The role of state constraints for turnpike behaviour and strict dissipativity of optimal control problems

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Mathematisches Institut, Universität Bayreuth

based on joint work with Roberto Guglielmi (GSSI, L'Aquila)

Control of state constrained dynamical systems
Padova, 25–29 September 2017

Outline

- The turnpike property
- Strict dissipativity
- Linear quadratic problems
- Main results

System class

We consider nonlinear discrete time control systems

$$x_{\mathbf{u}}(n+1) = f(x_{\mathbf{u}}(n), \mathbf{u}(n)), \quad x_{\mathbf{u}}(0) = x$$

with $x_{\mathbf{u}}(n) \in X$, $\mathbf{u}(n) \in U$, $\quad X$, U normed spaces

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$x_{\mathbf{u}}(n)$	$=$	state of the system at time t_n
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f	$=$	solution operator of a controlled ODE/PDE or of a discrete time model

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The turnpike property

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The turnpike property describes a behaviour of (approximately) optimal trajectories for a finite horizon optimal control problem

$$\underset{\mathbf{u}}{\text{minimise}} \quad J_N(x, \mathbf{u}) = \sum_{n=0}^{N-1} \ell(x_{\mathbf{u}}(n), \mathbf{u}(n))$$

with state and input constraints $x_{\mathbf{u}}(n) \in \mathbb{X}$, $\mathbf{u}(n) \in \mathbb{U}$

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We illustrate the property by two simple examples

Example 1: minimum energy control

Example: Keep the state of the system inside a given interval X minimising the quadratic control effort

$$\ell(x, u) = u^2$$

with dynamics

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$\rightsquigarrow$ optimal trajectory should stay near $x^e = 0$

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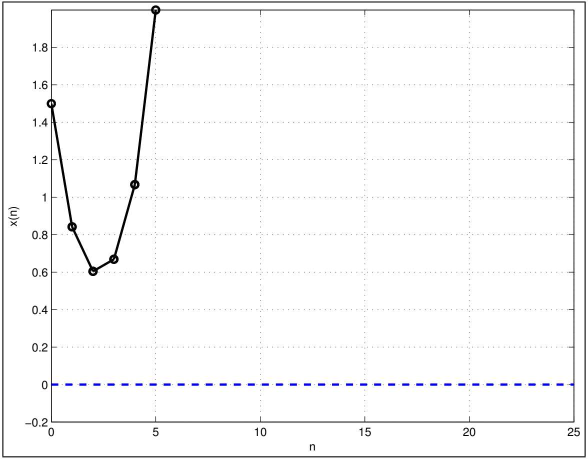

Optimal trajectory for $N = 5$

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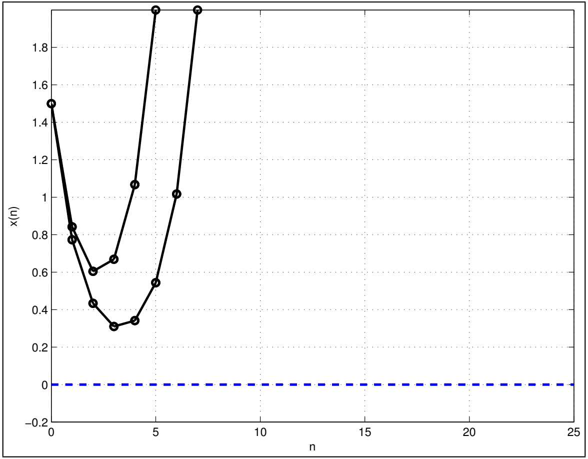

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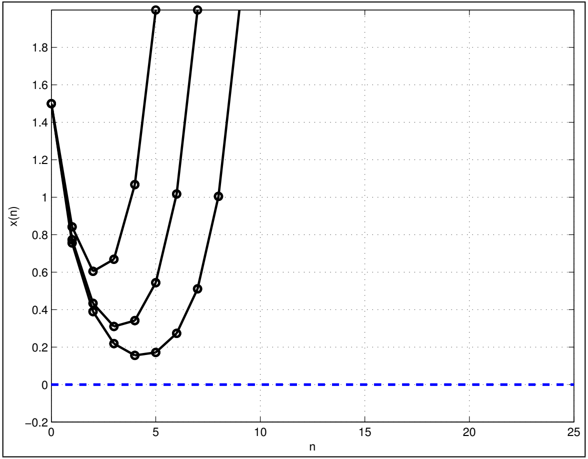

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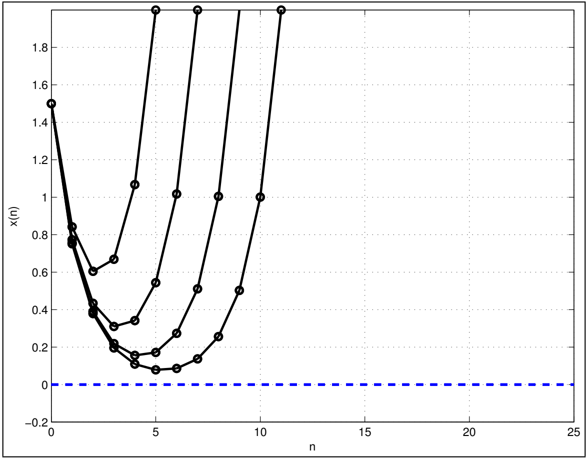

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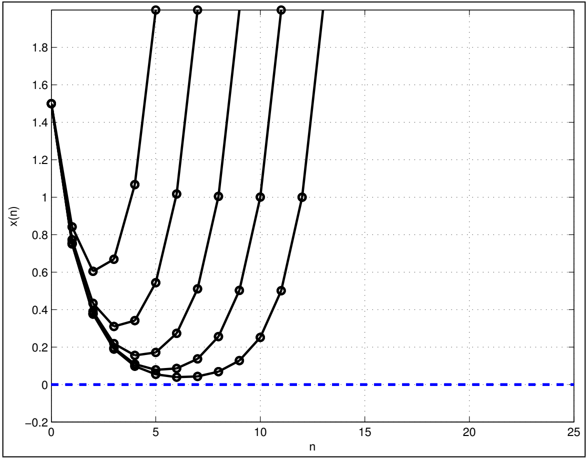

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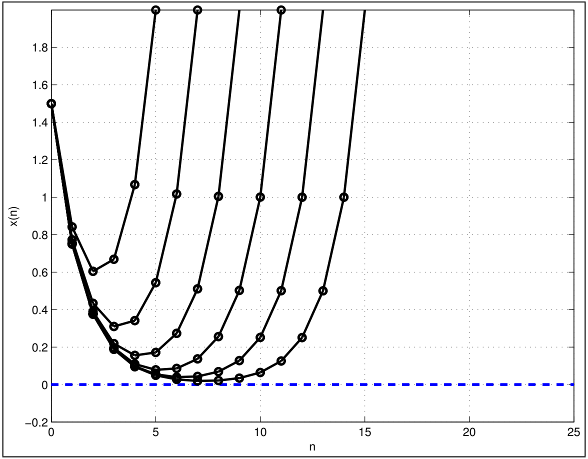

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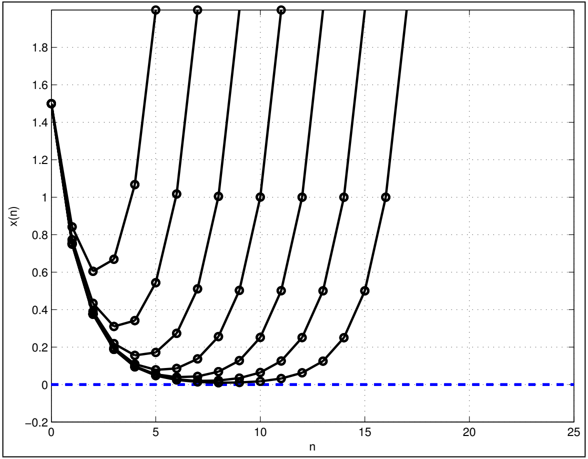

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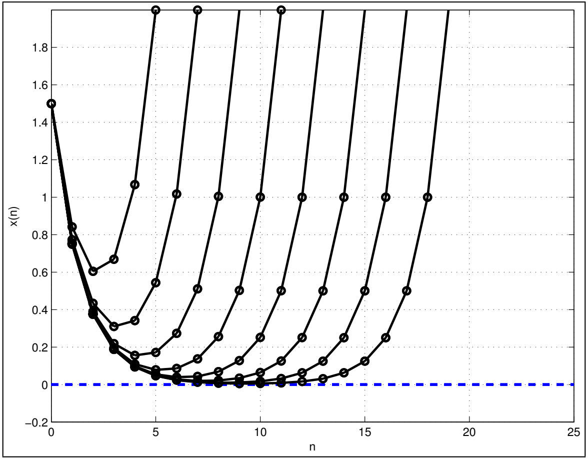

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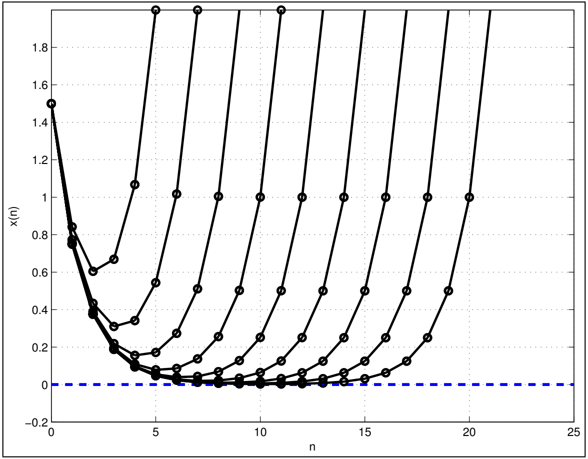

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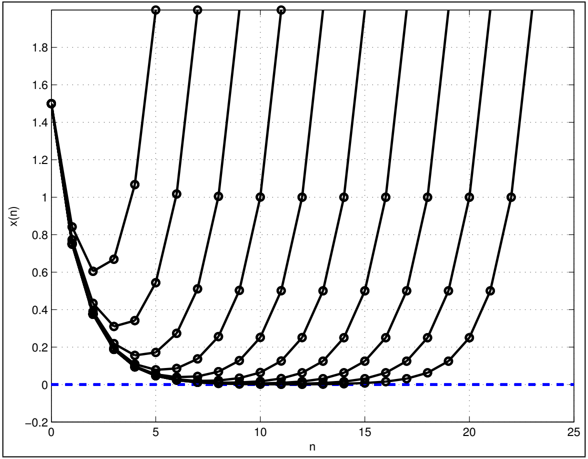

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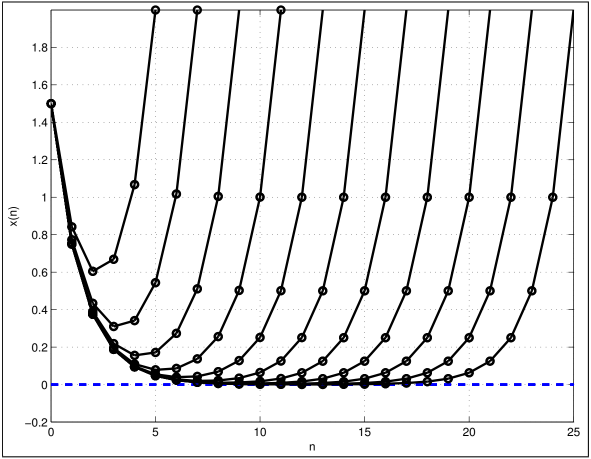

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Example 2: a macroeconomic model

The second example is a 1d macroeconomic model

[Brock/Mirman '72]

Minimise the finite horizon objective with

$$\ell(x,u) = -\ln(Ax^{\alpha} - u), \quad A = 5,\ \alpha = 0.34$$

with dynamics $\quad x^+ = u$

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$$x^e \approx 2.2344 \quad \text{with } \ell(x^e, u^e) \approx 1.4673$$

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One may thus expect that finite horizon optimal trajectories also stay for a long time near that equilibrium

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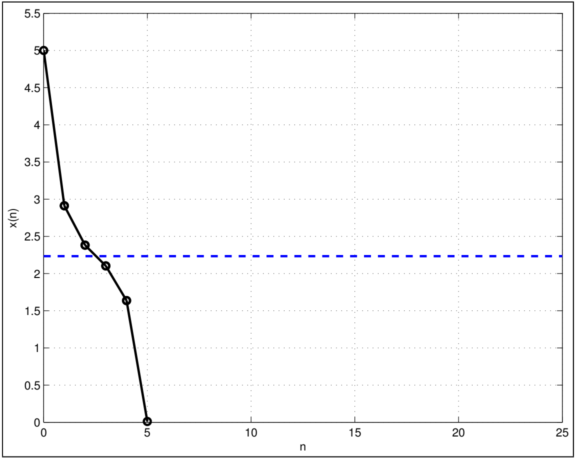

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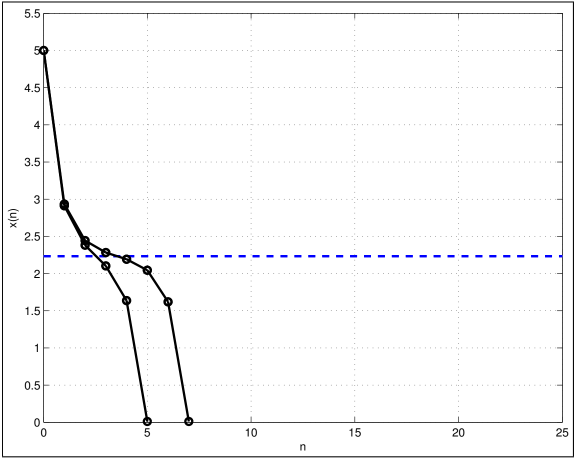

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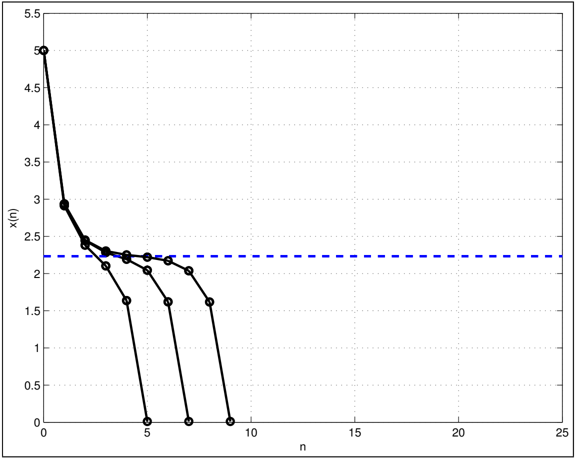

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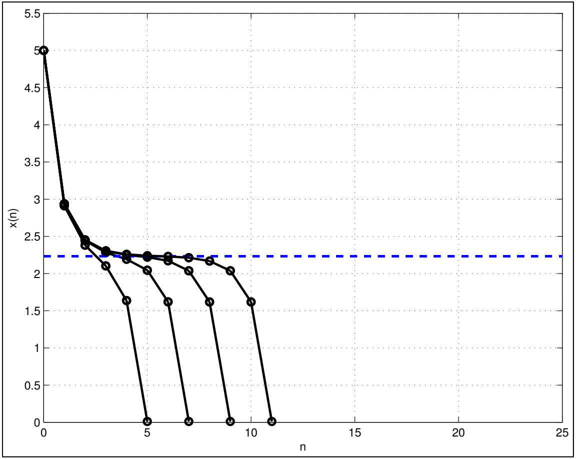

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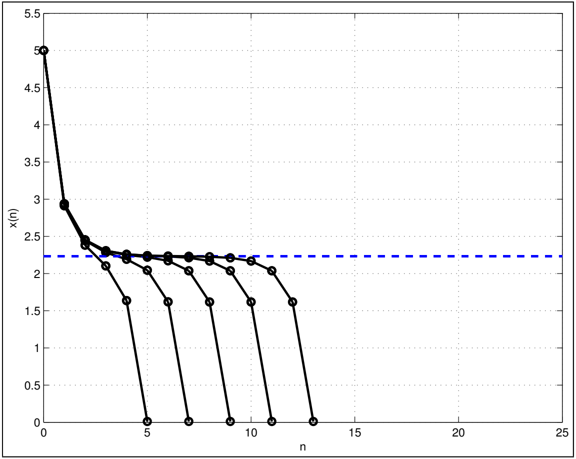

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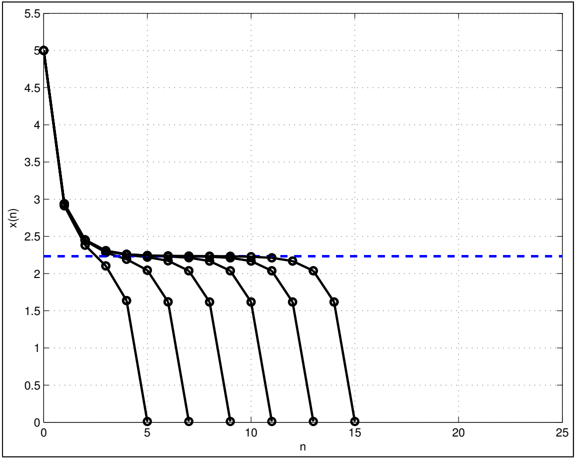

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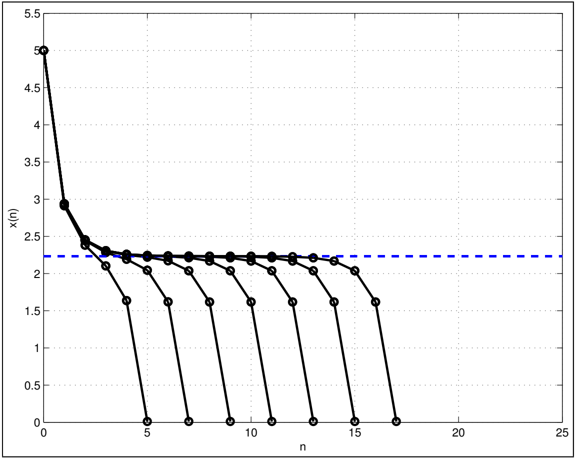

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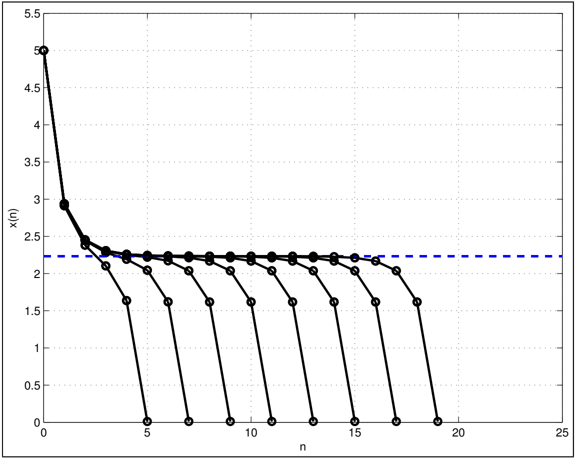

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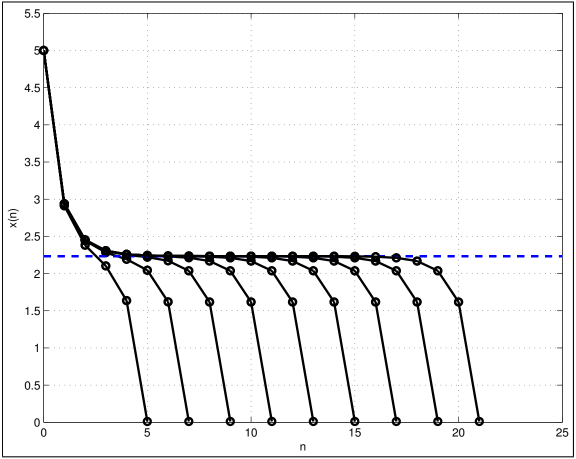

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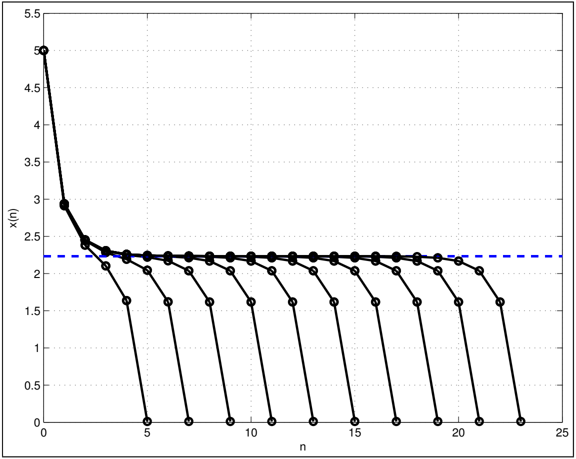

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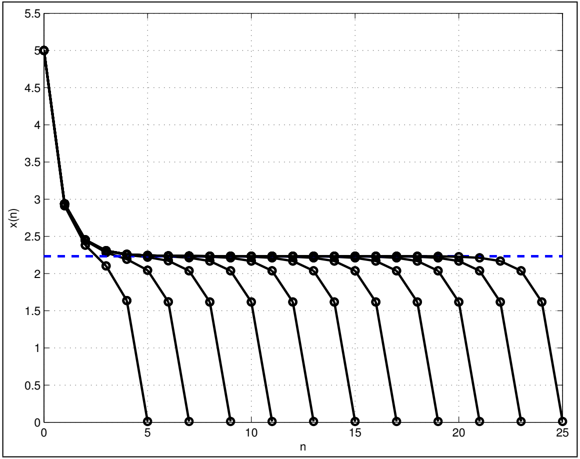

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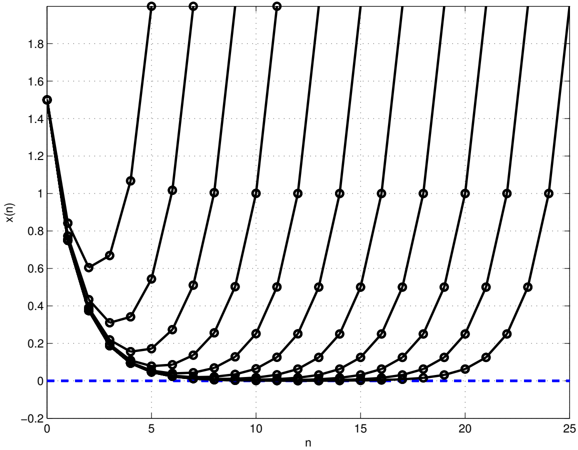
How to formalize the turnpike property?



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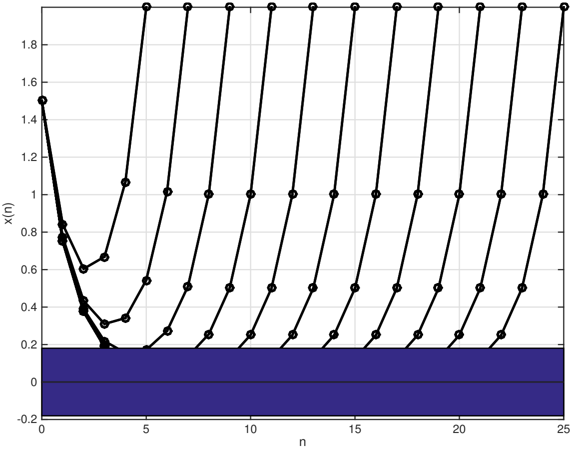

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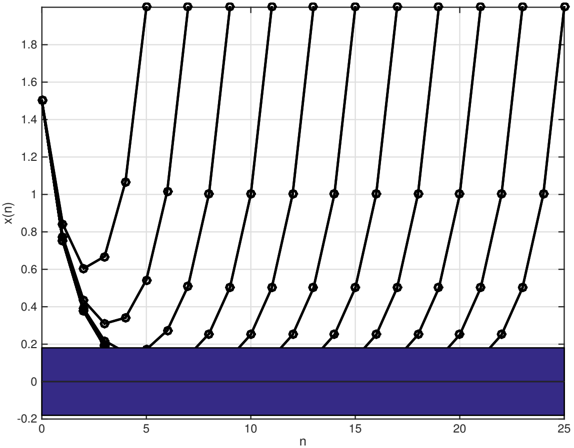

Number of points outside the blue neighbourhood is bounded by a number independent of N (here: by 8)

The turnpike property: formal definitions

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Turnpike property: For each $\varepsilon > 0$ and $\rho > 0$ there is $C_{\rho,\varepsilon} > 0$ such that for all $N \in \mathbb{N}$ all optimal trajectories $x^\star$ starting in $B_\rho(x^e)$ satisfy the inequality

$$\#\left\{k \in \{0, \dots, N-1\} \,\middle|\, \|x^\star(k) - x^e\| \geq \varepsilon\right\} \leq C_{\rho,\varepsilon}$$

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Near equilibrium turnpike property: For each $\varepsilon > 0$, $\delta > 0$ and $\rho > 0$ there is $C_{\rho,\varepsilon,\delta} > 0$ such that for all $x \in \mathbb{X}$ and $N \in \mathbb{N}$, all trajectories $x_\mathbf{u}$ with $x_\mathbf{u}(0) = x \in B_\rho(x^e)$ and $J_N(x, \mathbf{u}) \leq N\ell(x^e, u^e) + \delta$ satisfy the inequality

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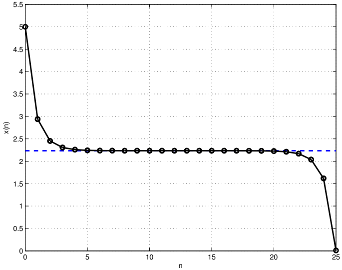

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- Many applications, e.g., structural insight in economic equilibria; synthesis of optimal trajectories [Anderson/Kokotovic '87]

Application: Model predictive control

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MPC is a method in which an optimal control problem on an infinite horizon

$$\operatorname*{minimise}_{\mathbf{u}} \; J_\infty(x,\mathbf{u}) = \sum_{n=0}^{\infty} \ell(x_{\mathbf{u}}(n), \mathbf{u}(n))$$

is approximated by the iterative solution of finite horizon problems

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with fixed $N \in \mathbb{N}$

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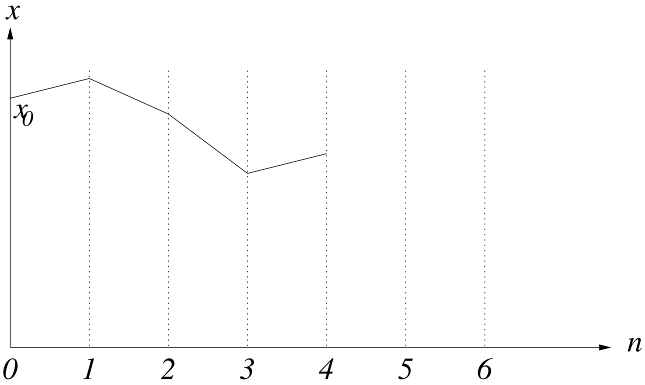

black = predictions (open loop optimization)

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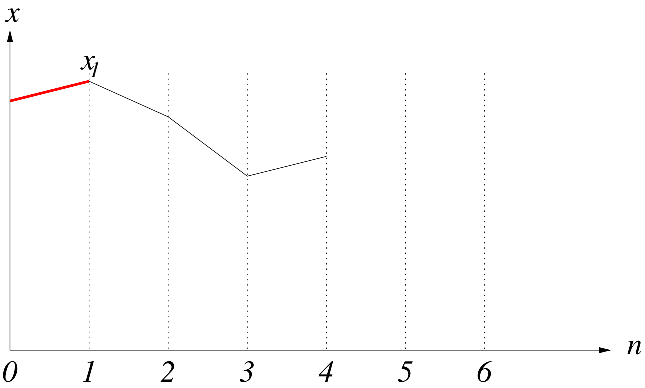

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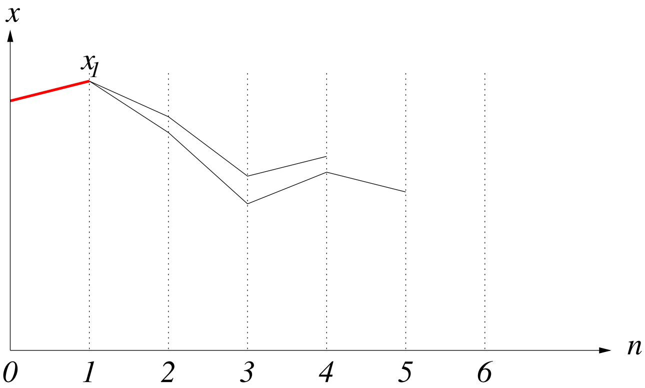

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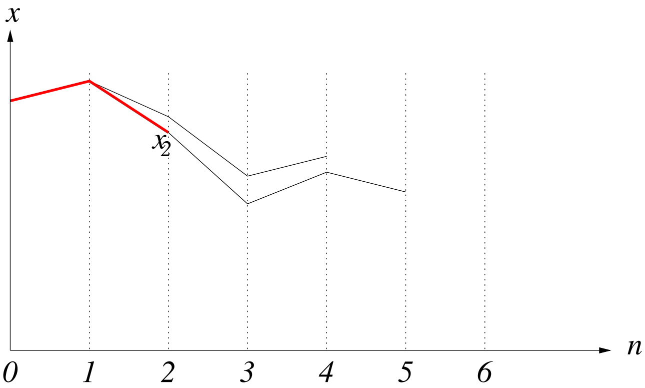

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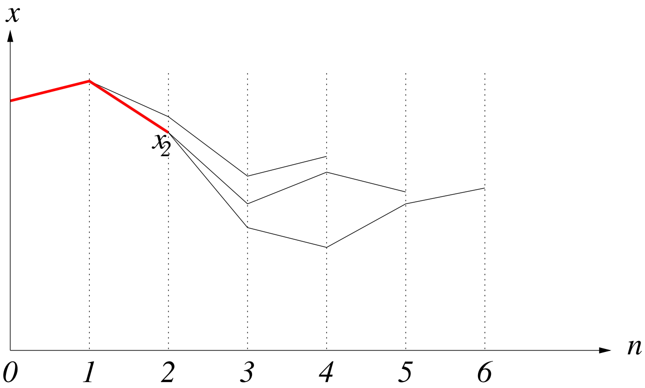

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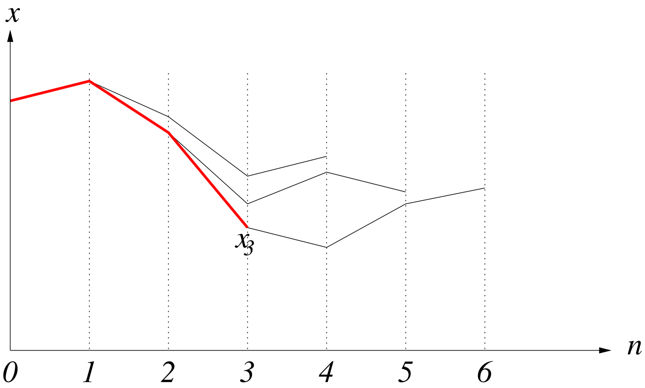

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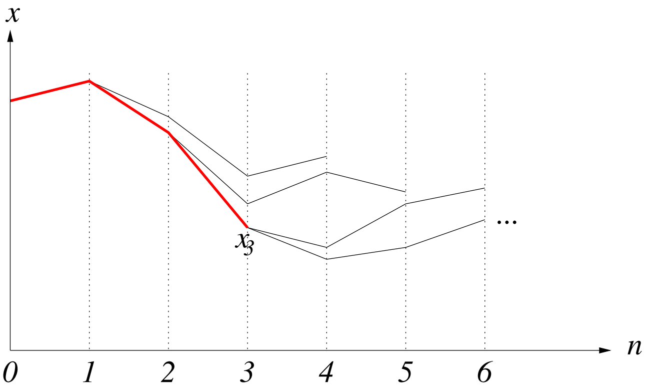

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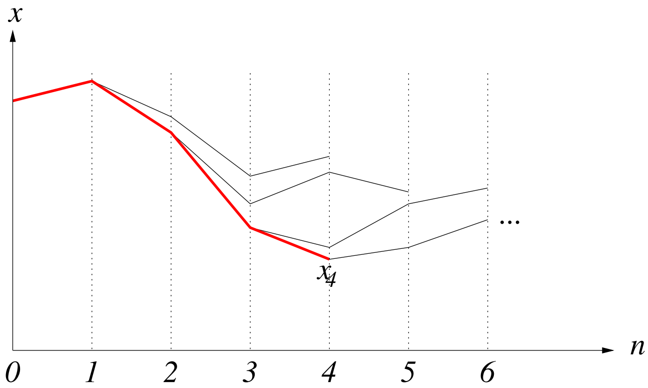

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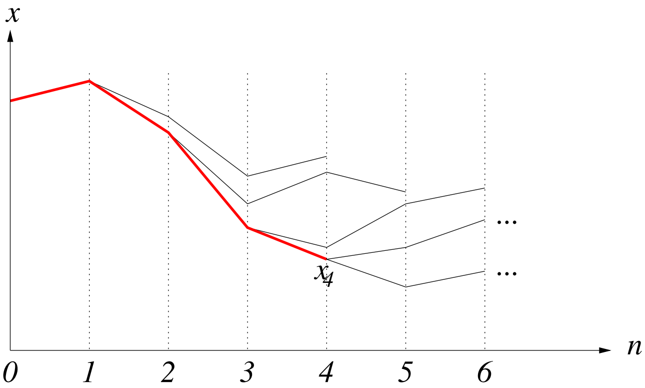

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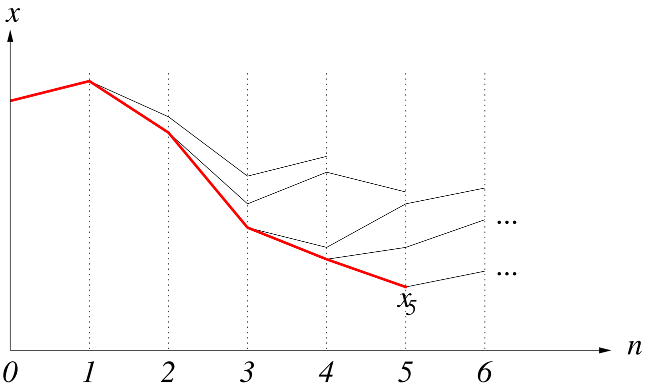

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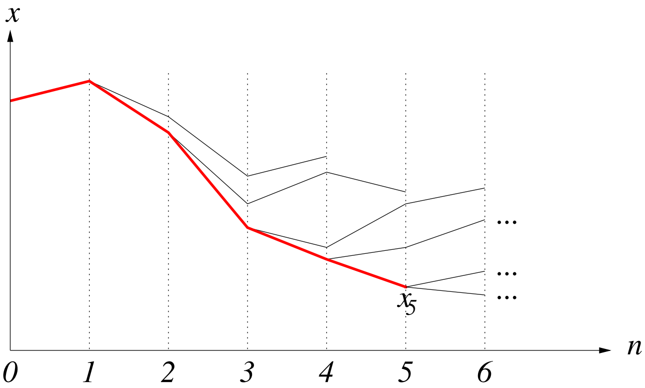

black = predictions (open loop optimization)

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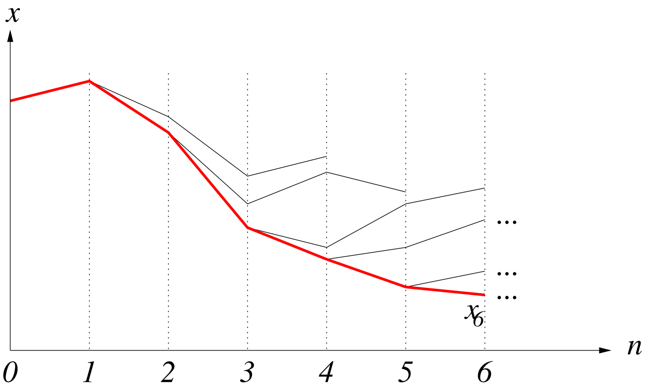

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Approximation result for MPC

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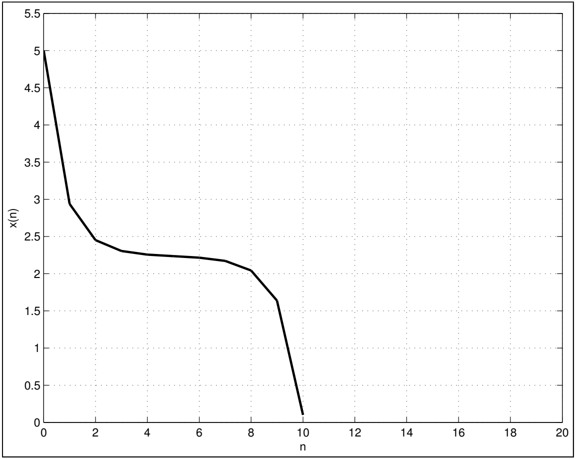
We illustrate this behaviour by our second example for $N = 10$

MPC for Example 2



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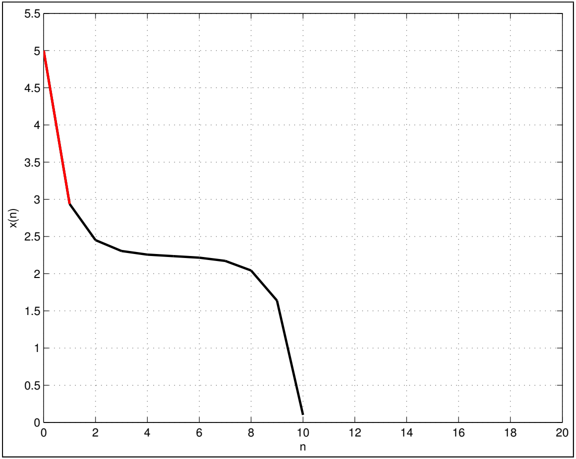

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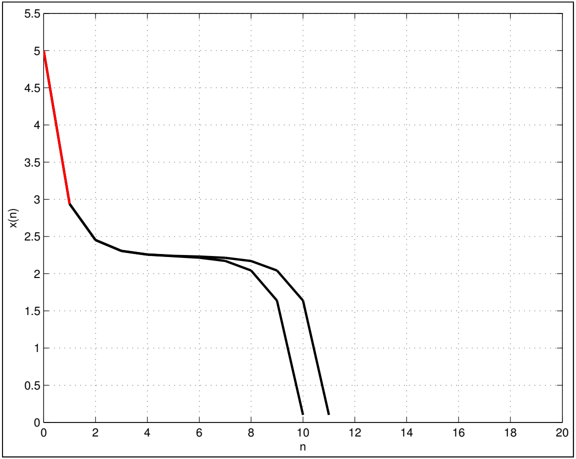

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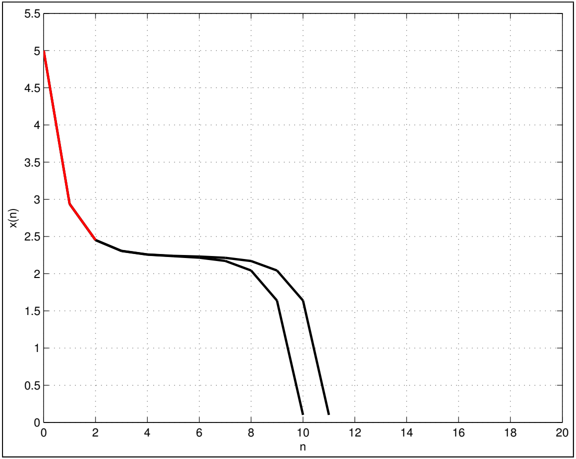

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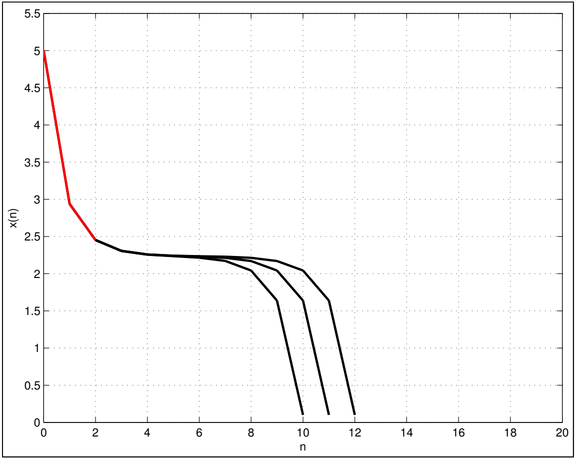

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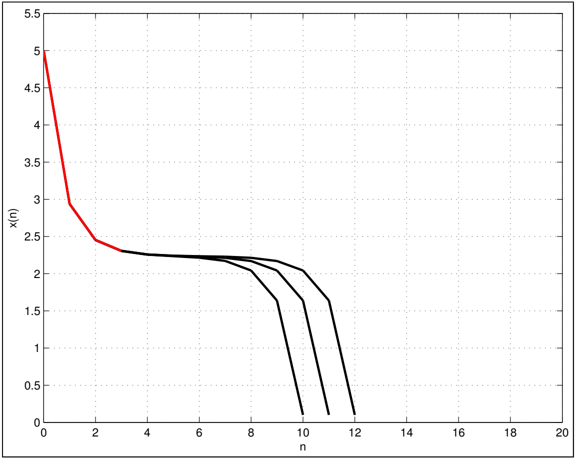

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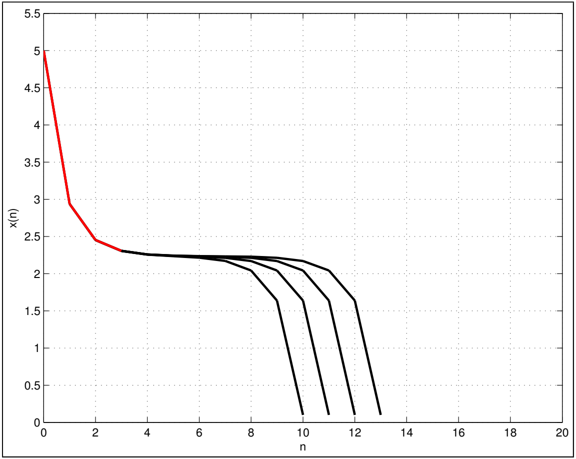

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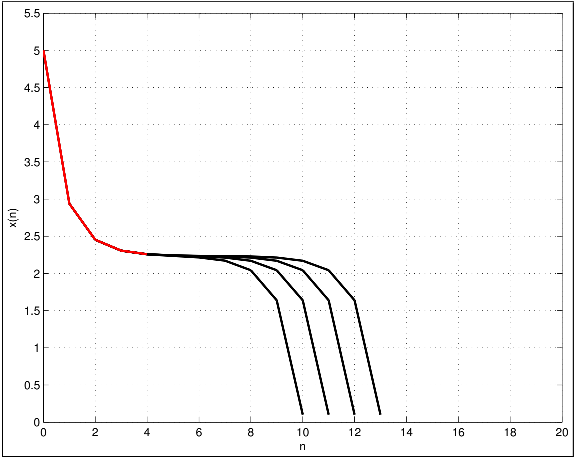

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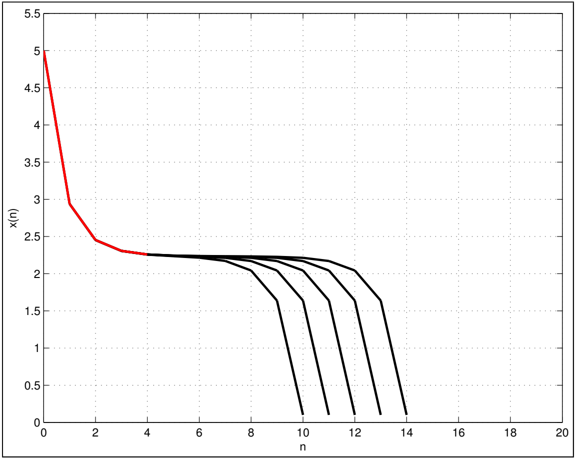

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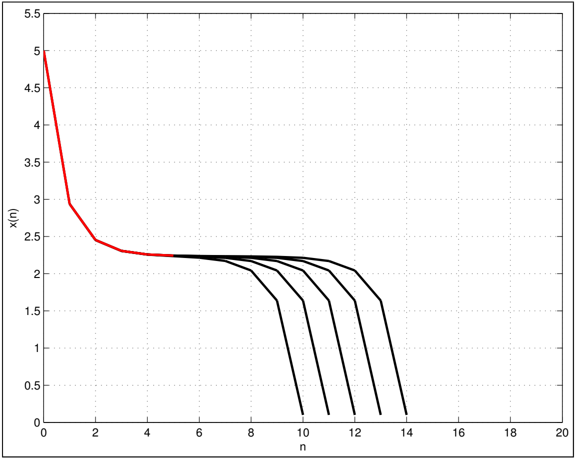

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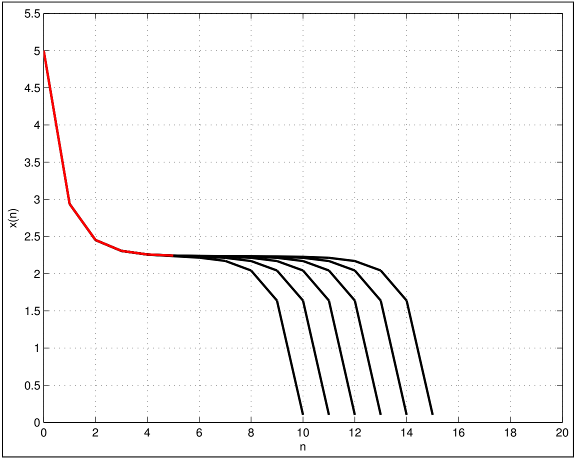

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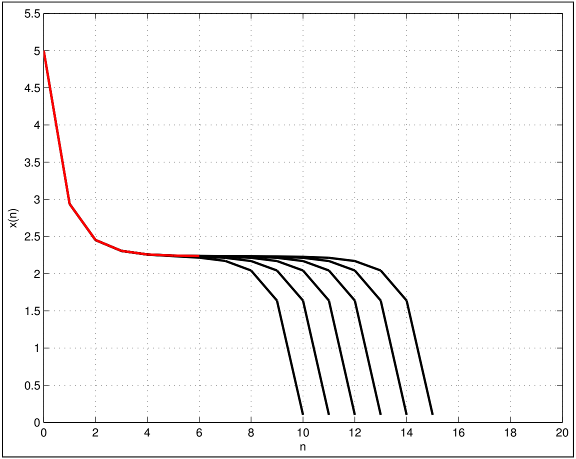

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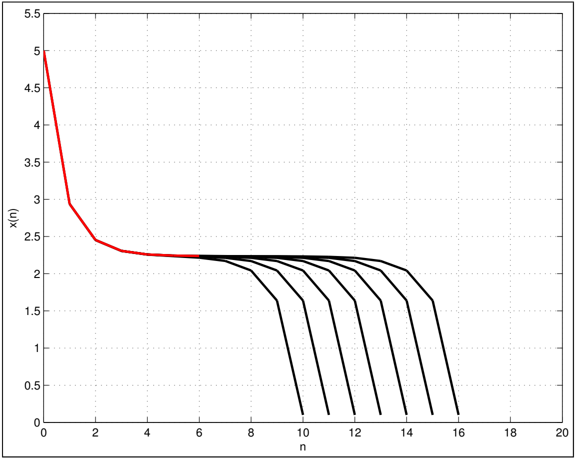

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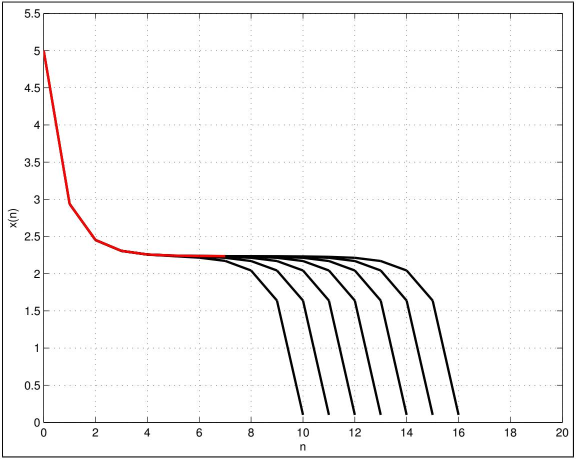

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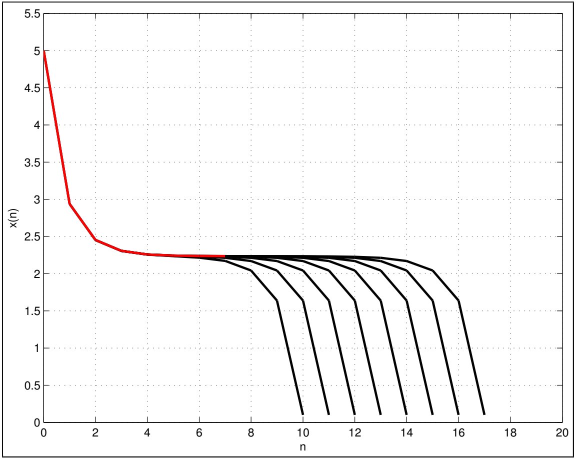

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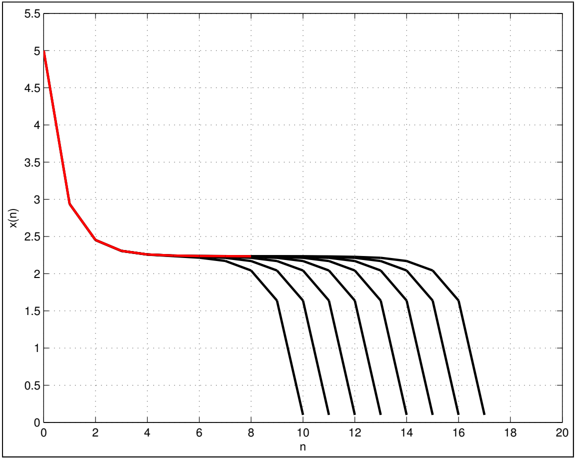

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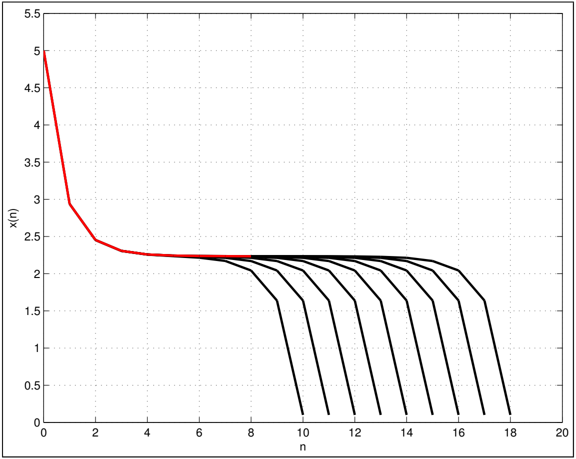

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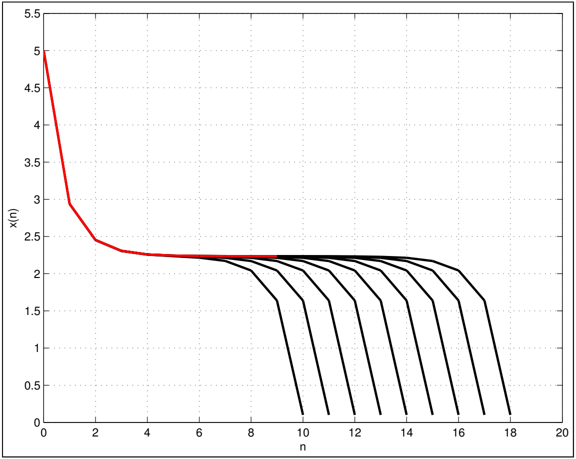

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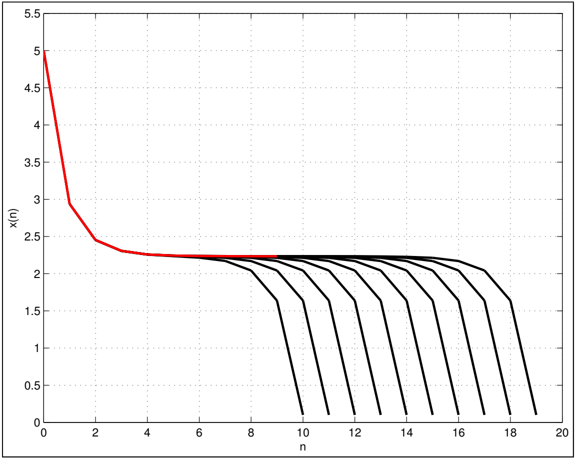

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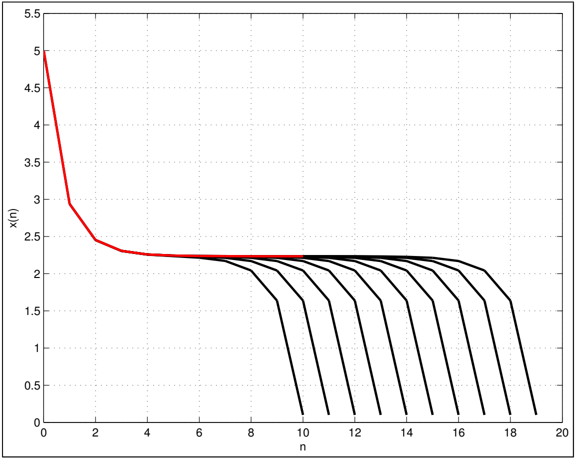

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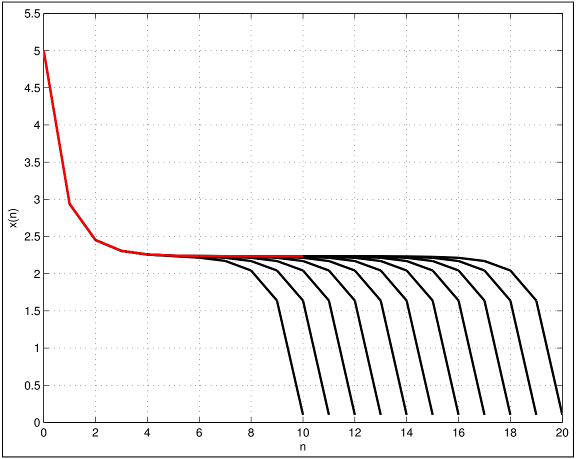

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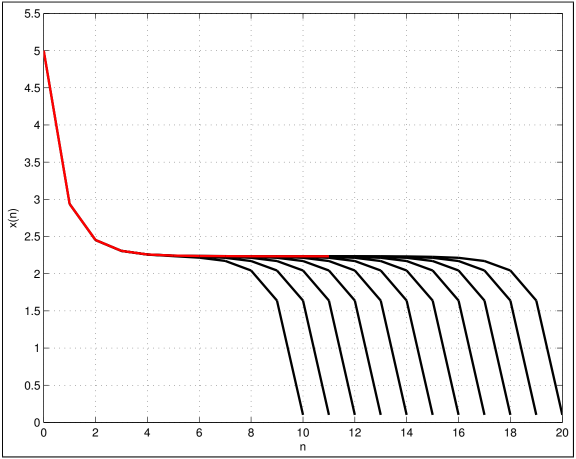

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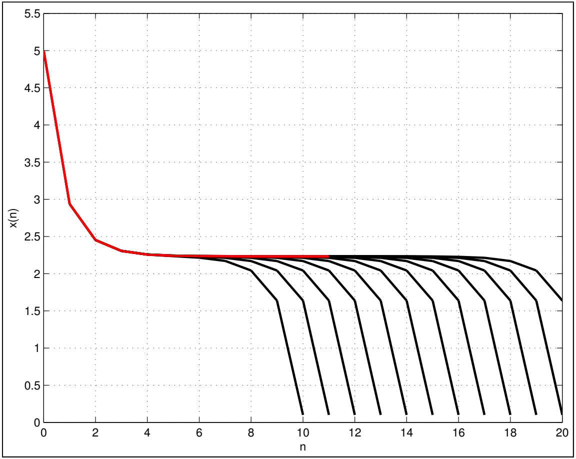

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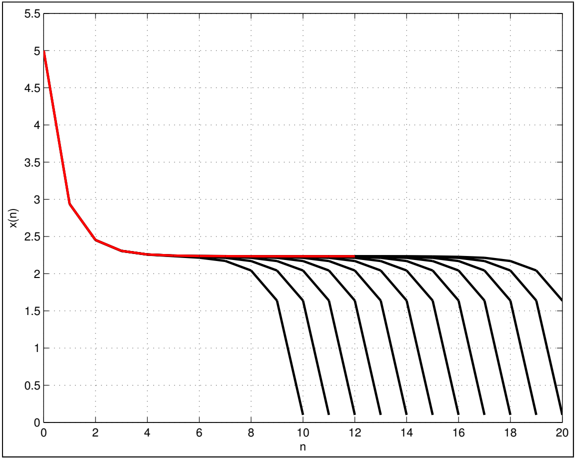

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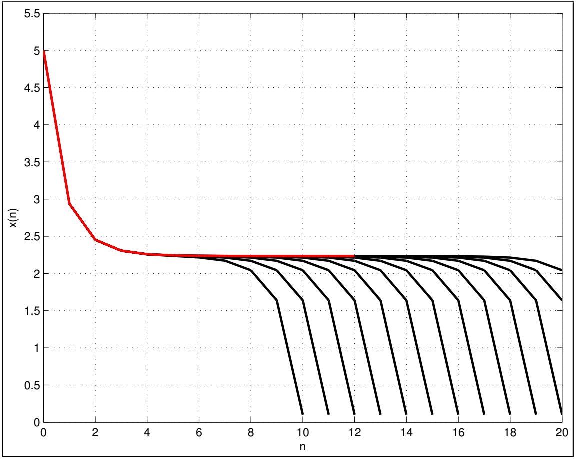

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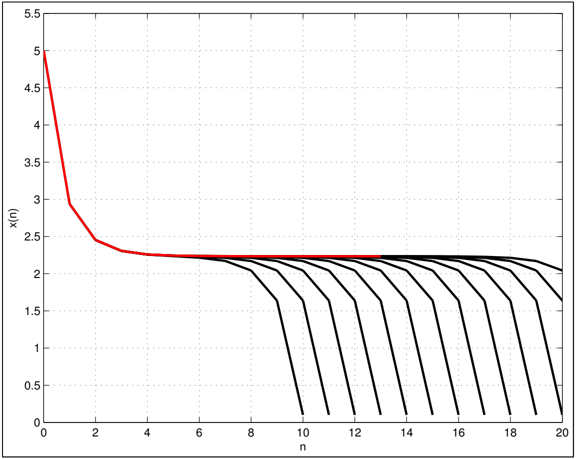

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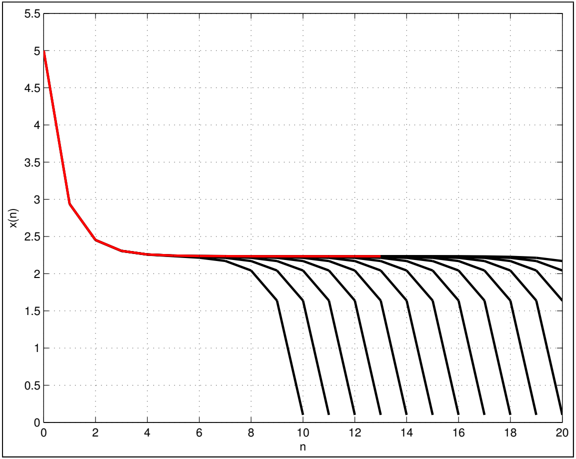

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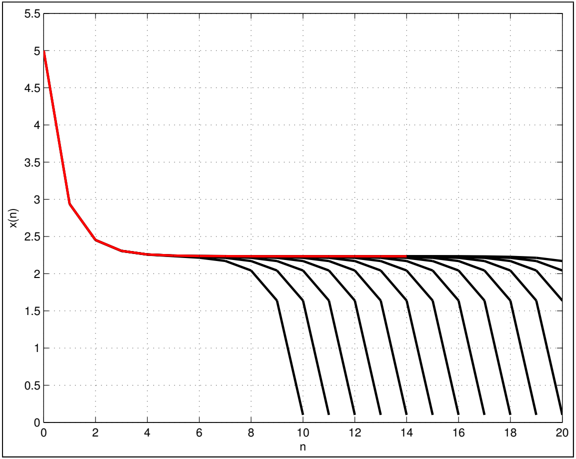

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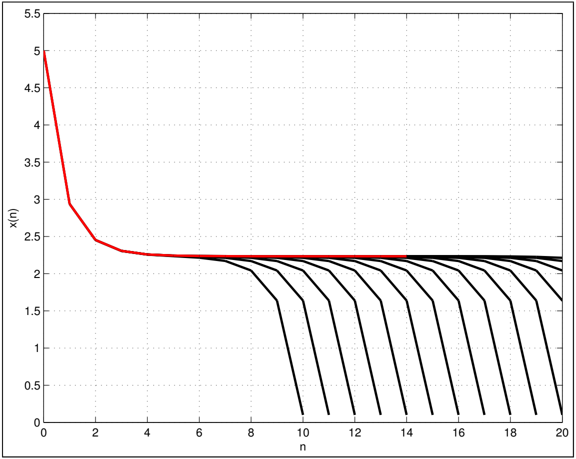

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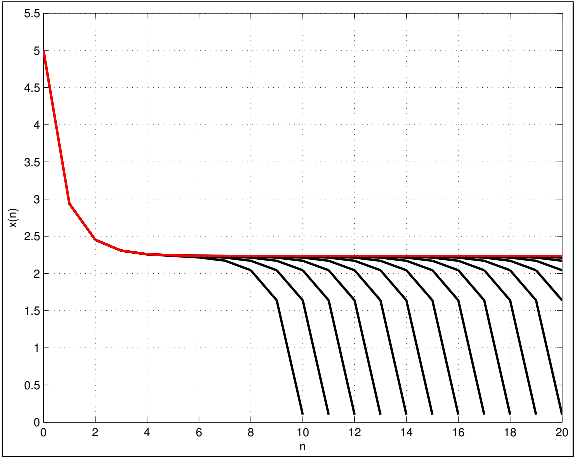

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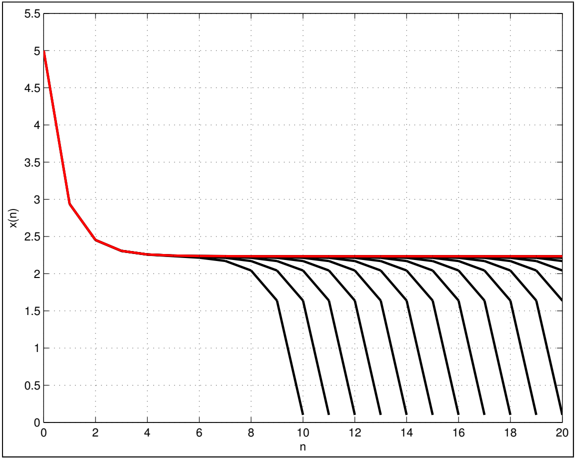

Extension to non equilibrium turnpikes possible
[Zanon/Gr./Diehl '17, Gr./Müller '17, Gr./Pirkelmann '17]

Strict dissipativity

Dissipativity

$$x^+ = f(x,u)$$

Introduce functions $s : X \times U \to \mathbb{R}$ and $\lambda : X \to \mathbb{R}$

$s(x,u)$ supply rate, measuring the (possibly negative) amount of energy supplied to the system via the input u in the next time step

$\lambda(x)$ storage function, measuring the amount of energy stored inside the system when the system is in state x

Dissipativity

Definition [cf. Willems '72] The system is called strictly pre-dissipative if there are $x^e \in \mathbb{X}$, $\alpha \in \mathcal{K}$ such that for all $x \in \mathbb{X}$, $u \in \mathbb{U}$ the inequality

$$\lambda(x^+) \leq \lambda(x) + s(x,u) - \alpha(\|x - x^e\|)$$

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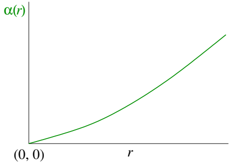
$\alpha \in \mathcal{K}$: $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, continuous, strictly increasing, $\alpha(0) = 0$



The system is called strictly dissipative if it is strictly pre-dissipative with λ bounded from below

Physical interpretation of dissipativity

$$\lambda(x^+) \leq \lambda(x) + s(x,u) - \alpha(\|x - x^e\|)$$

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physical interpretation of strict dissipativity:

$$\begin{aligned} \lambda(x) \quad &= \text{energy stored in the system} \\ s(x,u) &= \text{energy supplied to the system} \end{aligned}$$

strict dissipativity:

- energy can not be generated inside the system
- a certain amount of energy $\alpha(\|x - x^e\|)$ must be dissipated ($=$ given to the environment)

History

Dissipativity was defined for continuous time systems in
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Translation to discrete time systems is quite straightforward

[Byrnes/Lin '94]

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Then the following statements are equivalent

(a) The system is strictly dissipative with supply rate $s(x, u) = \ell(x, u) - \ell(x^e, u^e)$ and a bounded storage function

(b) The near equilibrium turnpike property holds

Linear quadratic problems

LQ problems

From now on we consider linear quadratic finite dimensional discrete time problems with $X = \mathbb{R}^n$, $U = \mathbb{R}^m$,

$$x^+ = Ax + Bu, \quad \ell(x,u) = x^T Q x + u^T R u + b^T x + d^T u$$

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The same holds for solutions $x(t)$ starting in v with $u(t) \equiv 0$

Storage functions for LQ problems

$$x^+ = Ax + Bu, \quad s(x,u) = x^T Q x + u^T R u + b^T x + d^T u$$

Lemma: For LQ problems, a storage function λ can always be chosen of the form

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Moreover, the solution satisfies $P > 0$ if and only if all unobservable eigenvalues satisfy $|\mu| < 1$ ("(A, C) detectable")

Strict dissipativity and pre-dissipativity

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"$\Rightarrow$" needs some more work in order to show that $P \geq 0, P \not> 0$ contradicts strict dissipativity for unbounded $\mathbb{X}$

Main results

Main result without state constraints

Without state constraints:

Theorem: Consider the LQ problem with (A, B) stabilizable, $Q = C^T C$ and state and control constraint sets $\mathbb{X} = \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$. Then the following properties are equivalent

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Moreover, if one of these properties holds, then the equilibria in (i) and (ii) coincide

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Main result with state constraints

With bounded state constraints:

Theorem: Consider the LQ-problem with $Q = C^T C$ and state and control constraint sets $\mathbb{X} \subset \mathbb{R}^n$ bounded and $\mathbb{U} \subseteq \mathbb{R}^m$. Then the following properties are equivalent

Main result with state constraints

With bounded state constraints:

Theorem: Consider the LQ-problem with $Q = C^T C$ and state and control constraint sets $\mathbb{X} \subset \mathbb{R}^n$ bounded and $\mathbb{U} \subseteq \mathbb{R}^m$. Then the following properties are equivalent

(i) The problem is strictly pre-dissipative at an equilibrium $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$

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Moreover, if one of these properties holds, then the equilibria in (i) and (ii) coincide. If, in addition, (A, B) is stabilizable then the turnpike property holds.

Discussion

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Is there an intuitive explanation for this fact?

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Cost function $\ell(x,u) = u^2$

Dynamics $x^+ = 2x + u$

Constraints $\mathbb{X} = [-2, 2]$, $\mathbb{U} = [-3, 3]$

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Dynamics $x^+ = 2x + u$ $\Rightarrow$ $\mu = 2$

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Indeed, in this case all optimal solutions grow exponentially, because $u \equiv 0$ is clearly the optimal control

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General principle for bounded constraints:

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- Analyse these relations for continuous time and infinite dimensional systems

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Outlook

- Analyse these relations for continuous time and infinite dimensional systems
- What happens if x^e is at the boundary of $\mathbb{X}$?

References

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