Second order optimality conditions of control-affine problems with a scalar state constraint

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Control of state constrained dynamical systems

September 25-29, 2017 - Dipartimento di Matematica "Tullio Levi-Civita"

Università di Padova

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Joint work with J.-F. Bonnans and B.S. Goh

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Goddard problem: vertical ascent of a rocket

[Goddard, 1919] [Seywald & Cliff, 1993]

State variables: (h, v, m) are the position, speed and mass.

Control variable: u is the normalized thrust (proportion of $\mathcal{T}_{max}$)

Cost: final mass of the rocket (minimize fuel consumption)

T : free final time, $D(h, v)$ is the **drag**

$$
\begin{aligned}
\max\ & m(T), \\
\text{s.t.}\ & \dot{h} = v, \\
& \dot{v} = -1/h^2 + 1/m(\mathcal{T}_{\mathsf{max}}\, u - D(h, v)), \\
& \dot{m} = -b\,\mathcal{T}_{\mathsf{max}}\, u, \\
& 0 \leq u \leq 1, \quad D(h, v) \leq D_{\mathsf{max}}, \\
& h(0) = 0,\ v(0) = 0,\ m(0) = 1,\ h(T) = 1.
\end{aligned}
$$

Goddard problem: Optimal trajectory

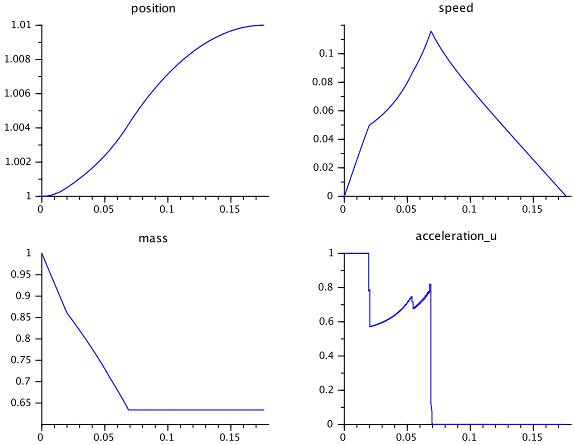

Figure: Thrust: full, constrained arc, singular arc, zero.

Regulator problem: statement

Consider the control-affine problem:

$$\min \ \frac{1}{2}\int_0^5 (x_1^2 + x_2^2)\mathrm{d}t;$$
$$s.t.\ \dot{x}_1 = x_2; \quad \dot{x}_2 = u; \quad x_1(0) = 0,\ x_2(0) = 1.$$

subject to the control bounds and state constraint:

$$-1 \le u(t) \le 1, \quad x_2 \ge -0.2,$$

To write the problem in the **Mayer form** (final cost), we define an extra state variable given by the dynamics

$$\dot{x}_3 = \frac{1}{2}(x_1^2 + x_2^2), \quad x_3(0) = 0.$$

Numerical solution

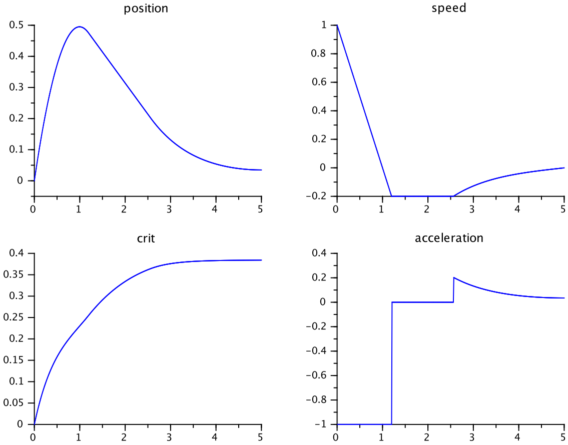

Figure: x_1 : position, x_2 : speed, x_3 : cost, u : acceleration

Some applications

Many other control-affine models with control bounds and state constraints appear in applications:

Maurer H, Gillessen W. *Application of multiple shooting to the numerical solution of optimal control problems with bounded state variables.* Computing. 1975

J.B. Keller, *Optimal velocity in a race*, Amer. Math. Monthly, 1974

A. Aftalion and J.F. Bonnans, *Optimization of running strategies based on anaerobic energy and variations of velocity*, SIAM J. Applied Math., 2014

P.-A. Bliman, M.S. Aronna, F.C. Coelho, and M.A. da Silva, *Ensuring successful introduction of* Wolbachia *in natural populations of* Aedes aegypti *by means of feedback control,* J. Math. Biology, 2017

M.H. Biswas, L.T. Paiva, M.R. De Pinho. *A SEIR model for control of infectious diseases with constraints.* Mathematical Biosciences and Engineering. 2014

Control bounds & state constraint: the problem

We consider the optimal control problem:

$$\begin{aligned}
&\min \varphi(x(0), x(T)) \\
&\quad \dot{x}(t) = f(u(t), x(t)) = f_0(x(t)) + u(t) f_1(x(t)), \quad \text{a.e. on } [0, T], \\
&\quad \Phi(x(0), x(T)) \in K_\Phi = \{0\}_{\mathbb{R}^{n_1}} \times \mathbb{R}^{n_2}_-, \\
&\quad u_{\min} \le u(t) \le u_{\max}, \quad \text{a.e. on } [0, T], \\
&\quad g(x(t)) \le 0, \quad t \in [0, T].
\end{aligned}$$

where $g : \mathbb{R}^n \to \mathbb{R}$.

Assuming feasibility, one has existence of optimal trajectory $(\hat{x}, \hat{u})$.

Previous second order conditions

Necessary conditions: **Kelley, Kopp, Moyer, Goh, Gabasov, Kirillova, Krener, Levitin, Milyutin, Osmolovskii, Agrachev, Sachkov, Gamkredlidze, Frankowska, Tonon,** others...

Sufficient conditions: **Moyer, Dmitruk, Sarychev, Poggiolini, Stefani, Zezza, Bonnard, Caillau, Trélat, Maurer, Osmolovskii, Sussmann, Schättler, Jankovic,** others...

Second order optimality condition under state-constraints: so many people...

Hamiltonian and Lagrangian functions

Let $\lambda := (\beta, \Psi, p, \mathrm{d}\mu) \in \mathbb{R}_+ \times \mathbb{R}^{(n_1+n_2)*} \times BV^{n*} \times \mathcal{M}$.
Let us define the (unmaximized) pre-Hamiltonian:

$$H[\lambda](x,u,t) := p(t)\big(f_0(x) + \sum_{i=1}^{m} u_i f_i(x)\big),$$

the endpoint Lagrangian:

$$\ell^{\beta,\Psi}(x(0),x(T)) := \beta\varphi(x(0),x(T)) + \Psi.\Phi(x(0),x(T)),$$

and the Lagrangian function

$$\mathcal{L}[\lambda](u,x) := \ell^{\beta,\Psi}(x(0),x(T)) + \int_0^T p\big(f(x,u) - \dot{x}\big)\mathrm{d}t + \int_{[0,T]} g(x)\mathrm{d}\mu(t).$$

First order optimality conditions

(I) Costate equation: for $p \in BV^{n*}$,

$$-\mathrm{d}p(t) = p(t) f_x(u(t), x(t))\mathrm{d}t + g'(x(t))\,\mathrm{d}\mu(t), \quad \text{for a.a. } t \in [0, T],$$

with endpoint conditions

$$(-p(0), p(T)) = D\ell^{\beta,\Psi}(x(0), x(T)).$$

(II) Minimization of Hamiltonian (affine w.r.t. u) for a.a. $t \in [0, T]$,

$$\begin{cases} u(t) = u_{\min}, & \text{if } p(t) f_1(x(t)) > 0, \\ u(t) = u_{\max}, & \text{if } p(t) f_1(x(t)) < 0, \\ p(t) f_1(x(t)) = 0, & \text{if } u_{\min} < u(t) < u_{\max}. \end{cases}$$

First order optimality conditions

(III) Final constraints multiplier: $\Psi \in \mathbb{R}^{(n_1+n_2)*}$;

$$\Psi_i \geq 0, \qquad \Psi_i \Phi_i(x(0), x(T)) = 0, \quad i = n_1 + 1, \dots, n_2.$$

(IV) State constraint multiplier: $\mathrm{d}\mu \in \mathcal{M}$;

$$\mathrm{d}\mu \geq 0; \qquad \int_0^T g(x(t))\mathrm{d}\mu(t) = 0.$$

$\Lambda(x, u) =$ set of Pontryagin multipliers associated to (x, u) :
$\lambda := (\beta, \Psi, p, \mathrm{d}\mu) \in \mathbb{R}_+ \times \mathbb{R}^{(n_1+n_2)*} \times BV^{n*} \times \mathcal{M}$ that satisfy (I)-(IV) and $|\beta| + |\Psi| = 1$.

A weak minimum is a feasible (x, u) such that $\phi(x(0), x(T)) \leq \phi(\tilde{x}(0), \tilde{x}(T))$ for any feasible $(\tilde{u}, \tilde{x})$ for which $\|(\tilde{x}, \tilde{u}) - (x, u)\|_\infty$ is small enough.

First order OC: Weak minimum $\Rightarrow \Lambda \neq \emptyset$.

Bang, contact and singular arcs

Bang sets for the control constraint:

$$\begin{cases} B_- := \{t \in [0,T] : \ \hat{u}(t) = u_{\min}\}, \\ B_+ := \{t \in [0,T] : \ \hat{u}(t) = u_{\max}\}, \end{cases}$$

and set $B := B_- \cup B_+$.

Contact set associated with the state constraint:

$$C := \{t \in [0,T] : \ g(\hat{x}(t)) = 0\},$$

Singular set

$$S := \{t \in [0,T] : u_{\min} < \hat{u}(t) < u_{\max} \text{ and } g(\hat{x}(t)) < 0\}.$$

$C, B, S-$arcs are maximal intervals contained in these sets.

Assumptions on control structure & Complementarity

(i) the interval $[0, T]$ is (up to a zero measure set) the disjoint union of finitely many arcs of type B, C and S,

(ii) the control $\hat{u}$ is at uniformly positive distance of the bounds, over C and S arcs and it is discontinuous at CS and SC junctions,

(iii) complementarity of control and state constraints, i.e., the corresponding multipliers do not vanish on the active arcs,

(iv) the state constraint is of first order, i.e.

$$g'(\hat{x}(t))f_1(\hat{x}(t)) \neq 0, \qquad \text{on } C.$$

Comments & consequences

1. These hypotheses are verified in the examples!

2. Jumping may be a necessary condition: [McDanell-Powers, *Necessary conditions for joining optimal singular and nonsingular arcs*, SIAM J. Control, 1971]

3. On $C-$arcs, $0 = \mathrm{d}H_u = p[f_1, f_0]\mathrm{d}t - g' f_1 \mathrm{d}\mu$. Then

$$\mathrm{d}\mu \text{ has a density } \nu = \frac{p[f_1, f_0](x)}{g'(x) f_1(x)}.$$

4. And on $C-$arcs, $0 = \frac{\mathrm{d}}{\mathrm{d}t} g(x) = g'(x)(f_0(x) + u f_1(x))$. Hence

$$u = -\frac{g'(x) f_0(x)}{g'(x) f_1(x)}.$$

The critical cone:

$$\mathcal{C} := \left\{ \begin{array}{l} (z[v,z_0],v) \in W^{1,\infty} \times L^\infty : \varphi'(\hat{x}(0),\hat{x}(T))(z(0),z(T)) \leq 0 \\ \Phi'(\hat{x}(0),\hat{x}(T))(z(0),z(T)) \in T_{K_\Phi},\ \ v(t) = 0 \text{ a.e. on } B \\ g'(\hat{x}(t))z(t) = 0 \text{ on } C \end{array} \right\}.$$

For every multiplier $\lambda \in \Lambda$, we have

$$D^2_{(x,u)^2}\mathcal{L}[\lambda](\hat{u},\hat{x})(z,v)^2 = Q[\lambda](z,v), \qquad \text{for } (z[v,z_0],v) \in W^{1,\infty} \times L^\infty,$$

where

$$Q := D^2\ell^{\beta,\Psi}(z(0),z(T))^2 + \int_0^T \left(z^\top H_{xx}\, z + 2vH_{ux}\, z\right)\mathrm{d}t + \int_{[0,T]} z^\top g''\, z\,\mathrm{d}\mu(t).$$

Theorem (Second order necessary condition)

Assume that $(\hat{x},\hat{u})$ is a weak minimum. Then

$$\max_{\lambda\in\Lambda} Q[\lambda](v,z) \geq 0, \qquad \textit{for all } (z,v) \in \mathcal{C}.$$

Q does not contain a term in H_{uu}!

Goh's transform

$$(z, v) \quad \mapsto \quad \left(y := \int v,\ \xi := z - f_u y \right).$$

$$\dot{z} = f_x\, z + f_u\, v \quad \mapsto \quad \dot{\xi} = f_x\, \xi + E y,$$

where $E := f_x f_u - \frac{\mathrm{d}}{\mathrm{d}t} f_u$.

Goh transformation on the Hessian of Lagrangian

Set for $(\xi, y, h) \in H^1 \times L^2 \times \mathbb{R}$ and $\lambda := (\beta, \Psi, p, \mathrm{d}\mu) \in \Lambda$,

$$\Omega := \Omega_T + \Omega^0 + \Omega^E + \Omega^g,$$

where

$$\begin{aligned}
\Omega_T[\lambda](\xi, y, h) :=& 2h\, H_{ux}(T)\xi(T) + h\, H_{ux}(T) f_1(\hat{x}(T))h, \\
\Omega^0[\lambda](\xi, y, h) :=& \int_0^T \big(\xi^\top H_{xx}\xi + 2yM\xi + yRy\big)\mathrm{d}t, \\
\Omega^E[\lambda](\xi, y, h) :=& D^2\ell^{\beta,\Psi}(\xi(0), \xi(T) + f_1(\hat{x}(T))h)^2, \\
\Omega^g[\lambda](\xi, y, h) :=& \int_0^T (\xi + f_1(\hat{x})y)^\top g''(\hat{x})(\xi + f_1(\hat{x})y)\mathrm{d}\mu(t).
\end{aligned}$$

Goh transformation of the critical cone

Remark: the final value of y matters

Set of (strict) primitive of critical directions:

$$\mathcal{P} := \left\{ \begin{array}{l} (\xi, y, h) \in W^{1,\infty} \times \mathbb{R} \times W^{1,\infty} : \\ y(0) = 0,\ y(T) = h,\ (\dot{y}, \xi + y f_1) \in \mathcal{C} \end{array} \right\}.$$

Closure of $\mathcal{P}$ in $H^1 \times L^2 \times \mathbb{R}$

Proposition

$(\xi, y, h) \in \overline{\mathcal{P}}$ *iff*

$$\begin{cases} g'(\hat{x}(t))(\xi(t) + y(t) f_1(\hat{x}(t))) = 0 \text{ on } C, \\ y \text{ is constant on each } B \text{ arc}, \\ \phi'(\hat{x}(0), \hat{x}(T))\big(\xi(0), \xi(T) + h f_1(\hat{x}(T))\big) \leq 0, \\ \Phi'(\hat{x}(0), \hat{x}(T))\big(\xi(0), \xi(T) + h f_1(\hat{x}(T))\big) \in T_\Phi, \\ y \text{ is continuous at the } BC, CB \text{ and } BB \text{ junctions}, \\ y(t) = 0, \text{ on } B_{0\pm} \text{ if a } B_{0\pm} \text{ arc exists}, \\ y(t) = h, \text{ on } B_{T\pm} \text{ if a } B_{T\pm} \text{ arc exists}, \\ \lim_{t \uparrow T} y(t) = h, \text{ if } T \in C. \end{cases}$$

Set $\mathcal{P}^2 := \overline{\mathcal{P}}$.

Second order necessary condition after Goh transformation

Let $(z[v, z_0], v) \in H^1 \times L^2$ and let $(\xi[y, z_0], y)$ be defined by Goh transformation. Then, for any $\lambda \in \Lambda$,

$$Q[\lambda](z[v, z_0], v) = \Omega[\lambda](y, y_T, \xi[y, z_0]).$$

Theorem (Second order necessary condition)

If $(\hat{x}, \hat{u})$ is a weak minimum, then

$$\max_{\lambda \in \Lambda} \Omega[\lambda](\xi, y, h) \geq 0, \quad \textit{for all } (\xi, y, h) \in \mathcal{P}^2.$$

Extended critical cone

Define $\mathcal{P}_*^2 \supseteq \mathcal{P}^2$ consisting of $(\xi, y, h) \in H^1 \times L^2 \times \mathbb{R}^m$ verifying

- $g'(\hat{x}(t))(\xi(t) + y(t) f_1(\hat{x}(t))) = 0$ on C,
- y is constant on each B arc,
- $\phi'(\hat{x}(0), \hat{x}(T))\big(\xi(0), \xi(T) + h f_1(\hat{x}(T))\big) \leq 0$,
- $\Phi'(\hat{x}(0), \hat{x}(T))\big(\xi(0), \xi(T) + h f_1(\hat{x}(T))\big) \in T_\Phi$,
- ~~y is continuous at the BC, CB and BB junctions,~~
- $y(t) = 0$, on $B_{0\pm}$ if a $B_{0\pm}$ arc exists,
- $y(t) = h$, on $B_{T\pm}$ if a $B_{T\pm}$ arc exists,
- ~~$\lim_{t \uparrow T} y_t = h$, if $T \in C$~~ (if $T \in C$, $[\mu(T)] = 0$ for all μ)

Sufficient condition: scalar control case

Pontryagin minimum: for any $N > 0$, $\exists\ \varepsilon_N > 0$ such that $(\hat{x}, \hat{u})$ is optimal on the set

$$\big\{(x,u) \text{ feasible} : \|u-\hat{u}\|_\infty < N, \quad \|u-\hat{u}\|_1 < \varepsilon_N\big\}.$$

Convergence in the Pontryagin sense: $\|v_k\|_1 \to 0,\ \|v_k\|_\infty < N.$

$\gamma-$order: $(\xi_0, y, h) \in \mathbb{R}^n \times L^2 \times \mathbb{R}$, let

$$\gamma(\xi_0, y, h) := |\xi_0|^2 + \int_0^T y(t)^2 \mathrm{d}t + |h|^2.$$

$\gamma-$growth condition in the Pontryagin sense: exists $\rho > 0$, for every $v_k \to 0$ in the Pontryagin sense,

$$J(\hat{u}+v_k) - J(\hat{u}) \geq \rho\gamma(y_k, y_k(T)),$$

where $y_k := \int v_k$.

Second order sufficient condition

Theorem

Suppose that for each $\lambda \in \Lambda$, $\Omega[\lambda](\cdot)$ *is a Legendre form in* $\{(\xi = \xi[y], y, h) \in H^1 \times L^2 \times \mathbb{R}\}$ *and there exists* $\rho > 0$ *such that*

$$\max_{\lambda \in \Lambda} \Omega[\lambda](\xi, y, h) \geq \rho\gamma(\xi_0, y, h), \quad \textit{for all } (\xi, y, h) \in \mathcal{P}_*^2.$$

Then $(\hat{x}, \hat{u})$ *is a Pontryagin minimum satisfying* $\gamma-$*growth.*

Remark: [Hestenes 1951] $\Omega[\lambda](\cdot)$ is a Legendre form if and only if,

$$R(\hat{x}(t)) + f_1(\hat{x}(t))^\top g''(\hat{x}(t)) f_1(\hat{x}(t))\nu(t) > \alpha > 0, \quad \text{on } [0, T],$$

where ν is the density of $\mathrm{d}\mu$.

Comments, work in progress & open problems

- Analytical proof of local optimality in some examples (see reference [Aronna, Bonnans & Goh])
- Convergence of an associated shooting algorithm
- **In progress:** "No-gap" conditions. Note that one may have $\mathcal{P}^2_* \supsetneq \mathcal{P}^2_S$.
- Possible extensions:
  - Optimal control continuous at the junction times. See e.g.: B.S. Goh, G. Leitmann, T.L. Vincent, *Optimal control of a prey-predator system,* Math. Biosci. 1974.
  - Apply second order sufficient conditions to show stability under data perturbation
  - General vector control and vector state constraint

References

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THANK YOU FOR YOUR ATTENTION