METHOD OF DISCRETE APPROXIMATIONS FOR CONTROLLED SWEEPING PROCESSES

BORIS MORDUKHOVICH

Wayne State University, USA

Based on joint work with Tan Cao (SUNY Korea)

Talk give at the International Conference on

Control of State-Constrained Dynamical Systems

Padua, Italy, September 2017

Supported by NSF grant DMS-1512846
and by Air Force grant 15RT0462

MOREAU SWEEPING PROCESS

The basic version of Moreau's sweeping process **[Moreau74]** is described by the discontinuous dissipative differential inclusion

$$\begin{cases} -\dot{x}(t) & \in N\Big(x(t);C(t)\Big) \quad \text{a.e.} \quad t \in [0,T] \\ x(0) & = x_0 \in C(0) \end{cases}$$

where $N(\cdot;\Omega)$ stands for the usual normal cone of convex analysis, and where $t \mapsto C(t)$ is a (Lipschitz) continuous set-valued mapping (moving set). The above model can be equivalently written in the form of evolution variational inequalities

Sweeping process theory establishes the existence and uniqueness of Lipschitzian solutions and the like for a given moving set $C(t)$, and thus doesn't leave any room for optimization

CONTROLLED SWEEPING PROCESS

In **[ColomHenHuangMor12]** we suggested to insert control functions into the moving set

$$C(t)=C\Big(u(t)\Big)\ \text{ for all }\ t\in[0,T]$$

and to choose an optimal control $\bar{u}(t)$ that minimizes an appropriate cost functional. In this way we formulated and started to study new classes of optimal control problems different from those considered in control theory and applications

Note that we always have the (implicit) state constraints

$$x(t)\in C\Big(u(t)\Big)\ \text{ for all }\ i=1,\ldots,m\ \text{ and }\ t\in[0,T]$$

POLYHEDRAL CONTROLLED SWEEPING SETS

In **[ColomHenHuangMor12,16]** we consider the structure

$$C(t) := \Big\{x \in I\!R^n \Big| \ \langle u_i(t), x\rangle \le b_i(t),\ i = 1,\ldots,m\Big\}$$
with $\|u_i(t)\| = 1$ for all $i = 1,\ldots,m$ and $t \in [0,T]$

where both $u_i(t)$ and $b_i(t)$ are (Lipschitz or absolutely) continuous functions on $[0,T]$

In **[CaoMor16,17]** we consider the perturbed sweeping process

$$-\dot{x}(t) \in N\Big(x(t); C(t)\Big) + f\Big(x(t), b(t)\Big) \text{ a.e. } t \in [0,T],\ x(0) = x_0 \in C(0)$$

with the given force f and with controls $b\colon [0,T] \to I\!R^d$ in perturbations and controls $u\colon [0,T] \to I\!R^n$ in the polyhedral moving set generated by the fixed vectors a_i as

$$C(t) := C + u(t),\ \ C := \Big\{x \in I\!R^n \Big| \ \langle a_i, x\rangle \le 0,\ i = 1,\ldots,m\Big\}$$

CROWD MOTION MODEL IN A CORRIDOR

Among various applications of necessary optimality conditions obtained in these papers we mention optimal control of the crowd motion model in a corridor **[CaoMor17]**

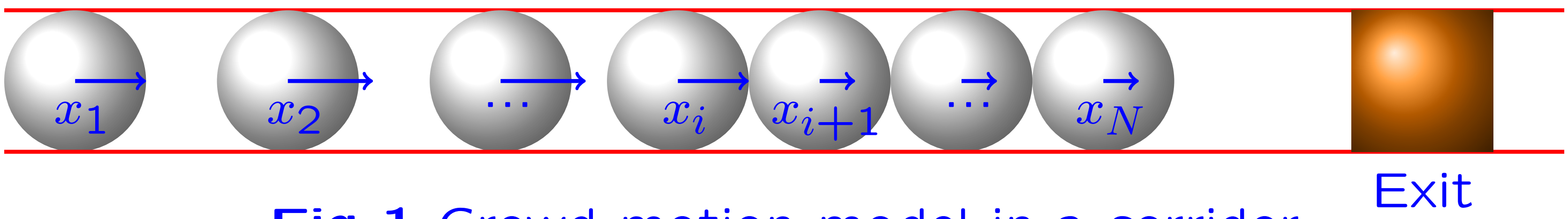

Fig 1 Crowd motion model in a corridor

The dynamic description of this model as a sweeping process was developed by Maury and Venel **[MauryVenel11]**

The polyhedral description of the moving set relates to the corridor version of the crowd motion model. We do not have it anymore (as well as the convexity of $C(t)$) for the more realistic and practical planar version

THE METHOD OF DISCRETE APPROXIMATIONS

for deriving necessary optimality conditions in optimal control of the differential inclusions

$$\dot{x}(t) \in F\Big(t, x(t)\Big) \;\text{ a.e. }\; t \in [0, T]$$

was developed in **[Mor95,06]** for bounded Lipschitzian mappings $F(t, \cdot)$. The three major steps of this approach are:

- Replace the time derivative $\dot{x}(t)$ by the finite differences

$$\dot{x}(t) \approx \frac{x(t+h) - x(t)}{h}, \quad h > 0$$

consider the family of discrete-time inclusions

$$(x(t+h) \in x(t) + hF\Big(x(t)\Big),\ t \in T_h := \Big\{0, h, \ldots, T-h\Big\},\ h \downarrow 0$$

with the corresponding approximations of cost functionals and constraints, and then establish well-posedness of the discrete approximation procedure in the sense of appropriate convergence of discrete (feasible and optimal) solutions to the ones for the original continuous-time control problems

• Reduce the discrete-time dynamic optimization problems for each fixed $h > 0$ to static problems of nonsmooth mathematical programming that contain, in particular, increasingly many geometric constraints. Then apply appropriate tools of variational analysis and generalized differentiation to derive necessary optimality conditions for each discrete-time problem

• Establish necessary conditions for appropriate classes of local minimizers of the original problem by passing to the limit as $h \downarrow 0$ from those for discrete approximations

The assumptions in **[Mor95,06]** fail for highly non-Lipschitzian sweeping control problems with intrinsic state constraints, and the method of discrete approximations requires further developments, particularly in implementing the limiting procedures

OPTIMAL CONTROL OF NONCONVEX SWEEPING PROCESS

Given a terminal cost function φ and a running cost ℓ, consider the optimal control problem (P): minimize

$$J[x,u,b]=\varphi\Big(x(T)\Big)+\int_0^T \ell\Big(t,x(t),u(t),b(t),\dot{x}(t),\dot{u}(t),\dot{b}(t)\Big)\,dt$$

over the controlled sweeping dynamics governed by the so-called play-and-stop operator appearing, e.g., in hysteresis

$$\begin{cases} \dot{x}(t)\in -N\Big(x(t);C(t)\Big)+f\Big(x(t),b(t)\Big) \\ \text{for a.e. } \ t\in[0,T], \quad x(0)=x_0\in C(0)\subset I\!R^n \\ \text{with } \ C(t)=C+u(t),\ C=\Big\{x\in I\!R^n\Big|\ g_i(x)\rangle\ge 0,\ i=1,\ldots,m\Big\} \\ 0<r_1\le\|u(t)\|\le r_2 \ \text{ for all } \ t\in[0,T] \end{cases}$$

where g_i are convex $\mathcal{C}^2$-smooth functions, the trajectory $x(t)$ and control $u(t) = (u_1(t), \ldots, u_n(t))$, $b(t) = (b_1(t), \ldots, b_n(t))$ functions are absolutely continuous on the fixed interval $[0, T]$

The normal cone in the nonconvex sweeping process is understood as the proximal one defined via the projections

$$N_P(\bar{x}; \Omega) := \Big\{ v \in I\!R^n \Big| \, \exists \, \alpha > 0 \; \text{ s.t. } \; \bar{x} \in \Pi(\bar{x} + \alpha v; \Omega) \Big\}, \; \bar{x} \in \Omega$$

with $N_P(\bar{x}; \Omega) := \emptyset$ for $\bar{x} \notin \Omega$. However, all the major normal cones agree under the assumptions made ensuring the uniform prox-regularity (or "positive reach") of the sweeping sets $C(t)$

Observe that we have the intrinsic/hidden state constraints

$$g_i\Big(x(t) - u(t)\Big) \geq 0 \quad \text{for all } \; t \in [0, T], \; i = 1, \ldots, m$$

due to the construction of the normal cone to $C(t) = C + u(t)$

DISCUSSION ON OPTIMAL CONTROL

The formulated optimal control problem for the sweeping process is not an optimization problem over a differential inclusion of the type $\dot{x} \in F(t,x)$. In our case the velocity set $F(t,x) = -N\big(x; C(t)\big) + f(x, b(t))$ is not fixed since the sweeping set $C(t) = C_{u(t)}(t)$ and the perturbation $f(x, b(t))$ are different for each control (u, b). Thus we optimize in the shape of $F(t,x)$ which somehow relates this problem to dynamic shape optimization. In fact there is no sense to formulate any optimization problem for the differential inclusion

$$\dot{x} \in F(t,x) := -N\Big(x; C(t)\Big) + f\Big(x, b(t)\Big), \quad t \in [0,T]$$

when $C(t)$ is fixed since, in major cases, the sweeping inclusion admits a unique solution for every initial point $x(0) = x_0 \in C(0)$

REFORMULATION

Denote $z := (x,u,b) \in I\!R^{3n},\ z(0) := (x_0, u(0), b(0))$

$G(z) := -N\Big(x; C(u)\Big) + f(x,b),\ \ C(u) := u + \Big\{x |\ g_i(x) \geq 0,\ i = 1,\ldots,m\Big\}$

Problem (P) can be reformulated as: minimize

$$J[z] = \varphi\Big(z(T)\Big) + \int_0^T \ell\Big(t, z(t), \dot{z}(t)\Big)\, dt \ \text{ s.t.}$$

$$\dot{z}(t) \in F\Big(z(t)\Big) := G\Big(z(t)\Big) \times I\!R^n \times I\!R^n \ \text{ a.e. } \ t \in [0,T]$$

$$g_i\Big(x(t) - u(t)\Big) \geq 0 \ \text{ for all } \ t \in [0,T],\ i = 1,\ldots,m$$

$$r_1 \leq \|u(t)\| \leq r_2 \ \text{ for all } \ t \in [0,T]$$

$F(z)$ is unbounded and highly non-Lipschitzian (discontinuous)

DISCRETE APPROXIMATIONS OF SWEEPING TRAJECTORIES

THEOREM Fix an arbitrary feasible solution $\bar{z}(\cdot)$ to (P) and consider discrete partitions

$$\Delta_k := \left\{0 = t_0^k < t_1^k < \ldots < t_k^k = T\right\},\ h_k := \max_{0 \le j \le k-1} \{t_{j+1}^k - t_j^k\} \downarrow 0$$

Then there is a sequence of piecewise linear functions $z^k(t) := (x^k(t), u^k(t), b^k(t))$ on $[0, T]$ satisfying the discretized inclusions

$$x^k(t) = x^k(t_j) + (t - t_j)v_j^k,\ x(0) = x_0,\ t_j^k \le t \le t_{j+1}^k,\ j = 0, \ldots, k-1$$

with $v_j^k \in G(z^k(t_j^k))$ on Δ_k and the perturbed constraints

$$r_1 - \varepsilon_k \le \|u^k(t_j^k)\| \le r_2 + \varepsilon_k,\ t_j^k \in \Delta_k,\ i = 1, \ldots, m$$

while exhibiting the $W^{1,2}[0, T]$ convergence

$$z^k(t) \to \bar{z}(t) \ \text{ uniformly } \ \text{ on } [0, T],\ \int_0^T \|\dot{z}^k(t) - \dot{\bar{z}}(t)\|^2\, dt \to 0$$

LOCAL RELAXATION OF OPTIMAL SWEEPING SOLUTIONS

If the running cost $\ell(t, z, \cdot)$ is convex with respect to velocity variables, then there exists an optimal solution $z(\cdot) \in W^{1,2}[0, T]$ to the controlled sweeping process (P). Furthermore, it $\bar{z}(\cdot)$ is a strong local minimizer to (P), it also gives strong local minimum to the relaxed problem: minimize

$$\widehat{J}[z] := \varphi\Big(x(T)\Big) + \int_0^T \widehat{\ell}\Big(t, x(t), u(t), b(t), \dot{x}(t), \dot{u}(t), \dot{b}(t)\Big)dt$$

subject to the convexified inclusion

$$\dot{x}(t) \in \text{cl co}\, G\Big(x(t), u(t), b(t)\Big)$$

under the same constraints, where $\widehat{\ell}$ stands for the convexification of ℓ with respect to velocity variables. This can be deduced from the recent results by Tolstonogov **[Tols16]**

DISCRETE OPTIMAL CONTROL PROBLEMS

Let $\bar{z}(\cdot) = \big(\bar{x}(\cdot), \bar{u}(\cdot), \bar{b}(\cdot)\big)$ be a strong local minimizer for (P). Consider discrete approximation problems (P_k) : minimize

$$J_k[z^k] := \varphi(x_k^k) + h_k \sum_{j=0}^{k-1} \ell\Big(z_j^k, \frac{z_{j+1}^k - z_j^k}{h_k}\Big)$$

$$+ \sum_{j=0}^{k-1} \int_{t_j^k}^{t_{j+1}^k} \left\| \frac{z_{j+1}^k - z_j^k}{h_k} - \dot{\bar{z}}(t) \right\|^2 dt$$

over $z^k := (x_0^k, \ldots, x_k^k, u_0^k, \ldots, u_k^k, b_0^k, \ldots, b_k^k)$ satisfying

$$x_{j+1}^k \in x_j^k + h_k G(x_j^k, u_j^k, b_j^k),\ j = 0, \ldots, k-1,\ \ x_0^k = x_0$$

$$g_i(x_j^k - u_j^k) \geq 0,\ r_1 - \varepsilon_k \leq \left\| u_j^k \right\| \leq r_2 + \varepsilon_k,\ j = 0, \ldots, k-1,\ \ i = 1, \ldots, m$$

Each problem (P_k) admits optimal solutions

STRONG CONVERGENCE OF DISCRETE APPROXIMATIONS

THEOREM Let $\bar{z}(\cdot) = (\bar{x}(\cdot), \bar{u}(\cdot), \bar{b}(\cdot))$ be a strong local minimizer for (P). Then any sequence of piecewise linearly extended to $[0, T]$ optimal solutions $\bar{z}^k(t)$ of the discrete problems (P_k) strongly converges to $\bar{z}(t)$ in the Sobolev space $W^{1,2}[0, T]$

PROOF Using the above result on the strong $W^{1,2}$ approximation of sweeping trajectories and local relaxation stability

GENERALIZED DIFFERENTIATION

See **[Mor06,RW98]**

Normal Cone to a closed set $\Omega \subset I\!R^n$ at $\bar{x} \in \Omega$

$$N(\bar{x};\Omega) := \Big\{ v \Big| \ \exists x_k \to \bar{x}, w_k \in \Pi(x_k;\Omega), \alpha_k \ge 0, \alpha_k(x_k - w_k) \to v \Big\}$$

Subdifferential of an l.s.c. function $\varphi\colon I\!R^n \to (-\infty,\infty]$ at $\bar{x}$

$$\partial\varphi(\bar{x}) := \Big\{ v \Big| \ (v,-1) \in N((\bar{x},\varphi(\bar{x})); \text{epi}\,\varphi) \Big\}, \quad \bar{x} \in \text{dom}\,\varphi$$

Coderivative of a set-valued mapping F

$$D^*F(\bar{x},\bar{y})(u) := \Big\{ v \Big| \ (v,-u) \in N((\bar{x},\bar{y}); \text{gph}\,F) \Big\}, \quad \bar{y} \in F(\bar{x})$$

Generalized Hessian of φ at $\bar{x}$

$$\partial^2\varphi(\bar{x}) := D^*(\partial\varphi)(\bar{x},\bar{v}), \quad \bar{v} \in \partial\varphi(\bar{x})$$

Enjoy **FULL CALCULUS** and **COMPLETELY COMPUTED** in terms of the given data of (P)

FURTHER STRATEGY

• For each k reduce problem (P_k) to a problem of mathematical programming (MP) with functional and increasingly many geometric constraints. The latter are given by graphs of the mapping $G(z) := -N(x; C(u)) + f(x, b)$, and so (MP) is intrinsically nonsmooth and nonconvex even for smooth initial data

• Use variational analysis analysis and generalized differentiation (first- and second-order) to derive necessary optimality conditions for (MP) and then discrete control problems (P_k)

• Explicitly calculate the coderivative of $G(z)$ entirely in terms of the given data of (P)

• By passing to the limit as $k \to \infty$, to derive necessary optimality conditions for the sweeping control problem (P)

NECESSARY OPTIMALITY CONDITIONS FOR (P)

For simplicity consider the case of smooth costs φ, ℓ

THEOREM Let $\bar{z}(\cdot)$ be a strong local minimizer for (P). Then there exist a multiplier $\lambda \geq 0$, an adjoint arc $p(t) = (p_x, p_u, p_b)(t) \in W^{1,2}$, subgradient functions $w(t) = (w^x, w^u, w^b) \in L^2$ and $v(t) = (v^x, v^u, v^b) \in L^2$ such that

$$\Big(w(t), v(t)\Big) \in \text{co}\, \partial\ell\Big(t, \bar{z}(t), \dot{\bar{z}}(t)\Big) \text{ a.e.}$$

and Borel measures $\gamma \in C^*$, $\xi^1 \in C^*_+$, $\xi^2 \in C^*_-$ satisfying

- **Primal-Dual Dynamic Relationships**

$$\dot{\bar{x}}(t) + f\Big(\bar{x}(t), \bar{b}(t)\Big) = \sum_{i=1}^{m} \eta_i(t) \nabla g_i\Big(\bar{x}(t) - \bar{u}(t)\Big) \text{ a.e.}$$

with the uniquely defined $\eta(t) \in L^2$ and

$$\dot{p}(t) = \lambda w(t) + \Big(\nabla_x f\big(\bar{x}(t), \bar{b}(t)\big)^* \big(\lambda v^x(t) - q^x(t)\big), 0, \nabla_b f\big(\bar{x}(t), \bar{b}(t)\big)^* \big(\lambda v^x(t) - q^x(t)\big)\Big)$$

$$q^u(t) = \lambda \nabla_{\dot{u}} \ell\big(t, \dot{\bar{u}}(t)\big),\ q^b(t) \in \lambda \partial_{\dot{b}} \ell\big(t, \dot{\bar{b}}(t)\big) \text{ a.e.}$$

where $q(t) = (q^x, q^u, q^b)$ is of bounded variation given by

$$q(t) := p(t) - \int_{[t,T]} \big(-d\gamma(s), 2\bar{u}(s)d(\xi^1(s) + \xi^2(s)) + d\gamma(s), 0\big)$$

Moreover, we have the implications

$$g_i\big(\bar{x}(t) - \bar{u}(t)\big) > 0 \Rightarrow \eta_i(t) = 0,\ \eta_i(t) > 0 \Rightarrow \langle \nabla g_i\big(\bar{x}(t) - \bar{u}(t), \lambda v^x(t) - q^x(t)\big)$$

- **Transversality Conditions**

$$-p^x(T) + \sum_{i\in I(\bar{x}(T)-\bar{u}(T))} \eta_i(T)\nabla g_i\Big(\bar{x}(T) - \bar{u}(T)\Big) \in \lambda\partial\varphi\big(\bar{x}(T)\big)$$

$$p^u(T) - \sum_{i\in I(\bar{x}(T)-\bar{u}(T))} \eta_i(T)\nabla g_i\Big(\bar{x}(T) - \bar{u}(T)\Big) \in$$

$$-2\bar{u}(T)\left(N_{[0,r_2]}\Big(\|\bar{u}(T)\|\Big) + N_{[r_1,\infty)}\Big(\|\bar{u}(T)\|\Big)\right)$$

$$p^b(T) = 0$$

where $I(y) \subset \{1, \ldots, m\}$ is the set of active constraint indices

- **Nontriviality Conditions**

$$\lambda + \|q^u(0)\| + \|p(T)\| + \|\xi^1\| + \|\xi^2\| > 0$$

Furthermore we have the implications

$$\Big[g_i(x_0 - \bar{u}(0)) > 0,\ i = 1, \ldots, m\Big] \Rightarrow \Big[\lambda + \|p(T)\| + \|\xi^1\| + \|\xi^2\| > 0\Big]$$

$$\Big[g_i(\bar{x}(T) - \bar{u}(T)) > 0,\ r_1 < \|\bar{u}(T)\| < r_2,\ i = 1, \ldots, m\Big] \Rightarrow$$
$$\Big[\lambda + \|q^u(0)\| + \|\xi^1\| + \|\xi^2\| > 0\Big]$$

REFERENCES

• **J. J. MOREAU**, On unilateral constraints, friction and plasticity, in: New Variational Techniques in Mathematical Physics, pp. 173–322, Rome, 1974

• **G. COLOMBO, R. HENRION, N. D. HOANG and B. S. MORDUKHOVICH**, Optimal control of the sweeping process, DCDIS–Ser B **19** (2012), 117–159

• **G. COLOMBO, R. HENRION, N. D. HOANG and B. S. MORDUKHOVICH**, Optimal control of the sweeping process over polyhedral controlled sets, J. Diff. Eqs. **260** (2016), 3397–3447

• **T. H. CAO and B. S. MORDUKHOVICH**, Optimal control of a perturbed sweeping process via discrete approximations, Disc. Contin. Dyn. Syst.–Ser B **21** (2016), 3331–3358

• **T. H. CAO and B. S. MORDUKHOVICH**, Optimality conditions for a controlled sweeping process with applications to the crowd motion model, Disc. Contin. Dyn. Syst.–Ser B **22** (2017), 267–306

• **B. MAURY and J. VENEL**, A discrete contact model for crowd motion, ESAIM-M2NA **45** (2011), 145–168

• **B. S. MORDUKHOVICH**, Discrete approximations and refined Euler-Lagrange conditions for differential inclusions, SIAM J. Control Optim. **33** (1995), 882–915

- **B. S. MORDUKHOVICH**, **VARIATIONAL ANALYSIS AND GENERALIZED DIFFERENTIATION, I: BASIC THEORY, II: APPLICATIONS**, Springer, 2006

- **A. A. TOLSTONOGOV**, Control sweeping process, J. Convex Anal. **23** (2016), 1099–1123

- **R. T. ROCKAFELLAR and R. J-B. WETS**, **VARIATIONAL ANALYSIS**, Springer, 1998