# Rate-independent unilateral evolutions for Ambrosio-Tortorelli functionals and applications (to fracture mechanics)

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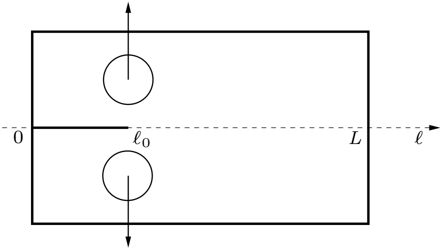
- sharp crack (general theory)
- phase-field

http://matematica.unipv.it/negri/

# Setting: ASTM Compact Tension




Admissible cracks: $K_\ell$ for $\ell \in [\ell_0, L]$

Admissible displacement: $\mathcal{U}(t,\ell) = \{u \in H^1(\Omega \setminus K_\ell, \mathbb{R}^2) : u = \pm t\hat{e}_2$ on $\partial_D \Omega\}$

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Linear elastic energy $\quad E(t,\ell,u) = \frac{1}{2} \int_{\Omega \setminus K_\ell} W(\boldsymbol{\varepsilon}(u))\, dx$ for $u \in \mathcal{U}(t,\ell)$

Dissipated energy $\quad \mathcal{D}(\ell) = G_c(\ell - \ell_0)$

# Energy

Potential energy functional $\boxed{\mathcal{F}(t,\ell) = \mathcal{E}(t,\ell) + \mathcal{D}(\ell)}$

"Static condensation of u" $\mathcal{E}(t,\ell) = min\,\{E(t,\ell,u) : u \in \mathcal{U}(t,\ell)\}$

Stored energy $\mathcal{E}(t,\cdot)$ is non-increasing, non-convex and of class $W^{2,\infty}_{loc}$

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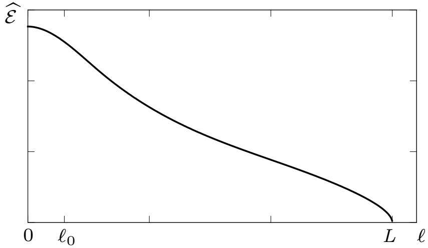
Behaviour of $\widehat{\mathcal{E}}(\ell)$ (i.e. for $u = \pm\hat{e}_2$ on $\partial_D\Omega$)



# Energy

Potential energy functional $\boxed{\mathcal{F}(t,\ell) = \mathcal{E}(t,\ell) + \mathcal{D}(\ell)}$

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By irreversibility equilibrium is characterized by

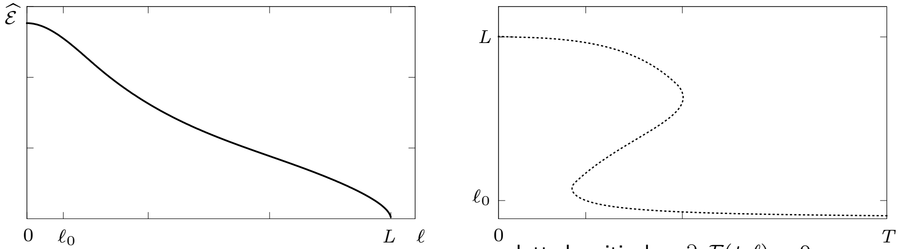
$$\partial_\ell \mathcal{F}(t,\ell) \geq 0 \quad \Leftrightarrow \quad [\partial_\ell \mathcal{F}(t,\ell)]_- = 0 \quad \Leftrightarrow \quad G(t,\ell) \leq G_c$$

where $G(t,\cdot) = -\partial_\ell \mathcal{E}(t,\cdot)$ is the energy release

# Equilibrium configurations

Behaviour of $\widehat{\mathcal{E}}(\ell)$ and set $\partial_\ell \mathcal{F}(t,\ell) = 0$

(interpolated numerical computation)




dotted: critical $\partial_\ell \mathcal{F}(t,\ell) = 0$

stable $\partial_\ell \mathcal{F}(t,\ell) > 0$ (left)

unstable $\partial_\ell \mathcal{F}(t,\ell) < 0$ (right)

$\boxed{\text{Griffith's (selection) criterion}}$

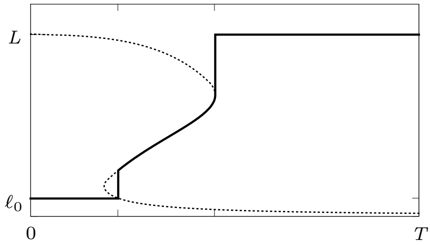
*"the* **equilibrium position** *[...] must be one in which [...] the system can pass from the unbroken to the broken condition by a process involving a* **continuous decrease of potential energy***."*

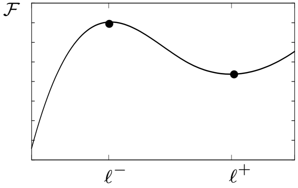
# “Mechanical” characterization

An evolution $\ell : [0, T] \to [\ell_0, L]$ s.t. (for $\ell$ left-continuous) [N.-Ortner (08)]

- $\ell$ is non-decreasing, thus in $BV(0, T)$ (irreversibility)
- $\partial_\ell \mathcal{F}(t, \ell(t)) \geq 0$ (equilibrium)
- $\partial_\ell \mathcal{F}(t, \ell(t)) d\ell(t) = 0$ (in the sense of measures) (stability)
- $\partial_\ell \mathcal{F}(t, l) \leq 0$ for $l \in [\ell^-(t), \ell^+(t)]$ and $t \in J_\ell$ (instability)

Jumps are characterized by unstable propagations.



# Characterization by equilibrium and energy balance

An evolution $\ell : [0, T] \to [\ell_0, L]$ s.t. (for $\ell$ left-continuous)

- $\ell$ is non-decreasing, thus in $BV(0, T)$ (irreversibility)
- $[\partial_\ell \mathcal{F}(t, \ell)]_- = 0$ $\boxed{\text{equilibrium}}$
- $\mathcal{F}(t, \ell(t)) = \mathcal{F}(0, \ell_0) + \int_0^t \mathcal{P}_{ext}(s, \ell(s))\, ds - \sum_{s \in J_\ell} \big| [\![ \mathcal{F}(s, \cdot) ]\!] \big|$

$\boxed{\text{energy balance}}$

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Stability and instability $\Leftrightarrow$ energy balance $\boxed{\text{by the chain rule in BV}}$

Writing $d\ell = d\ell^{ac} + \sum_{t \in J_\ell} [\![ \ell(t) ]\!] \, \delta_t$ by the chain rule in $BV$

$$d\mathcal{F}(t,\ell(t)) = \partial_t \mathcal{F}(t,\ell(t)) \, dt \; + \; \partial_\ell \mathcal{F}(t,\ell(t)) \, d\ell^{ac}(t) \; + \; \sum_{t \in J_\ell} [\![ \mathcal{F}(t,\cdot) ]\!] \delta_t$$

since $\partial_t \mathcal{F}(t,\ell(t)) = \mathcal{P}_{ext}(t,\ell(t))$

$\partial_\ell \mathcal{F}(t,\ell(t)) \, d\ell^{ac}(t) = 0$

$[\![ \mathcal{F}(s,\cdot) ]\!] = -\big| [\![ \mathcal{F}(s,\cdot) ]\!] \big|$

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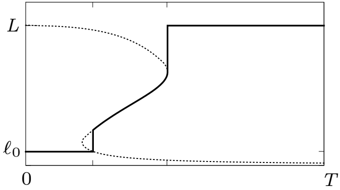
Viceversa in a similar way on subintervals of $[0,\, T]$

# BV-evolution by graph characterization

Parametrization of the extended graph

$$s \mapsto (t(s), \ell(s)) \text{ in } W^{1,\infty}(0, S)$$

- discontinuity intervals where $t'(s) = 0$
- continuity points if $t'(s) > 0$

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A parametrization $s \mapsto (t(s), \ell(s))$ with $0 \le t' \le 1$ and $0 \le \ell' \le 1$ s.t.

- $[\partial_\ell \mathcal{F}(t(s), \ell(s))]_- = 0$ for every $s$ with $t'(s) > 0$ $\boxed{\text{equilibrium}}$

- $$\mathcal{F}(t(s), \ell(s)) = \mathcal{F}(0, \ell_0) + \int_0^s \mathcal{P}_{ext}(t(r), \ell(r))\, t'(r)\, dr + \\ - \int_0^s [\partial_\ell \mathcal{F}(t(r), \ell(r))]_- \, \ell'(r)\, dr$$ $\boxed{\text{energy balance}}$

[Efendiev-Mielke (06), N. (14), Mielke-Rossi-Savarè (16)]

# BV-evolution by graph characterization

Parametrization of the extended graph

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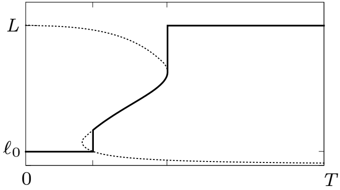


A parametrization $s \mapsto (t(s), \ell(s))$ with $t' \geq 0$, $\ell' \geq 0$ and $t' + \ell' \leq 1$

- $$\mathcal{F}(t(s), \ell(s)) = \mathcal{F}(0, \ell_0) + \int_0^s \mathcal{P}_{ext}(t(r), \ell(r))\, t'(r)\, dr + \\ - \int_0^s |\partial_\ell \mathcal{F}(t(r), \ell(r))|_-\, dr$$

energy balance

[Efendiev-Mielke (06), N. (14), Mielke-Rossi-Savarè (16)]

A single scalar equations, suitable for the metric setting

# PDE approach

**A rate independent dissipation functional**

$$\Psi(\dot\ell) = \begin{cases} +\infty & \text{for } \dot\ell < 0 \\ G_c \dot\ell & \text{for } \dot\ell \geq 0 \end{cases} \qquad \partial\Psi(\dot\ell) = \begin{cases} \emptyset & \text{for } \dot\ell < 0 \\ (-\infty, G_c] & \text{for } \dot\ell = 0 \\ G_c & \text{for } \dot\ell > 0 \end{cases}$$

$$\mathcal{D}(\ell(t)) = \int_0^t \Psi(d\ell) = G_c(\ell(t) - \ell_0)$$

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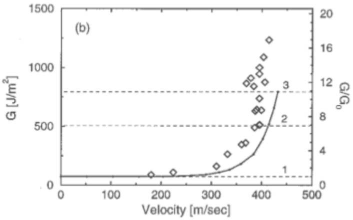
- $-\partial_\ell \mathcal{E}(t, \ell(t)) \in \partial\Psi(0)$ for a.e. $t \in (0, T)$ **equilibrium**
- $\mathcal{E}(t, \ell(t)) = \mathcal{E}(0, \ell_0) + \int_0^t \mathcal{P}_{ext}(s, \ell(s))\, ds - \int_0^t \Psi(d\ell) - \sum_{s \in J_\ell} diss_J(s)$ **energy balance**

$$diss_J(s) = \int_{\ell^-}^{\ell^+} dist\big(-\partial_\ell \mathcal{E}(s, l), \partial\Psi(0)\big)\, dl = \int_{\ell^-}^{\ell^+} [\partial_\ell \mathcal{F}(s, l)]_+ \, dl = |[\![\mathcal{F}(s, \cdot)]\!]|$$

[Mielke etc. (..)]

$$\Psi_\varepsilon(\dot\ell) = \begin{cases} +\infty & \text{for } \dot\ell < 0 \\ G_c \dot\ell + \varepsilon \frac{1}{2} \dot\ell^{\,2} & \text{for } \dot\ell \geq 0 \end{cases}$$

$$\partial\Psi_\varepsilon(\dot\ell) = \begin{cases} \emptyset & \text{for } \dot\ell < 0 \\ (-\infty, G_c] & \text{for } \dot\ell = 0 \\ G_c + \varepsilon\,\dot\ell & \text{for } \dot\ell > 0 \end{cases}$$




Rate dependence in Homalite [Hauch-Marder (94)]

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Crack evolution is approximated by the (doubly non-linear) differential inclusion

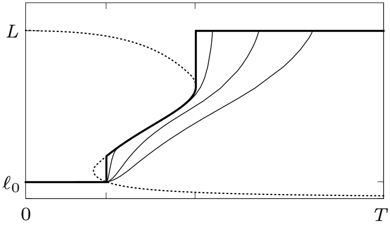
$$\begin{cases} \partial\Psi_\varepsilon(\dot\ell_\varepsilon(t)) \ni -\partial_\ell \mathcal{E}(t, \ell_\varepsilon(t)) \\ \ell_\varepsilon(0) = \ell_0 \end{cases}$$

- existence of a $C^1$ solution
- convergence (at least pointwise) of $\ell_\varepsilon$ to a BV solution $\ell$

[Dal Maso-DeSimome-Mora-Morini (08), Knees-Mielke-Zanini (08), Mielke-Rossi-Savarè (..), N. (10)]

# The limit parametrized BV evolution (I)

Convergence of the evolutions




Equivalent formulations

$$\partial\Psi_\varepsilon(\dot\ell_\varepsilon(t)) \ni -\partial_\ell \mathcal{E}(t, \ell_\varepsilon(t))$$

$$\Updownarrow$$

$$\varepsilon \dot\ell_\varepsilon(t) = [\partial_\ell \mathcal{F}(t, \ell_\varepsilon(t))]_-$$

$$\Downarrow$$

$$\mathcal{F}_\varepsilon(t, \ell_\varepsilon(t)) = \mathcal{F}(0, \ell_0) + \int_0^t \partial_\ell \mathcal{F}(\tau, \ell_\varepsilon(\tau))\, \dot\ell_\varepsilon(\tau)\, dt + \int_0^t \mathcal{P}_{ext}(\tau, \ell_\varepsilon(\tau))\, d\tau$$

$$\Downarrow$$

$$\mathcal{F}_\varepsilon(t, \ell_\varepsilon(t)) = \mathcal{F}(0, \ell_0) - \int_0^t \varepsilon^{-1} [\partial_\ell \mathcal{F}(\tau, \ell_\varepsilon(\tau))]_-^2\, dt + \int_0^t \mathcal{P}_{ext}(\tau, \ell_\varepsilon(\tau))\, d\tau$$

Length $s_\varepsilon(t) = t + \int_0^t |\dot\ell_\varepsilon(\tau)| \, d\tau$ Inverse $t_\varepsilon(s)$ and $\ell_\varepsilon(s) = \ell_\varepsilon \circ t_\varepsilon(s)$

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By change of variable and by $t'_\varepsilon(s) + |\ell'_\varepsilon(s)| = 1$

$$\mathcal{F}_\varepsilon(t_\varepsilon(s), \ell_\varepsilon(s)) = \mathcal{F}(0, \ell_0) - \int_0^s \Phi_\varepsilon\big([\partial_\ell \mathcal{F}(t_\varepsilon(r), \ell_\varepsilon(r))]_-\big) \, dr + \int_0^s \mathcal{P}_{ext}(\dots) \, dr$$

where $\Phi_\varepsilon(z) = \dfrac{z^2}{\varepsilon + z} \ \to \ \Phi(z) = z$ uniformly

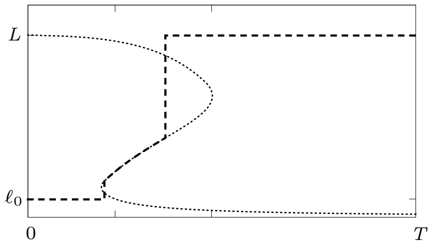
$$\mathcal{F}_\varepsilon(t(s), \ell(s)) = \mathcal{F}(0, \ell_0) - \int_0^s [\partial_\ell \mathcal{F}(t(r), \ell(r))]_- \, dr + \int_0^s \mathcal{P}_{ext}(\dots) \, dr$$

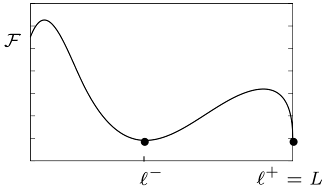
# (Quasi-static) energetic evolutions

Find $\ell : [0, T] \to [\ell_0, L]$ monotone:

- $\ell(t) \in argmin\,\{\mathcal{F}(t, l) : l \geq \ell^-(t)\}$
- $\mathcal{E}(t, \ell(t)) = \mathcal{E}(0, \ell_0) + \int_0^t \mathcal{P}_{ext}(s, \ell(s))\, ds - \mathcal{D}(\ell(t))$

[Francfort-Marigo (98), Dal Maso-Toader (02), Mielke-Roubíček (15)]




Without jumps are BV evolutions (e.g. strictly convex energies)

Choice of a norm $\|\cdot\|_V$ (or a metric) in the state space $V$

Formally same characterizations and constructions

Equilibrium and (normalized) gradient flow w.r.t. $\|\cdot\|_V$ on jumps

- Parametrized BV solutions [N. (14)]

$$|\partial_v \mathcal{F}(t,v)| = \limsup_{w \to v} \frac{[\mathcal{F}(t,v) - \mathcal{F}(t,w)]_-}{\|v-w\|_V} \qquad \text{(slope w.r.t. } \|\cdot\|_V)$$

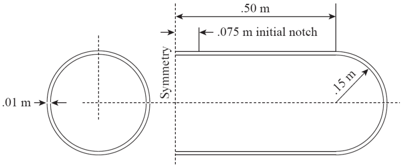
$$= \sup\left\{ -d\mathcal{F}(t,v)[\xi] : \|\xi\|_V \le 1 \right\} \qquad \text{(if } \mathcal{F} \text{ regular)}$$

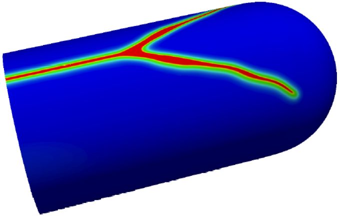
By chain rule $d\mathcal{F}(t(s),v(s))[v'(s)] = |\partial_v \mathcal{F}(t(s),v(s))|$

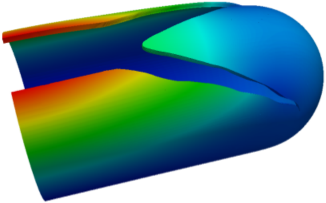
$v'(s)$ is the steepest descent direction

- Vanishing viscosity $\Psi_\varepsilon(\dot v) = \Psi(\dot v) + \varepsilon \frac{1}{2}\|\dot v\|_V^2$ [Mielke-Rossi-Savarè (..)]

# Phase-field $v$ and (equivalent) displacement $u$



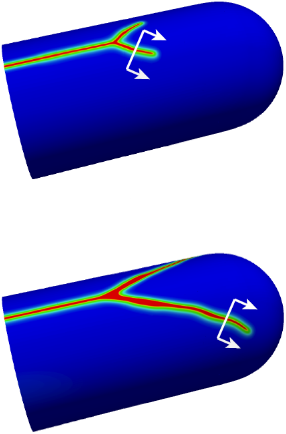


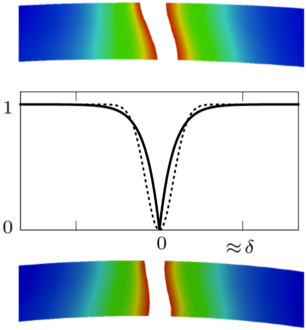
Pictures are courtesy of [Borden-Verhoosel-Scott-Hughes-Landis (11)]

# Phase-field function $v$

Phase field variable $v$ plotted in the reference configuration

Pictures are courtesy of [Borden-Verhoosel-Scott-Hughes-Landis (11)]





# Phase-field setting

Energy functional $\boxed{\mathcal{F}(t,u,v) = \mathcal{E}(t,u,v) + G_c \mathcal{L}(v)}$

$$\mathcal{U} = \{u \in H^1_0(\Omega, \mathbb{R}^2)\} \qquad \mathcal{V} = \{v \in H^1(\Omega),\, 0 \le v \le 1\}$$

Linear elastic energy $\mathcal{E}(t,u,v) = \frac{1}{2} \int_\Omega (v^2 + \eta)\, W(\varepsilon)\, dx - \int_\Omega f(t) \cdot u \, dx$

Dissipated energy and dissipation

$$\mathcal{L}(v) = \frac{1}{2} \int_\Omega (v-1)^2 + |\nabla v|^2 \, dx \qquad \mathcal{D}(v, \dot v) = \begin{cases} d\mathcal{L}(v)[\dot v] & \dot v \le 0 \\ +\infty & \text{other.} \end{cases}$$

$\boxed{\mathcal{F}(t,\cdot,\cdot) \text{ is non-convex but separately convex (quadratic)}}$

$\boxed{\text{Which quasi-static evolution ?}}$

$i)$ alternate minimization

$ii)$ gradient flow (vanishing viscosity)

# From phase-field to sharp crack

$\Gamma$-convergence of the energies (convergence holds in the setting of $SBV$ .... $GSBD^2$ )

$$\mathcal{F}_\delta(t,u,v) = \int_\Omega (v^2 + \eta_\delta)\, W(\boldsymbol{\varepsilon})\, dx + \tfrac{1}{2} \int_\Omega (\tfrac{1}{2}\delta^{-1})(v-1)^2 + (2\delta)|\nabla v|^2\, dx$$

$$\downarrow$$

$$\mathcal{F}_0(t,u) = \int_{\Omega \setminus J_u} W(\boldsymbol{\varepsilon})\, dx + \mathcal{H}^{n-1}(J_u)$$

[Ambrosio & Tortorelli (90)] [Chambolle (03)] [Iurlano (12)]

$\Gamma$-convergence implies convergence of (global) minimizers

$\boxed{\text{Slope for } \mathcal{F}_0 \text{ is not known}}$

# Time discrete evolution by alternate minimization

Time discretization $t_k = k\triangle t$. Set $u(t_0)$ and $v(t_0)$

Define by recursion, $u_m = u(t_{k-1})$ and $v_m = v(t_{k-1})$ for $m = 0$

$$\text{for } m \geq 1 \qquad \begin{cases} u_m \in \text{argmin}\,\big\{\mathcal{F}\big(t_k\,, \bullet\,, v_{m-1}\big) \text{ in } \mathcal{U}\,\big\} \\ v_m \in \text{argmin}\,\big\{\mathcal{F}(t_k\,, u_m\,, \bullet\,) \text{ in } \mathcal{V} \text{ with } v \leq v_{m-1}\big\} \end{cases}$$

[Bourdin-Francfort-Marigo (00)]

Let $u(t_k) = \lim\limits_{m\to+\infty} u_m$ and $v(t_k) = \lim\limits_{m\to+\infty} v_m$

Then $u(t_k), v(t_k)$ are equilibrium points for $\mathcal{F}(t_k, \cdot, \cdot)$ and separate minimizers

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Piecewise-affine interpolation $u_{\triangle t}$ and $v_{\triangle t}$

$\boxed{\text{Which evolution for } \triangle t \to 0\,?}$

# Families of intrinsic norms for the phase field energy (I)

Write the energy

$$\begin{aligned}\mathcal{F}(t,u,v) &= \tfrac{1}{2}\int_\Omega (v^2+\eta)\,W(\boldsymbol{\varepsilon})\,dx - \int_\Omega f(t)\cdot u\,dx + G_c\mathcal{L}(v)\\ &= \qquad \tfrac{1}{2}\|u\|_v^2 \qquad\quad + \qquad \beta_t(u) \qquad + \quad c_v\end{aligned}$$

$\|\cdot\|_v$ is equivalent to $\|\cdot\|_{H^1}$ (by Korn-Poincaré)

The corresponding $\boxed{\text{'slope'}}$

$$|\partial_u\mathcal{F}(t,u,v)|_v = \Big|\min\big\{\partial_u\mathcal{F}(t,u,v)[\phi] \,:\, \|\phi\|_v \le 1\big\}\Big|$$

---

If $t_n \to t$, $u_n \to u$ in $H^1$ and $v_n \rightharpoonup v$ in $H^1$ (with $0 \le v_n \le 1$) then

$$\liminf_n |\partial_u\mathcal{F}(t_n,u_n,v_n)|_{v_n} \ge |\partial_u\mathcal{F}(t,u,v)|_v$$

# Families of intrinsic norms for the phase field energy (II)

Write the energy

$$\begin{aligned}\mathcal{F}(t,u,v) &= \tfrac{1}{2}\int_\Omega (v^2+\eta)\,W(\boldsymbol{\varepsilon})\,dx + \tfrac{1}{2}G_c\int_\Omega (v-1)^2 + |\nabla v|^2\,dx - \dots \\ &= \tfrac{1}{2}\|v\|_u^2 + b(v) + c_{t,u}\end{aligned}$$

$\|\cdot\|_u$ is equivalent to $\|\cdot\|_{H^1}$

Regularity and continuous dependence for

$u \in \operatorname{argmin}\,\{\mathcal{E}(t,\cdot,v) : u \in \mathcal{U}\}$ in $W^{1,p}(\Omega)$ for $p \gtrsim 2$ and $\Omega \subset \mathbb{R}^2$

[Herzog-Meyer-Wachsmuth (11) cf. also Knees-Rossi-Zanini (14)]

The corresponding $\boxed{\text{'unilateral slope'}}$

$$|\partial_v\mathcal{F}(t,u,v)|_u = \Big|\min\Big\{\partial_v\mathcal{F}(t,u,v)[\xi] \,:\, \xi \le 0\,,\, \|\xi\|_u \le 1\Big\}\Big|$$

---

If $t_n \to t$, $u_n \to u$ in $W^{1,p}$ and $v_n \rightharpoonup v$ in $H^1$, then

$$\liminf_n |\partial_v\mathcal{F}(t_n,u_n,v_n)|_{u_n} \ge |\partial_v\mathcal{F}(t,u,v)|_u$$

# Back to alternate minimization

Let $u(t_k) = \lim_{m \to +\infty} u_m$ and $v(t_k) = \lim_{m \to +\infty} v_m$ where

$$\text{for } m \geq 1 \qquad \begin{cases} u_m \in \text{argmin} \left\{ \mathcal{F}\big(t_k \,, \bullet \,, v_{m-1}\big) \text{ in } \mathcal{U} \right\} \\ v_m \in \text{argmin} \left\{ \mathcal{F}(t_k \,, u_m \,, \bullet \,) \text{ in } \mathcal{V} \text{ with } v \leq v_{m-1} \right\} \end{cases}$$

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- In the limit configuration

discrete equilibrium

$$|\partial_u \mathcal{F}(t_k, u(t_k), v(t_k))|_{v(t_k)} = |\partial_v \mathcal{F}(t_k, u(t_k), v(t_k))|_{u(t_k)} = 0$$

# Back to alternate minimization

Let $u(t_k) = \lim_{m\to+\infty} u_m$ and $v(t_k) = \lim_{m\to+\infty} v_m$ where

$$\text{for } m \geq 1 \qquad \begin{cases} u_m \in \text{argmin}\,\big\{\mathcal{F}\big(t_k\,, \bullet\,, v_{m-1}\big) \text{ in } \mathcal{U}\,\big\} \\ v_m \in \text{argmin}\,\big\{\mathcal{F}(t_k\,, u_m\,, \bullet\,) \text{ in } \mathcal{V} \text{ with } v \leq v_{m-1}\big\} \end{cases}$$

---

- In the limit configuration

$\boxed{\text{discrete equilibrium}}$

$$|\partial_u \mathcal{F}(t_k, u(t_k), v(t_k))|_{v(t_k)} = |\partial_v \mathcal{F}(t_k, u(t_k), v(t_k))|_{u(t_k)} = 0$$

- For every $m$ (minimization as a gradient flow)

$\boxed{\text{discrete energy balance}}$

$$\begin{aligned} \mathcal{F}(t_k, u_{m+1}, v_{m+1}) = \mathcal{F}(t_k, u_m, v_m) \; + & \\ & - \int_0^1 |\partial_u \mathcal{F}(t_k, u_{m+r}, v_m)|_{v_m} \, \|u_{m+1} - u_m\|_{v_m} \, dr \; + \\ & - \int_0^1 |\partial_v \mathcal{F}(t_k, u_{m+1}, v_{m+r})|_{u_{m+1}} \, \|v_{m+1} - v_m\|_{u_{m+1}} \, dr \end{aligned}$$

# Arc-length parametrization

Finite length of the path ($m$ and $k$)

$$\sum_m \Big( \| u_{m+1} - u_m \|_{H^1} + \| v_{m+1} - v_m \|_{H^1} \Big) < L_{\Delta t}(t_k) \qquad \sum_k L_{\Delta t}(t_k) \le L$$

---

Arc-length interpolation of the “stair-like” path w.r.t. $\| \cdot \|_u$ and $\| \cdot \|_v$

$$[0, S] \ni s \mapsto (t_{\Delta t}(s), u_{\Delta t}(s), v_{\Delta t}(s)) \in [0, T] \times H^1_0 \times H^1$$

Then $(t_{\Delta t}, u_{\Delta t}, v_{\Delta t})$ is bounded in $W^{1,\infty}([0,S]; [0,T] \times H^1_0 \times H^1)$

# A parametrized BV evolution for $\triangle t \to 0$

A Lipschitz parametrization $s \mapsto (t(s), u(s), v(s))$ with [Knees-N. (17)]

$$0 \leq t'(s) \leq 1 \qquad v'(s) \leq 0$$

- for every $s$ with $t'(s) > 0$ equilibrium

$$|\partial_u \mathcal{F}(t(s), u(s), v(s))|_{v(s)} = |\partial_v \mathcal{F}(t(s), u(s), v(s))|_{u(s)} = 0$$

- for every $s$ energy balance

$$\begin{aligned}\mathcal{F}(t(s), u(s), v(s)) = \mathcal{F}(0, u_0, v_0) &+ \int_0^s \mathcal{P}_{ext}(t(r), u(r), v(r))\, t'(r)\, dr \\ &- \int_0^s |\partial_u \mathcal{F}(t(r), u(r), v(r))|_{v(r)}\ \|u'(r)\|_{v(r)}\, dr \\ &- \int_0^s |\partial_v \mathcal{F}(t(r), u(r), v(r))|_{u(r)}\ \|v'(r)\|_{u(r)}\, dr\end{aligned}$$

Scheme of parametrized BV solutions [N. (14)] with "gradient flow arguments"

# Convergence for any stopping criterion

Incremental problem: $u_m = u(t_{k-1})$ and $v_m = v(t_{k-1})$ for $m = 0$

$$\text{for } m \geq 1 \qquad \begin{cases} u_m \in \text{argmin}\left\{\mathcal{F}\big(t_k\,, \bullet\,, v_{m-1}\big) \text{ in } \mathcal{U}\right\} \\ v_m \in \text{argmin}\left\{\mathcal{F}(t_k\,, u_m\,, \bullet\,) \text{ in } \mathcal{V} \text{ with } v \leq v_{m-1}\right\} \end{cases}$$

$$\boxed{\text{Set } \; u(t_k) = u_{\bar{m}} \; \text{ and } \; v(t_k) = v_{\bar{m}} \text{ for some } \bar{m} \geq 1 \text{ according to ...}}$$

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---

Piecewise-affine interpolation $u_{\triangle t}$ and $v_{\triangle t}$

$\boxed{u_{\triangle t} \text{ and } v_{\triangle t} \text{ converge for } \triangle t \to 0 \text{ to a parametrized BV evolution}}$

All the previous analysis still works.

But in general there is no uniqueness of such evolutions.

# Discontinuity points: a "normalized gradient flow"

Up to normalization factors (depending on $s$)

Steepest descent (normalized gradient flow)

$$u'(s) \in \operatorname{argmin} \{ \partial_u \mathcal{F}(t(s), u(s), v(s))[\phi] : \|\phi\|_{v(s)} = 1 \}$$

$$\Updownarrow$$

$$\operatorname{div} \big( \boldsymbol{\sigma}_{v(s)}(u(s) + u'(s)) \big) = f \quad \text{in } H^{-1}(\Omega) \qquad \boxed{\text{phase field visco-elastic flow}}$$

for the phase-field stress $\boldsymbol{\sigma}_v(w) = (v^2 + \eta)\, \boldsymbol{\sigma}(w)$

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---

Steepest descent (normalized gradient flow)

$$v'(s) \in \text{argmin}\,\{\partial_v \mathcal{F}(t(s), u(s), v(s))[\xi] : \|\xi\|_{u(s)} = 1\,, \xi \leq 0\}$$

Not so clear in terms of a PDE ...

Phase field energy release?

$$\tilde{\mathcal{E}}(t,v) = \mathcal{E}(t,u_{t,v},v) \quad \text{for} \quad u_{t,v} \in \text{argmin}\,\{\mathcal{E}(t,\cdot,v) : u \in \mathcal{U}\}.$$

$$\partial_v \tilde{\mathcal{E}}(t,v)[\xi] = \int_\Omega v\,\xi\, W(\boldsymbol{\varepsilon}_{t,v}) - f(t)v\,dx = \partial_v \mathcal{E}(t,u_{t,v},v)[\xi]$$

Normalized admissible variations $\quad \widehat{\Xi} = \{\xi \in H^1 : \xi \le 0 \text{ and } d\mathcal{L}(v)[\xi] = 1\}$

$$\boxed{\mathcal{G}(t,v) = -\inf\left\{\partial_v \tilde{\mathcal{E}}(t,v)[\xi] : \xi \in \widehat{\Xi}\right\}}$$

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---

Griffith's criterion in KT form. If $t'(s) > 0$ then $\mathcal{L}'(v(s)) \ge 0$ and

- $\mathcal{G}(t(s),v(s)) \le G_c$ equilibrium
- $(\mathcal{G}(t(s),v(s)) - G_c)\,\mathcal{L}'(v(s)) = 0$ stability

# A "Ginzburg-Landau" approach

Evolution governed by a system of PDEs [Kuhn-Müller (10), N. (16)]

$$\boxed{\begin{cases} \varepsilon \dot{v}(t) = -\big[ v(t)\, W(Du(t)) + (v(t)-1) - \Delta v(t) \big]_+ \\ \operatorname{div}\big( \boldsymbol{\sigma}_{v(t)}(u(t)) \big) = 0 \end{cases}}$$

where $\boldsymbol{\sigma}_v(u) = (v^2 + \eta)\, \boldsymbol{\sigma}(u)$ is the phase-field stress.

Couples stationarity of $u$ with a unilateral $L^2$-gradient flow:

$$\varepsilon \dot{v}(t) = -\big[ \partial_v \mathcal{F}(t, u(t), v(t)) \big]_+$$

Technically $\partial_v \mathcal{F}(t, u(t), v(t))$ is a Radon measure with positive part in $L^2$

Similar "Ginzburg-Landau" systems ...

[Hakim-Karma (09), Miehe-Welschinger-Hofacker (10), Abdollahi-Arias (12)]

[Gianazza-Savaré (94), Takaishi-Kimura (09), Knees-Rossi-Zanini (13)]

Time discrete alternate minimization scheme: for $t_n = n\triangle t$

$$\begin{cases} u_n \in \operatorname{argmin} \{\mathcal{F}(t_n, u, v_{n-1}) : u = g(t_n) \text{ on } \partial\Omega\}, \\ v_n \in \operatorname{argmin} \{\mathcal{F}(t_n, u_n, v) + \frac{1}{2}\varepsilon\tau^{-1}\|v - v_{n-1}\|^2_{L^2} : v \leq v_{n-1}\}, \end{cases}$$

"Euler-Lagrange equations provide an implicit discretization of the PDEs"

---

Energy bound

$$\mathcal{F}(t_n, u_n, v_n) \leq \mathcal{F}(t_{n-1}, u_{n-1}, v_{n-1}) + \int_{t_{n-1}}^{t_n} \partial_t \mathcal{F}(t, u_{n-1}, v_{n-1})\, dt +$$
$$- \frac{1}{2} \int_{t_{n-1}}^{t_n} \|\varepsilon \dot{v}_n\|^2_{L^2} + |\partial_v^- \mathcal{F}(t_n, u_{n-1}, v_n)|^2_{L^2}\, dt.$$

for $|\partial_v \mathcal{F}(t,u,v)|_{L^2} = |\inf\{\partial_v \mathcal{F}(t,u,v)[\xi] : \xi \in H^1,\, \xi \leq 0,\, \|\xi\|_{L^2} \leq 1\}|_-$.

# Limit evolution: energy balance and gradient flow

Compactness: $v_{\Delta t} \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$.

By lower semicontinuity of energy the slope and by an upper gradient inequality

$$\mathcal{F}(t,u(t),v(t)) = \mathcal{F}(0,u_0,v_0) + \int_0^t \partial_t \mathcal{F}(r,u(r),v(r))\,dr + \\ - \tfrac{1}{2}\int_0^t \|\varepsilon \dot v(r)\|^2_{L^2} + |\partial_v^- \mathcal{F}(r,u(r),v(r))|^2_{L^2}\,dr.$$

(De Giorgi's representation of "gradient flows")

---

- Evolutions $v_{\Delta t}$ bounded in $W^{1,2}(0,T;H^1(\Omega))$

a discrete Gronwall argument from [Nocetto-Savaré-Verdi (00)]

- $\partial_v \mathcal{F}(t,u(t),v(t))$ is a measure with positive part in $L^2$ and

$$\begin{cases} \varepsilon \dot v(t) = -\big[v(t)\,W(Du(t)) + G_c(v(t)-1) - G_c \Delta v(t)\big]^+ \\ \mathrm{div}\big(\boldsymbol{\sigma}_{v(t)}(u(t))\big) = 0 \end{cases}$$

[Gianazza-Savaré (94)]

# Parametrization and quasi-static limit (I)

Arc-length $t \mapsto s_\varepsilon(t) = t + \displaystyle\int_0^t \|\dot v_\varepsilon(r)\|_{L^2}\, dr$ with inverse $t_\varepsilon(s)$

Parametrized evolution $s \mapsto (t_\varepsilon(s), u_\varepsilon \circ t_\varepsilon(s), v_\varepsilon \circ t_\varepsilon(s))$

---

Then $0 \le t'_\varepsilon + \|v'_\varepsilon\|_{L^2} \le 1$ and for every $\lambda \in [0,1]$

$$\mathcal{F}(t_\varepsilon(s), u_\varepsilon(s), v_\varepsilon(s)) = \mathcal{F}(0, u_0, v_0) + \int_0^s \partial_t \mathcal{F}(t_\varepsilon(r), u_\varepsilon(r), v_\varepsilon(r))\, t'_\varepsilon(r)\, dr +$$
$$- \int_0^s \lambda \Psi_\varepsilon\big(\|v'_\varepsilon(r)\|_{L^2}\big) + (1-\lambda)\Phi_\varepsilon\big(|\partial_v^- \mathcal{F}(t_\varepsilon(r), u_\varepsilon(r), v_\varepsilon(r))|_{L^2}\big)\, dr$$

where

$$\Psi_\varepsilon(\xi) = \begin{cases} \varepsilon\xi^2/(1-\xi) & 0 \le \xi < 1 \\ +\infty & \xi \ge 1, \end{cases} \qquad \Phi_\varepsilon(\xi) = \xi^2/(\varepsilon + \xi).$$

Then $t'_\varepsilon \to t$, $v_\varepsilon \to v$ in $W^{1,\infty}$ and (for $\lambda = 0$)

$$\mathcal{F}(t(s), u(s), v(s)) = \mathcal{F}(0, u_0, v_0) + \int_0^s \partial_t \mathcal{F}(t(r), u(r), v(r))\, t'(r)\, dr + \\ - \int_0^s |\partial_v^- \mathcal{F}(t(r), u(r), v(r))|_{L^2}\, dr$$

---

Properties of the parametrized BV-evolution:

- equilibrium in continuity points ($t' > 0$)
- normalized unilateral $L^2$-gradient flow in discontinuity points ($t' = 0$)