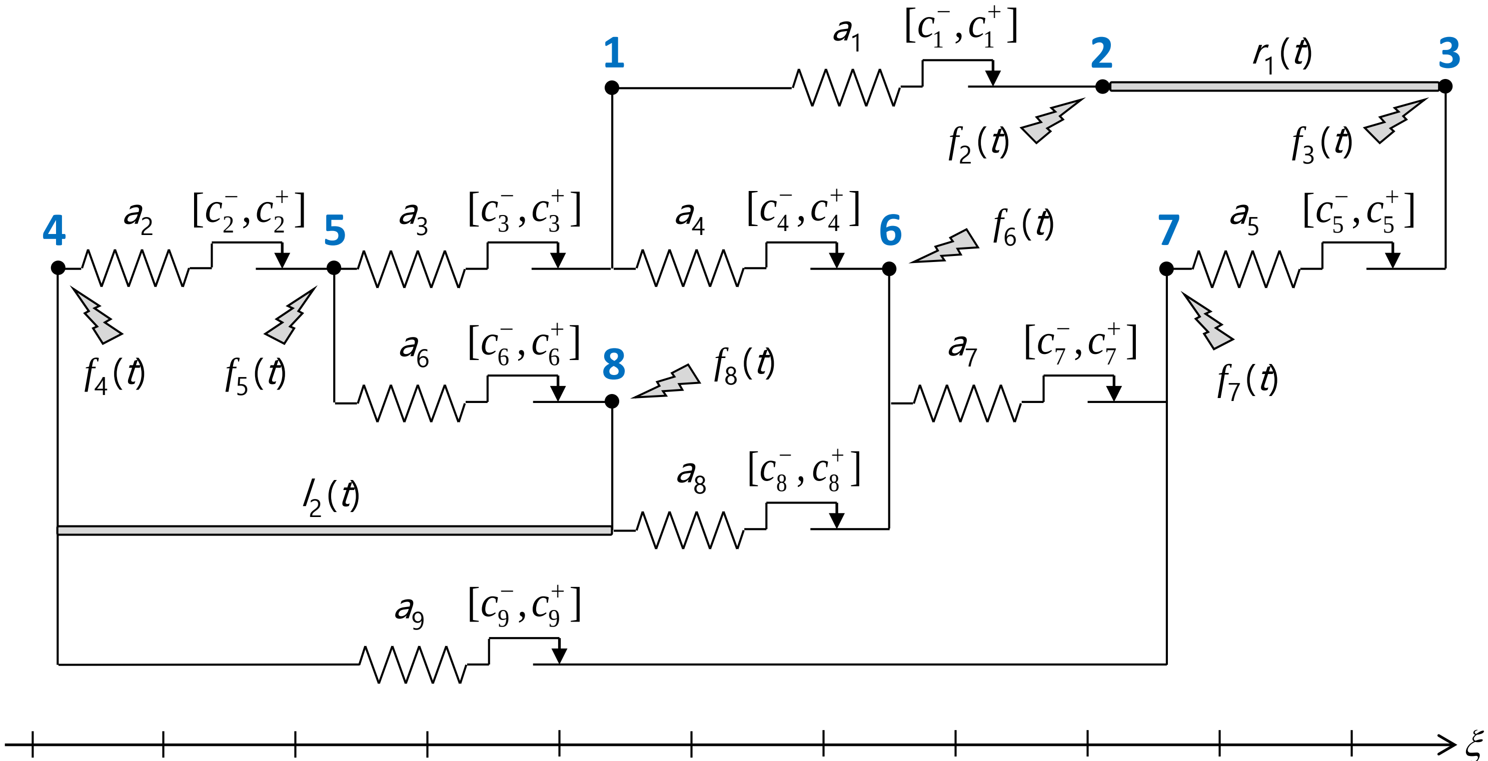
$-\dot{y} \in N^A_{\left(A^{-1}C + h(t) - g(t)\right) \cap V}(y)$

$\dot{z} \in \left(N^A_{\left(A^{-1}C + h(t) - g(t)\right)}(y) + \dot{y}\right) \cap U$

$z(0) \in U$

$$\mathrm{rank}\left(D^T R\right) = q$$

Moreau sweeping process



$(i_1, j_1) = (1,2)$
$(i_2, j_2) = (4,5)$
$(i_3, j_3) = (5,1)$
$(i_4, j_4) = (1,6)$
$(i_5, j_5) = (7,3)$
$(i_6, j_6) = (5,8)$
$(i_7, j_7) = (6,7)$
$(i_8, j_8) = (8,6)$
$(i_9, j_9) = (4,7)$

$$-\dot{y} \in N^A_{\left(A^{-1}C + h(t) - g(t)\right)\cap V}(y), \qquad \dim V = m - n + q + 1$$

$$U = \left\{x \in D(\Re^n) : R^T x = 0\right\} \qquad g(t) = \left(D\bar{\xi} l(t)\right)\Big|_V \qquad f(t) = -D^T \bar{h}(t)$$

$$V = A^{-1} U^{\perp} \qquad h(t) = \left(A^{-1}\bar{h}(t)\right)\Big|_U \qquad y = e + h(t) - g(t)$$

Geometry of the moving constraint

$$-\dot{y} \in N^{A}_{C(t)}(y), \qquad \Pi(t) = A^{-1}C + h(t) - g(t)$$

$$C(t) = \Pi(t) \bigcap V$$

$$U \otimes V = \mathrm{R}^{m}$$

$$(u, v)_A = \langle u, Av \rangle$$

$$h(t) \in U$$

$$g(t) \in V$$

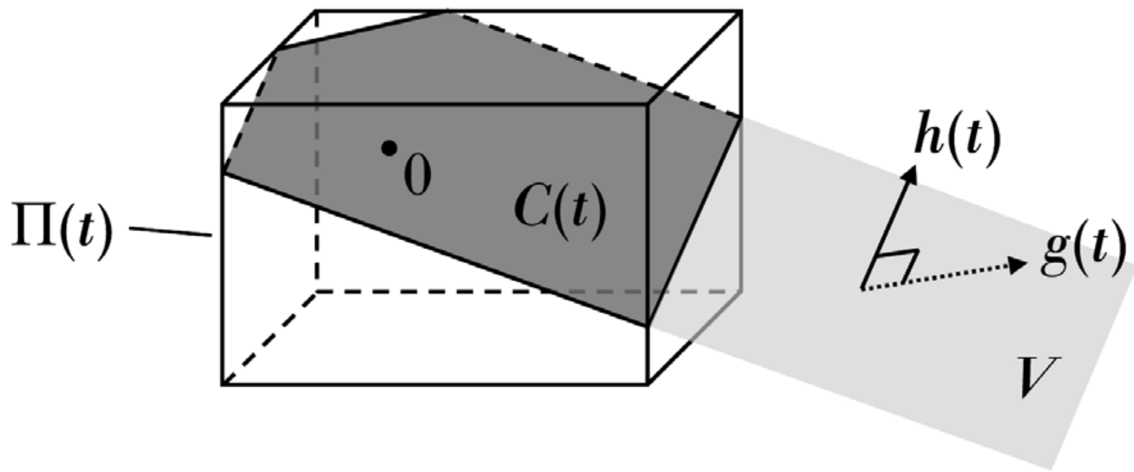
$$\dim U = n - q - 1$$

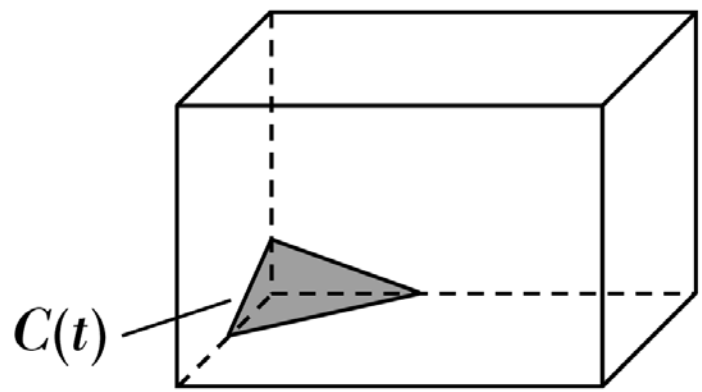
$$\dim V = m - n + q + 1$$

n = number of nodes

m = number of springs

q = number of tension/compression constraints





A criterion for the safe load condition to hold

$$-\dot{y} \in N^{A}_{C(t)}(y), \qquad \Pi(t) = A^{-1}C + h(t) - g(t)$$

$$C(t) = \Pi(t) \bigcap V$$

$$U \otimes V = \mathrm{R}^{m}$$

$$(u,v)_A = \langle u, Av \rangle$$

$$h(t) \in U$$

$$g(t) \in V$$

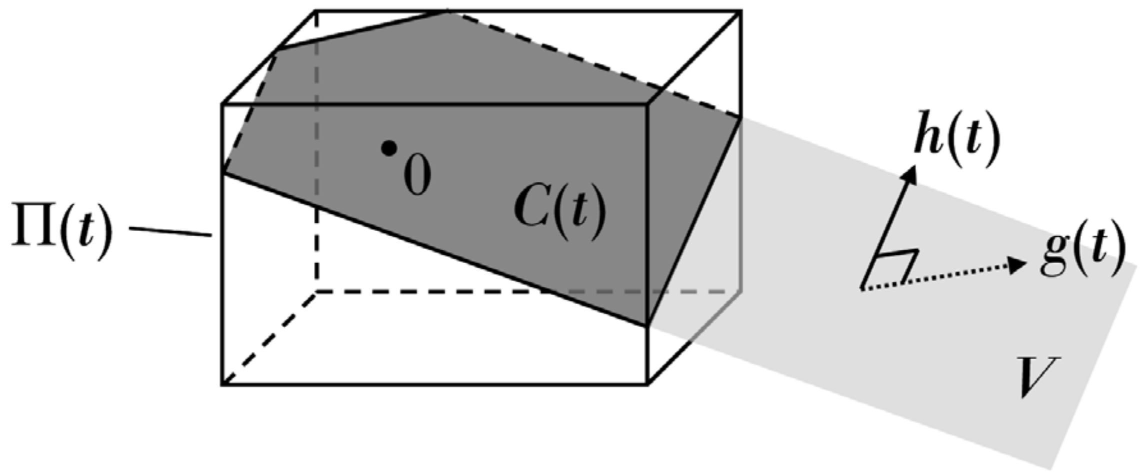
$$\dim U = n - q - 1$$

$$\dim V = m - n + q + 1$$

n = number of nodes

m = number of springs

q = number of tension/compression constraints

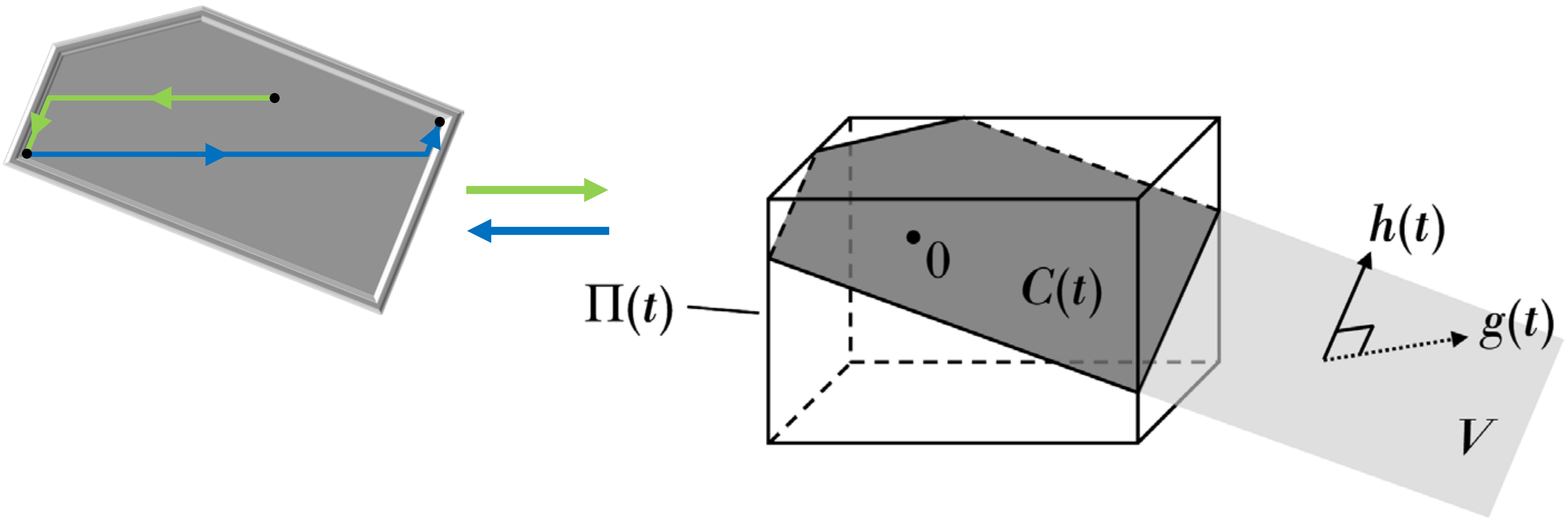


Proposition 1 (safe load):

$-Ah(t) \in C \Rightarrow C(t) \neq \varnothing$

$C(t) = \varnothing \Rightarrow$ plastic collapse

A criterion for plastic shakedown to occur

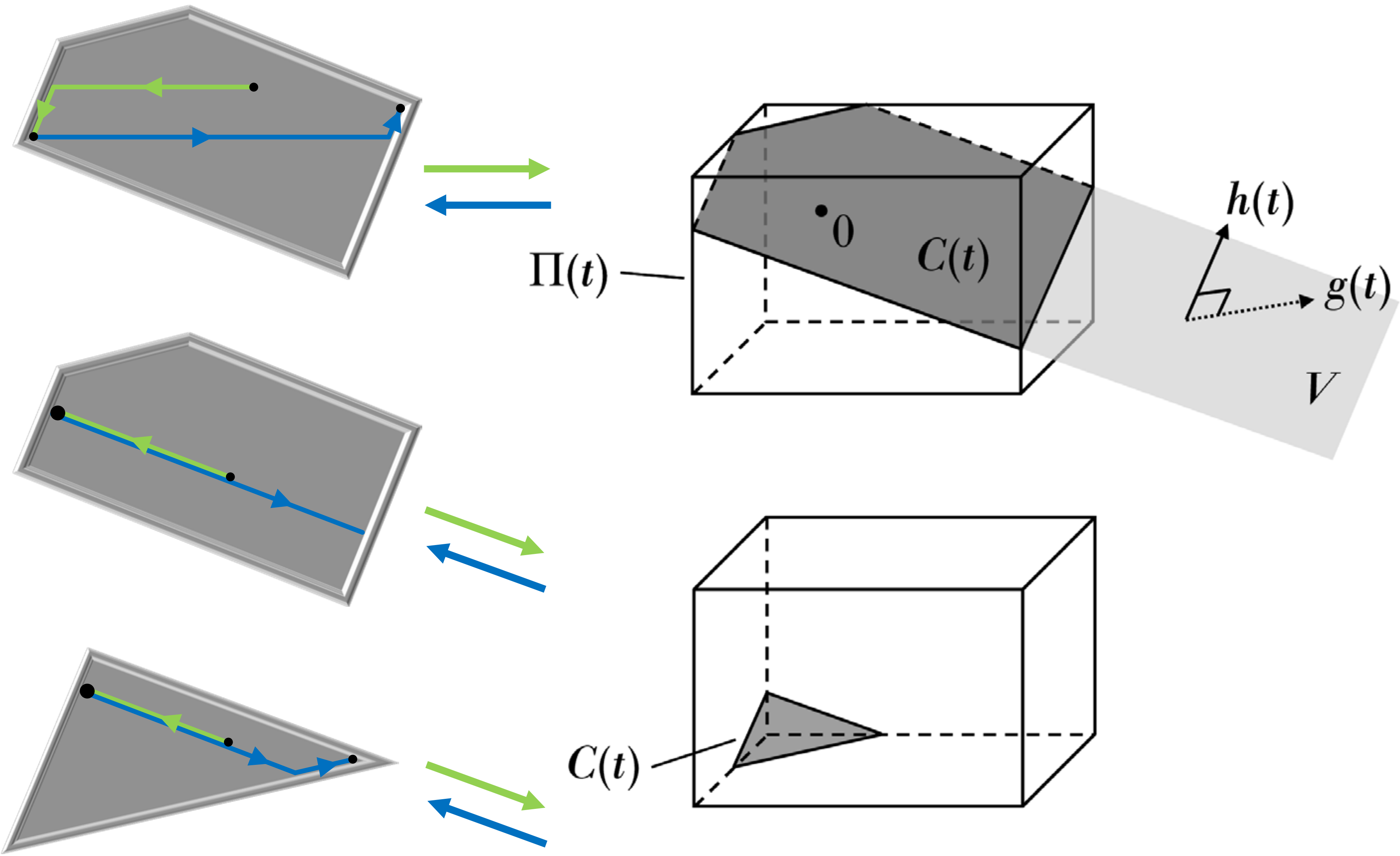


Proposition 2 (plastic shakedown):
Assume that the safe load condition holds. If

$$\|A^{-1}c^{-} - A^{-1}c^{+}\|_A < \|g(t_1) - g(t_2)\|_A \,,$$

then the sweeping process doesn't have any solutions that are constant on the interval $[t_1, t_2]$.

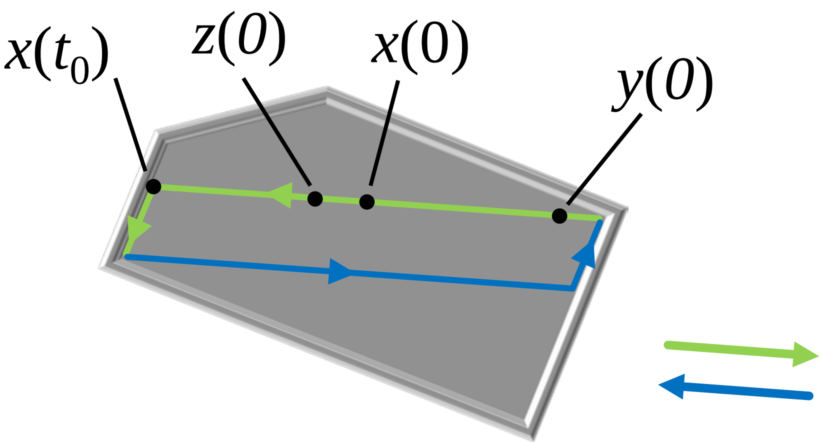
Dynamics under *T*-periodic loading



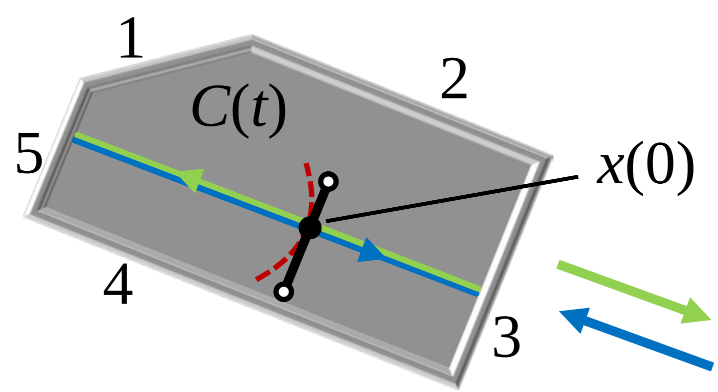
Existence of a periodic attractor

$$-\dot{y} \in N^{A}_{C(t)}(y)$$

Theorem 1 (existence of periodic attractor, Krejci): If $C(t)$ is T-periodic, then the set of all T-periodic solutions is a global attractor. For each fixed $t \in [0, t]$, the active set $J(t, x(t))$ is the same for all $x \in \mathrm{ri}(X)$.



$J(t_0, x(t_0)) = \{5\}$, $J(t_0, y(t_0)) = \varnothing$
$x(t)$ is T-periodic $\Rightarrow$
$\Rightarrow y(t)$ is not T-periodic
and $z(t)$ is not T-periodic

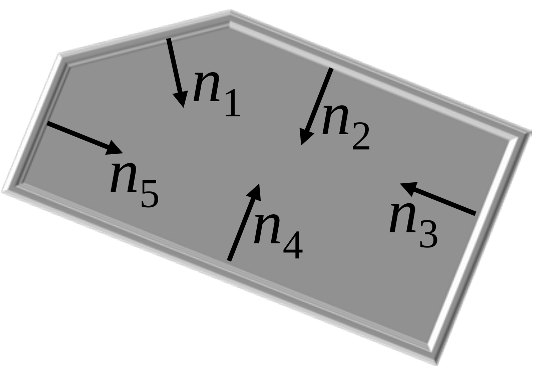


if $x(0)$ is an initial condition of a T-periodic solution $x(t)$ then the set of initial conditions of all other T-periodic solutions is a straight line.

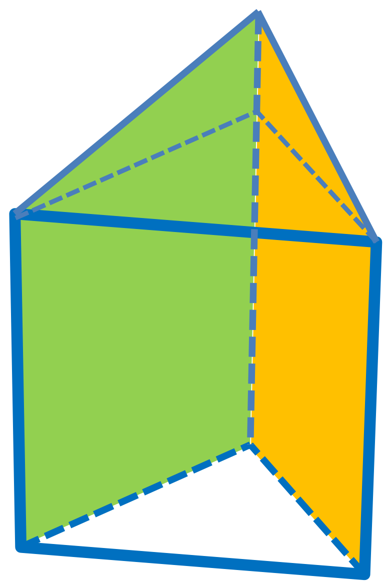
Uniqueness of non-constant *T*-periodic solutions

$$-\dot{y} \in N^{A}_{C(t)}(y)$$

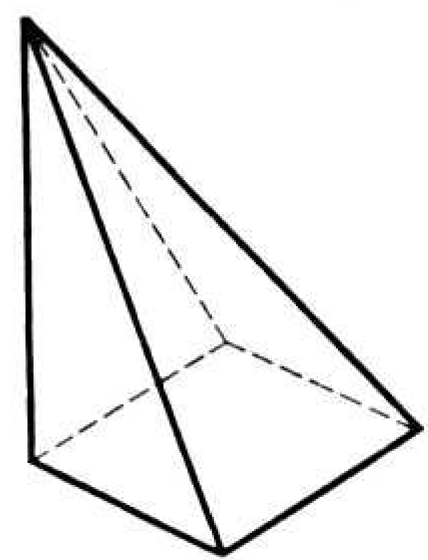
Theorem 2 (uniqueness of *T*-periodic solutions): Let $C(t) \subset \mathrm{R}^m$ be *T*-periodic. Assume that any m vectors out the collection $\{n_i\}$ are linearly independent and the number of adjoin facets doesn't exceed m. Then the sweeping process has at most one non-constant *T*-periodic solution.



no



no

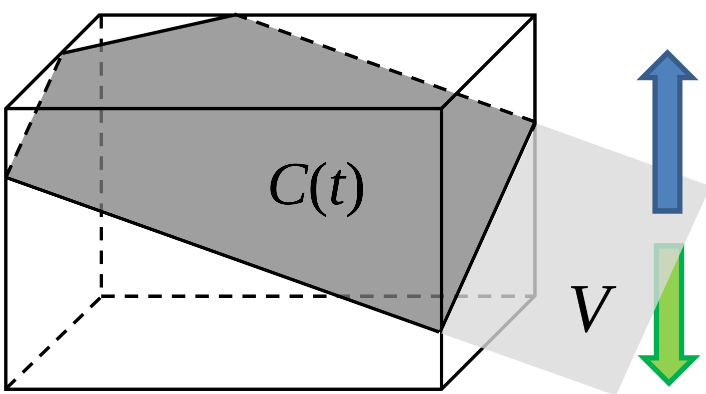


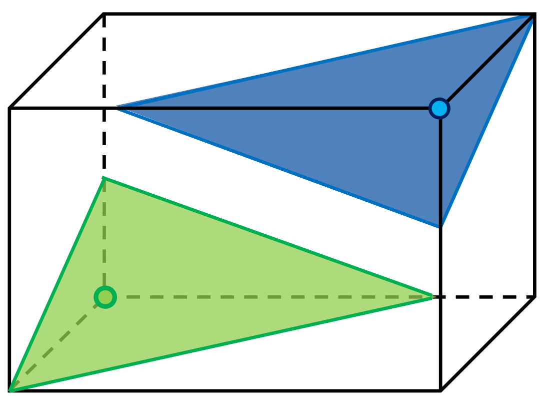
yes

Uniqueness of non-constant *T*-periodic solutions

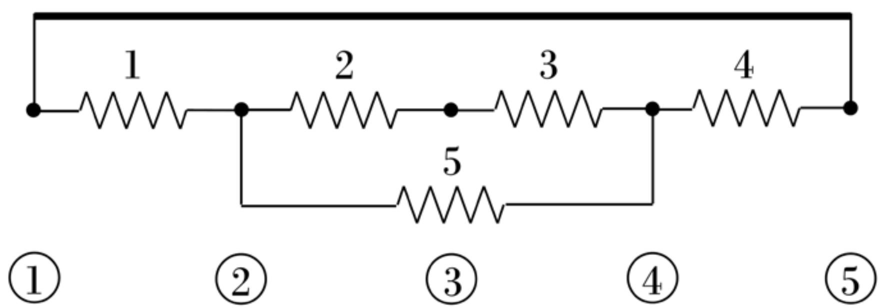
$$-\dot{y} \in N^{A}_{C(t)}(y), \qquad C(t) = \left(A^{-1}C + h(t) - g(t)\right) \bigcap V$$

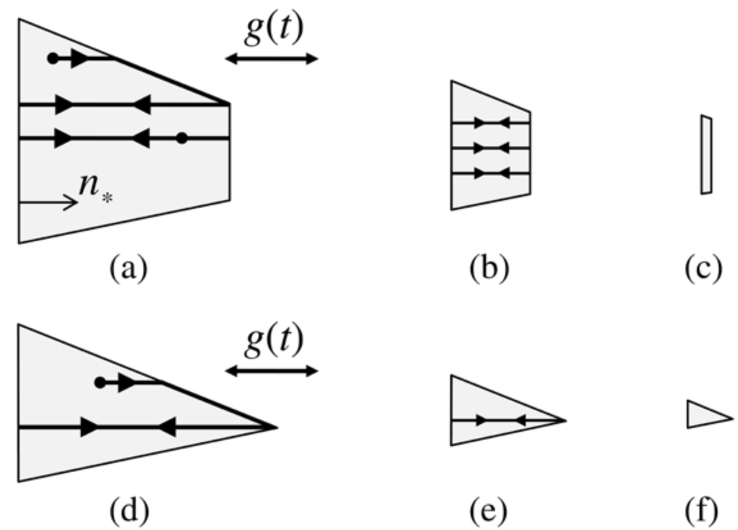
Theorem 3 (uniqueness of non-constant *T*-periodic solutions):
Let $h(t)$ and $g(t)$ be T-periodic. Assume that $\dim V = m$-1. If V is either between the blue vertex and the blue triangle or between the green vertex and the green triangle, then $C(t)$ is a simplex and the sweeping process has at most one non-constant T-periodic solution.





Structurally stable family of periodic solutions





$$R^T V_{basis} \bar{L} = I_{q\times q}$$

$$(D^{\perp})^T V_{basis} \bar{L} = 0$$

$\dim U = 3$

$\dim V = 2$

$$g(t) = V_{basis} \bar{L} l(t)$$

$$\bar{L} = \left(\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \end{pmatrix} V_{basis} \right)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\bar{n}_i = \left(\begin{pmatrix} R^T \\ (D^{\perp})^T \end{pmatrix} V_{basis} \right)^{-1} \begin{pmatrix} R^T \\ (D^{\perp})^T \end{pmatrix} e_i.$$

$$n_* = V_{basis} \left(\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \end{pmatrix} V_{basis} \right)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$m \times (m-n+1)$−matrix that solves

$$(D^{\perp})^T D = 0_{(m-n+1)\times(m-n+1)}$$

$$D^{\perp} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Thank you for your attention !!!

References:

[1] S. Adly, M. Ait Mansour, L. Scrimali, Sensitivity analysis of solutions to a class of quasi-variational inequalities. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 8 (2005), no. 3, 767-771.

[2] I. Gudoshnikov, O. Makarenkov, Stabilization of quasistatic evolution of elastoplastic systems subject to periodic loading, submitted, https://arxiv.org/abs/1708.03084

[3] P. Krejci, Hysteresis, Convexity and Dissipation in Hyperbolic Equations. Gattotoscho, 1996.

[4] J. J. Moreau, On unilateral constraints, friction and plasticity. New variational techniques in mathematical physics (Centro Internaz. Mat. Estivo (C.I.M.E.), II Ciclo, Bressanone, 1973), pp. 171–322. Edizioni Cremonese, Rome, 1974.

Static balance law

Elastic deformation: $s = Ae,$

Plastic deformation: $\dot{p} \in N_C(s),$

Geometric constraint: $e + p \in D\Re^n,$

Enforced constraint: $R^T(e + p) = l(t),$

Static balance: $s^1 + \ldots + s^m + r^1 + \ldots + r^q + f(t) = 0.$

Geometric constraint and enforced constraint combined

Elastic deformation: $s = Ae,$

Plastic deformation: $\dot{p} \in N_C(s),$

Geometric constraint: $e + p \in D\Re^n,$

Enforced constraint: $R^T(e + p) = l(t),$

$\Longleftrightarrow$

$e + p \in U^{l(t)},$

$U^l = \{x \in D\Re^n : R^T x = l\}$

Static balance: $s^1 + \ldots + s^m + r^1 + \ldots + r^q + f(t) = 0.$

Geometric constraint and enforced constraint combined

$$\text{Elastic deformation}: \quad s = Ae,$$

$$\text{Plastic deformation}: \quad \dot{p} \in N_C(s),$$

$$\text{Geometric constraint}: \quad e + p \in D\Re^n,$$

$$\text{Enforced constraint}: \quad R^T(e+p) = l(t),$$

$\Longleftrightarrow$

$$e + p \in U^{l(t)},$$

$$U^l = \{x \in D\Re^n : R^T x = l\}$$

$\Longleftrightarrow$

$$\text{Static balance}: \quad s^1 + \ldots + s^m + r^1 + \ldots + r^q + f(t) = 0.$$

$$R^T D\bar{\xi} = I_{q\times q}$$

$\Longleftrightarrow$

$$x = x\big|_U + x\big|_V$$

$$e + p \in U + g(t),$$

$$U = \{x \in D\Re^n : R^T x = 0\}, \; g(t) = \left(D\bar{\xi}\, l(t)\right)\Big|_V, \; V = A^{-1}U^{\perp}$$

Static balance

$$\text{Elastic deformation}: \quad s = Ae,$$

$$\text{Plastic deformation}: \quad \dot{p} \in N_C(s),$$

$$\text{Geometric constraint}: \quad e + p \in D(\Re^n),$$

$$\text{Enforced constraint}: \quad R^T(e+p) = l(t),$$

$\Leftrightarrow$

$$e + p \in U^{l(t)},$$

$$U^l = \{x \in D(\Re^n) : R^T x = l\}$$

$\Leftrightarrow$

$$\text{Static balance}: \quad s^1 + \ldots + s^m + r^1 + \ldots + r^q + f(t) = 0.$$

$$R^T D\bar{\xi} = I_{q\times q}$$

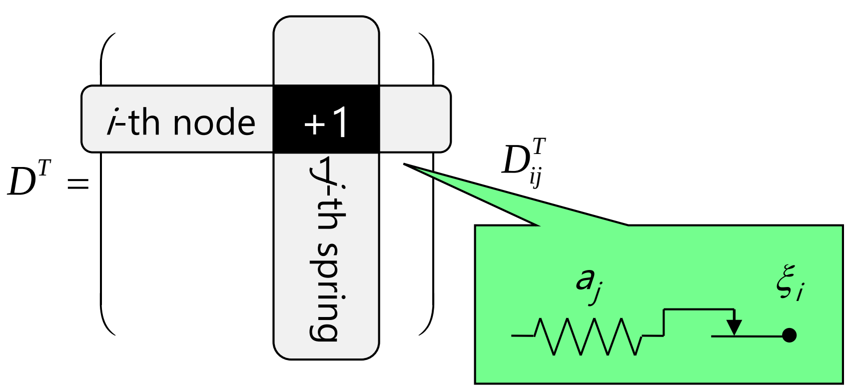
$\Leftrightarrow$

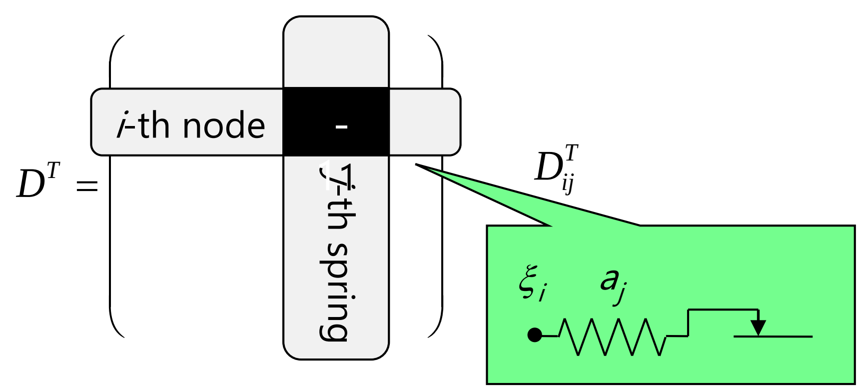
$$x = x|_U + x|_V$$

$$e + p \in U + g(t),$$

$$U = \{x \in D(\Re^n) : R^T x = 0\},\ g(t) = \left(D\bar{\xi}l(t)\right)\Big|_V,\ V = A^{-1}U^{\perp}$$

$$s^1 + \ldots + s^m = -D^T s$$





Static balance

Elastic deformation : $s = Ae,$

Plastic deformation : $\dot{p} \in N_C(s),$

Geometric constraint : $e + p \in D(\Re^n),$

Enforced constraint : $R^T(e+p) = l(t),$

$\Leftrightarrow$

$e + p \in U^{l(t)},$

$U^l = \{x \in D(\Re^n) : R^T x = l\}$

$\Leftrightarrow$

Static balance : $s^1 + \ldots + s^m + r^1 + \ldots + r^q + f(t) = 0.$

$R^T D\bar{\xi} = I_{q\times q}$

$\Longleftrightarrow$

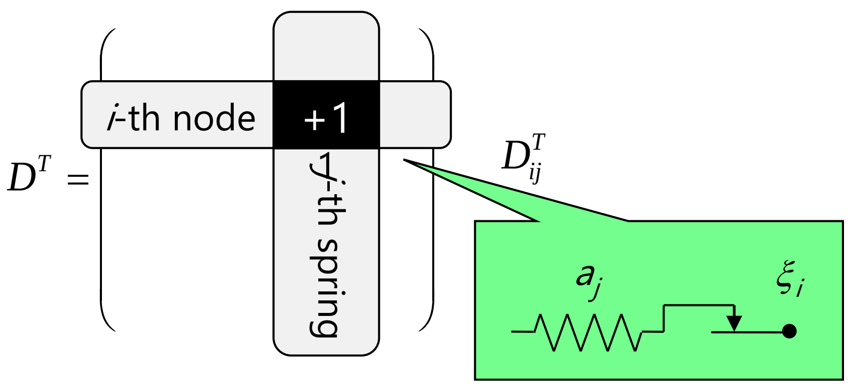
$x = x|_U + x|_V$

$e + p \in U + g(t),$

$U = \{x \in D(\Re^n) : R^T x = 0\},\ g(t) = \left(D\bar{\xi}l(t)\right)\big|_V,\ V = A^{-1}U^{\perp}$

$s^1 + \ldots + s^m = -D^T s$

$r^1 + \ldots + r^q = -D^T Rr$



Graph theory:

$-D^T R^k = (0,\ldots,0,1,0,\ldots,0,-1,0,\ldots,0)^T$

I^k-th component $\quad$ J^k-th component

Static balance

Elastic deformation: $s = Ae,$

Plastic deformation: $\dot{p} \in N_C(s),$

Geometric constraint: $e + p \in D(\Re^n),$

Enforced constraint: $R^T(e+p) = l(t),$

$\Leftrightarrow$

$e + p \in U^{l(t)},$

$U^l = \{x \in D(\Re^n) : R^T x = l\}$

$\Leftrightarrow$

Static balance: $s^1 + \ldots + s^m + r^1 + \ldots + r^q + f(t) = 0.$ $\Leftrightarrow$ $-D^T s - D^T Rr + f(t) = 0$

$R^T D\overline{\xi} = I_{q\times q}$

$\Leftrightarrow$

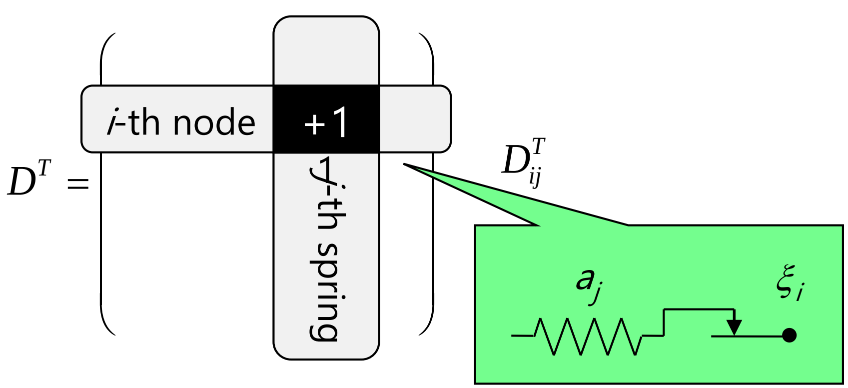
$x = x|_U + x|_V$

$e + p \in U + g(t),$

$U = \{x \in D(\Re^n) : R^T x = 0\},\ g(t) = \left(D\overline{\xi}l(t)\right)\Big|_V,\ V = A^{-1}U^{\perp}$

$s^1 + \ldots + s^m = -D^T s$

$r^1 + \ldots + r^q = -D^T Rr$



Graph theory:

$-\ D^T R^k = (0,\ldots,0,1,0,\ldots,0,-1,0,\ldots,0)^T$

I^k-th component $\qquad$ J^k-th component

Static balance

Elastic deformation: $s = Ae,$

Plastic deformation: $\dot{p} \in N_C(s),$

Geometric constraint: $e + p \in D(\Re^n),$

Enforced constraint: $R^T(e+p) = l(t),$

$\Leftrightarrow$

$e + p \in U^{l(t)},$

$U^l = \{x \in D(\Re^n) : R^T x = l\}$

$\Leftrightarrow$

Static balance: $s^1 + \ldots + s^m + r^1 + \ldots + r^q + f(t) = 0.$ $\Leftrightarrow$ $-D^T s - D^T Rr + f(t) = 0$

$R^T D\bar{\xi} = I_{q\times q}$

$\Leftrightarrow$

$x = x|_U + x|_V$

$e + p \in U + g(t),$

$U = \{x \in D(\Re^n) : R^T x = 0\},\ g(t) = \left(D\bar{\xi} l(t)\right)\big|_V,\ V = A^{-1}U^{\perp}$

$s^1 + \ldots + s^m = -D^T s$

Graph theory: $r^1 + \ldots + r^q = -D^T Rr$

$\Leftrightarrow$

$s + Rr + \bar{h}(t) \in \mathrm{Ker} D^T$

$f(t) = -D^T \bar{h}(t)$

$\Leftrightarrow$

$s + \bar{h}(t) \in U^{\perp}$

$\Leftrightarrow$

$s + h(t) \in U^{\perp}$

$h(t) = A\left(A^{-1}\bar{h}(t)\right)\big|_U$

$\Leftrightarrow$

$\Leftrightarrow$

By applying A^{-1}

$e + A^{-1}h(t) \in V$