SOME REMARKS ON STATE CONSTRAINTS AND MIXED CONSTRAINTS

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CONTENTS

Abstract

We discuss some optimal control problems with both state and control constraints, or general mixed constraints. In the setting of state constrained control problems, we consider an approximation technique involving variational inequalities. The constraints may be automatically satisfied in this procedure. For control problems with mixed constraints, a relaxation of classical interiority assumptions is presented together with a recent approach based on implicit parametrization results and yielding global type algorithms.

Approximation and equivalence

We consider $V, H, \mathcal{U}$ to be Hilbert spaces with dense and compact imbedding $V \subset H \subset V^*$ (the dual space of V - H is identified with its own dual space) and $A : V \to V^*$, $B : \mathcal{U} \to H$ to be linear bounded operators with the assumptions:

$$(Av, v)_{V^* \times V} \geq \omega |v|_V^2, \omega > 0, \forall\ v \in V, \tag{1}$$

$$(Ay, v)_{V^* \times V} = (y, Av)_{V \times V^*}, \forall\ y, v \in V. \tag{2}$$

Approximation and equivalence

The state constrained optimal control problem is defined by

$$\mathrm{Min}\{\mathrm{g(y)} + \mathrm{h(u)}\}, \tag{3}$$

$$y' + Ay = Bu + f \text{ a.e. in } [0, T], \tag{4}$$

$$y(0) = y_0, \tag{5}$$

$$y(t) \in C,\ \ t \in [0, T], \tag{6}$$

Approximation and equivalence

Above, $C \subset H$ is a nonvoid, closed, convex subset, $y_0 \in C$, $Ay_0 \in H$, $f \in L^2(0, T; H)$, $g : L^2(0, T; H) \to R$ is convex, continuous, majorized from below by a constant and $h : L^2(0, T; \mathcal{U}) \to]-\infty, +\infty]$ is convex, proper, lower semicontinuous and coercive:

$$\lim_{|u|_{L^2(0,T;\mathcal{U})} \to \infty} h(u) = \infty. \qquad (7)$$

For any $u \in L^2(0, T; H)$, the equation (4), (5) has a unique solution $y \in C(0, T; V)$, $y' \in L^2(0, T; H)$ due to (1), (2). Under the usual admissibility hypothesis the control problem (3) - (6) has an optimal pair $[y^*, u^*]$ due to (7) and unique if strict convexity is assumed for the cost functional (3).

Approximation and equivalence

One variant of the variational inequality approximation technique is to associate with the constrained control problem (3) - (6), the approximate problem without state constraints:

$$\mathrm{Min}\{g(y) + h(u) + \frac{1}{2}|w|^2_{L^2(0,T;V^*)}\}, \tag{8}$$

$$y' + Ay + \varepsilon\omega = Bu + f, \varepsilon > 0, w \in \partial\varphi(y), \tag{9}$$

$$y(0) = y_0, \tag{10}$$

where $\varphi : V \to]-\infty, +\infty]$ is convex, lower semicontinuous, proper

$$\varphi(v) = \begin{cases} 0 & v \in C \cap V, \\ +\infty & \text{otherwise.} \end{cases} \tag{11}$$

Approximation and equivalence

The variational inequality (9), (10) has a unique solution $y \in C(0, T; H) \cap L^2(0, T; V)$, $y' \in L^2(0, T; H)$ by standard existence results in the literature. Moreover $w \in L^2(0, T; V^*)$ as well (related to the section of $\partial\varphi(y)$ occuring in (9)).
Using standard techniques involving minimizing sequences, it is possible to show that the unconstrained control problem (8) - (10) has at least one optimal pair $[y_\varepsilon, u_\varepsilon]$ for any $\varepsilon > 0$.

Approximation and equivalence

Theorem

If h is strictly convex and superquadratic and $\mathcal{U} = L^2(\Omega)$, then

$$u_\varepsilon \to u^*, \ \textit{strongly in } L^2(0,T;H),$$

$$y_\varepsilon \to y^*, \ \textit{strongly in } C(0,T;H).$$

If we denote by y^ε the solution of (4), (5) corresponding to u_ε, then

$$\mathrm{dist}(y^\varepsilon, C \cap V)_{C(0,T;H)\cap L^2(0,T;V)} \leq k\varepsilon, \tag{12}$$

where $k > 0$ is independent of $\varepsilon > 0$.

Approximation and equivalence

This shows the suboptimal character of the control u_ε, including an explicit uniform estimate of the possible violation of the state constraint (6). Theorem 1 can be strengthened to include a regularization of the nonlinear operator $\partial\varphi$ that appears both in the cost (8) and in the state equation (9).
Slightly weaker estimates as in (12) may be obtained as well.
For the regularized problems, usual gradient methods may be applied on numerical results. The variational inequality approach is a refinement of the penalization method with better estimates.

Approximation and equivalence

Assume now that $B : \mathcal{U} \to V^*$ linear bounded, $f \in L^2(0, T; V^*)$ and $C \subset V$. Then, $y \in C(0, T; H) \cap L^2(0, T; V)$, $y' \in L^2(0, T; V^*)$ as defined in (4), (5). Let $B^* : V \to \mathcal{U}^*$ be the adjoint operator and $\widetilde{C} = \{v \in V, B^* v \in B^*(C)\}$. We introduce the unconstrained problem

$$\mathrm{Min}\{g(y) + h(u - w) + \frac{1}{2}|w|^2_{L^2(0,T;\mathcal{U})}\}, \tag{13}$$

$$y' + Ay + Bw = Bu + f, \ \ w \in \partial\psi(B^* y), \tag{14}$$

$$y(0) = y_0, \tag{15}$$

Approximation and equivalence

where ψ is the indicator function of $B^*(C)$ in $\mathcal{U}^*$. Under certain condition on $\mathrm{dom}(\psi) \cap \mathrm{range}(B^*)$ of interiority type , the equation (14) can be rewritten in the form

$$y' + Ay + \partial\widetilde{\phi}(y) \ni Bu + f, \tag{16}$$

where $\widetilde{\phi}$ is the indicator of $\widetilde{C}$ in V.

Theorem

The control problems (13) - (15) and (3) - (6) are equivalent in the sense that they have the same optimal values and optimal pairs.

Interiority assumptions and generalized optimality conditions

We discuss here a more general optimal control problem involving abstract mixed constraints:

$$\mathrm{Min}\{L(y,u)+l(y(T))\}, \tag{17}$$

$$y'(t)+A(t)y(t)=Bu(t)+f(t) \text{ a.e. in } [0,T], \tag{18}$$

$$[y,u]\in D\subset[L^2(0,T;V)\cap W^{1,2}(0,T;V^*)]\times L^2(0,T;\mathcal{U}). \tag{19}$$

Here D is a convex closed nonvoid subset, $f\in L^2(0,T;V^*)$, $L: L^2(0,T;H\times\mathcal{U})\to R$, $l: H\to R$ are convex, continuous mappings, with the coercivity property:

$$L(y,u)\geq c_1|u|^2_{L^2(0,T;\mathcal{U})}-c_2,\ c_i>0 \text{ constants}. \tag{20}$$

Interiority assumptions and generalized optimality conditions

The family of operators $A(t)$ is V^* - measurable on $]0, T[$ and satisfies conditions like (1), (2) with uniform in $t \in [0, T]$ constants. The solution of (18), with initial condition $y(0) = y_0 \in H$ is unique in $L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$.
Under the admissibility condition, hypothesis (20) ensures the existence of at least one optimal pair $[y^*, u^*] \in D$ for the problem (17) - (19).
To obtain the optimality conditions for the problem (17) - (19), it is usual to impose Slater type assumptions

$$\exists\ [\bar{y}, \bar{u}] \text{ feasible } : \bar{y} \in \operatorname{int}\{y \in C(0, T; H); [y, \bar{u}] \in D\}. \qquad (21)$$

Interiority assumptions and generalized optimality conditions

We denote the operator

$$\forall\ [y,u] \in [L^2(0,T;V) \cap W^{1,2}(0,T;V^*)] \times L^2(0,T;V),$$
$$T(y,u) = y' + A(t)y - Bu - f \in L^2(0,T;V^*). \tag{22}$$

We impose the following weak constraint qualification

$$\exists\ \mathcal{M} \in D,\ \text{bounded in } C(0,T;H) \times L^2(0,T;\mathcal{U}):$$
$$0 \in \ \text{int}\ T(\mathcal{M}) \text{ in } L^2(0,T;V^*). \tag{23}$$

Interiority assumptions and generalized optimality conditions

It is easy to show that (21), (22) give (23). One can also infer that (23) is, as well, a consequence of

$$\exists\ [\bar{y}, \bar{u}] \text{ feasible } : \bar{u} \in \operatorname{int}\{u \in L^2(0, T; V); [\bar{y}, u] \in D\}. \tag{24}$$

Clearly (21) may not be valid if (24) is imposed. That is, constraint qualification (23) is strictly weaker than the Slater condition.

By using an adapted regularization and penalization of (17) combined with the penalization of (18), rather delicate duality-type estimates give the following generalized optimality system:

Interiority assumptions and generalized optimality conditions

Theorem

If the pair $[y^*, u^*]$ is optimal for the problem (17) - (19), then:

$$\int_0^T (y^{*'} - y' + Ay^* - Ay, p^* + r^*)_{V^* \times V} dt \leq 0, \tag{25}$$

$$\langle w_2, u^* - u \rangle_{L^2(0,T;\mathcal{U})} - \int_0^T \langle u^* - u, B^* J^{-1}(r^*) \rangle_{\mathcal{U}} dt \leq 0, \tag{26}$$

for any $[y, u]$ such that $[y, u^*] \in D$ and $[y^*, u] \in D$.
Moreover, summing (25) and (26) is valid for any $[y, u] \in D$ and it is also sufficient for the optimality of $[y^*, u^*]$.

Interiority assumptions and generalized optimality conditions

Here, p^* is the solution of the adjoint system

$$-p^{*'} + A^* p^* = w_1, \tag{27}$$

$$p^*(T) = w, \tag{28}$$

where $w \in \partial l(y^*(T))$, $[w_1, w_2] \in \partial L(y^*, u^*]$ and $J : V \to V^*$ is the canonical isomorphism, $r^* \in L^2(0, T; V)$ is the weak limit (on a subsequence) of

$$r_\varepsilon = \frac{1}{\varepsilon} J^{-1}(y_\varepsilon' + A(t)y_\varepsilon - Bu_\varepsilon - f)$$

with $[y_\varepsilon, u_\varepsilon]$ being the unique optimal pair of the approximating regularized/penalized optimal control problem.

Interiority assumptions and generalized optimality conditions

Remark

The form (25), (26) decouples the adjoint system (27), (28) from the constraints. In case, $D = \mathcal{K} \times \mathcal{U}_{ad}$ (separate state and control constraints), one can easily reobtain from (25) - (28) the usual form of the optimality conditions.

We briefly comment now on an example from that shows that even in the case of separate constraints, their interior may be void, while hypothesis (22), (23) is satisfied. We consider the following optimal control problem governed by a parabolic equation:

$$\mathrm{Min}\{\frac{1}{2}\int_Q (y - z_d)^2 dx + \frac{N}{2}\int_Q u^2 dx\}, \tag{29}$$

Interiority assumptions and generalized optimality conditions

$$\frac{\partial y}{\partial t} - \Delta y = f + u \text{ in } Q = \Omega\times]0, T[, \tag{30}$$

$$y(x,t) = 0 \text{ on } \Sigma = \partial\Omega \times [0,T], \tag{31}$$

$$y(x,0) = y_0(x) \text{ in } \Omega, \tag{32}$$

$$e(x,t) \leq y(x,t) \leq g(x,t) \text{ a.e. in } Q, \tag{33}$$

$$a(x,t) \leq u(x,t) \leq b(x,t) \text{ a.e. in } Q, \tag{34}$$

Interiority assumptions and generalized optimality conditions

$\Omega \subset R^d$ is a smooth bounded domain, $z_d \in L^2(Q)$, $N \geq 0$, $y_0 \in L^2(\Omega)$, f, a, b are in $L^\infty(Q)$ and e, g are in $C(\bar{Q})$. From (33), (34) one can immediately infer the form of D from (19). We ask the "rich" admissibility hypothesis: $\exists\ \alpha > 0, \exists\ \widetilde{u}$ satisfying (34) such that if Y denotes the operator $u \to y$ defined by (30) - (32), we have:

$$e \leq Y(\widetilde{u} - \alpha) \leq Y(\widetilde{u} + \alpha) \leq g \text{ a.e. in } Q. \tag{35}$$

Relation (35) is not an interiority condition since, we may have $e(x,t) = g(x,t)$ in certain points, for instance $e(x,t) = g(x,t) = 0$ on $\partial\Omega \times [0,T]$. Moreover $\widetilde{u} + \alpha$, $\widetilde{u} - \alpha$ need not satisfy (34) and we may as well have $a(x,t) = b(x,t)$ on some subset.

Interiority assumptions and generalized optimality conditions

Taking the spaces $V = H_0^1(\Omega)$, $H = \mathcal{U} = L^2(\Omega)$ and the operators $A(t) = -\Delta$, $B : H \to V^*$, $B = i$, the canonical injection and the cost $L(y, u)$ as given in (29), while $l = 0$, one can put the example (29) - (34) in the abstract form (17) - (19). The condition (23) may be checked with

$$\mathcal{M} = \mathrm{conv}\{[y_\xi, u_\xi]; \xi \in L^\infty(Q), \|\xi\|_{L^\infty(Q)} = 1, y_\xi = y(u_\xi), u_\xi = \widetilde{u} + \alpha\xi\}$$

(notice that the space $C(0, T; H) \times L^2(0, T; \mathcal{U})$ is replaced by $L^\infty(Q)$ in this example). The arguments are similar and take advantage that we work in a functional setting and we can use comparison theorems.

Interiority assumptions and generalized optimality conditions

Other examples of mixed constraints, in connection with an optimal investment problem may be discussed. The setting is similar with the above problem governed by parabolic partial differential equations:

$$\frac{1}{2}\int_{\Omega} y(x,t)^2 dx \leq C(u)(t), t \in [0,T],$$

where $C(\cdot) : \mathcal{U} \to L^1(0,T)$ is some given operator.

$$0 \leq u(x,t) \leq Cy(x,t) \text{ a.e. in } Q.$$

One can establish generalized bang-bang properties for such applications.

An implicit parametrization approach in the control for ODE's

We briefly review first the Hamiltonian approach to implicit systems.
In the Euclidean space R^d, we consider the general implicit functions system:

$$\begin{aligned} F_1(x_1,\ldots,x_d) &= 0,\\ F_2(x_1,\ldots,x_d) &= 0,\\ \ldots\ldots&\ldots\ldots\\ F_l(x_1,\ldots,x_d) &= 0, \end{aligned} \tag{36}$$

where $1 \leq l \leq d-1$ and $F_j \in C^1(\Omega)$, $F_j(x^0) = 0$, $j = \overline{1,l}$, $x_0 \in \Omega \subset R^d$, given bounded domain.

An implicit parametrization approach in the control for ODE's

We assume the standard independence assumption

$$\frac{D(F_1, F_2, \ldots, F_l)}{D(x_1, x_2, \ldots, x_l)} \neq 0 \text{ in } x^0, \tag{37}$$

however this hypothesis can be dropped and the notion of generalized solution can be then used.
We introduce in V the undetermined system of linear algebraic equations

$$v(x) \cdot \nabla F_j(x) = 0, j = \overline{1, l} \tag{38}$$

and we shall use $d - l$ solutions of (38) obtained by fixing successively the last $d - l$ components of the vector $v(x) \in R^d$ to be the rows of the identity matrix in R^{d-l} multiplied by $\Delta(x) = \det A(x) \neq 0$. Other choices of interest appear in the sequel.

An implicit parametrization approach in the control for ODE's

In this way we obtain $d - l$ independent solutions of (38) denoted by $v_1(x), \ldots, v_{d-l}(x)$, in some arbitrary order. Their first l components are obtained from (38) by inverting $A(x)$, $x \in V$, due to (37). The vector fields $v_k(x)$, $k = \overline{1, d-l}$ are also continuous in V since $F_j, j = \overline{1, l}$ are of class $C^1(\Omega)$.

We associate to them the following iterated type Hamiltonian system of differential equations (weakly coupled just via the initial conditions - that's why we call it iterated):

$$\frac{\partial y_1(t_1)}{\partial t_1} = v_1(y_1(t_1)), t_1 \in I_1 \subset R, \qquad (39)$$
$$y_1(0) = x^0,$$

$$\frac{\partial y_2(t_1, t_2)}{\partial t_2} = v_2(y_2(t_1, t_2)), t_2 \in I_2(t_1) \subset R, \qquad (40)$$
$$y_2(t_1, 0) = y_1(t_1),$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\frac{\partial y_{d-l}(t_1, t_2, \dots, t_{d-l})}{\partial t_{d-l}} = v_{d-l}(y_{d-l}(t_1, t_2, \dots, t_{d-l})), t_{d-l} \in I_{d-l}(t_1, t_2, \dots$$
$$y_{d-l}(t_1, t_2, \dots, t_{d-l-1}, 0) = y_{d-l-1}(t_1, t_2, \dots, t_{d-l-1}). \qquad (41)$$

An implicit parametrization approach in the control for ODE's

Although partial differential notations are used in (39) - (41), each of the above subsystems may be interpreted as an ordinary differential system since just one derivative appears. The Hamiltonian character of (39) - (41) will be obvious from their properties listed in what follows, and from the example. Existence is valid by the Peano theorem.

Theorem

Under assumption (37), if $l = d - 1$*, the system (39) - (41) has the uniqueness property.*
If $1 \leq d \leq d - 2$*, every subsystem of (39) - (41) has the uniqueness property. Moreover, the intervals* $I_j(t_1, t_2, \ldots, t_{j-1})$*,* $j = \overline{1, d - l}$ *may be choosen independent of the parameters.*

An implicit parametrization approach in the control for ODE's

Proposition

The above differential systems have solutions $y_j \in C^1(I_1 \times I_2 \times \ldots \times I_j)$, $j = \overline{1, d-l}$.

Proposition

For every $k = \overline{1, l}$, $j = \overline{1, d-l}$, *we have the conservation property*

$$F_k(y_j(t_1, t_2, \ldots, t_j)) = 0, \quad \forall\, (t_1, t_2, \ldots, t_j) \in I_1 \times I_2 \times \ldots \times I_j.$$

An implicit parametrization approach in the control for ODE's

Proposition

If $F_k \in C^1(\Omega)$, $k = \overline{1, l}$, and I_j are sufficiently small, $j = \overline{1, d-l}$, then the mapping $y_{d-l} : I_1 \times I_2 \times \ldots \times I_{d-l} \to R^d$ is regular and one-to-one on its image.

The solution y_{d-l} is a parametrization of the manifold on $I_1 \times I_2 \times \ldots \times I_{d-l}$. If the last $d-l$ components of the algebraic solutions $v_j(x) \in R^d$ are chosen as the rows of the identity matrix in R^{d-l}, we obtain more :

Proposition

The last $d-l$ components of y_{d-l} have the form $(t_1 + x^0_{l+1}, t_2 + x^0_{l+2}, \ldots, t_j + x^0_{l+j}, x^0_{l+j+1}, \ldots, t_{d-l} + x^0_d)$, that is the first l components of y_{d-l} give the unique solution of the implicit system on $x^0 + (I_1 \times I_2 \times \ldots \times I_{d-l})$.

An implicit parametrization approach in the control for ODE's

We introduce now the optimal control problem with equality mixed constraints:

$$\mathrm{Min}\{l(x(0), x(1))\}, \tag{42}$$

$$x'(t) = f(t, x(t), u(t)), t \in [0, 1], \tag{43}$$

$$h(x(t), u(t)) = 0, t \in [0, 1]. \tag{44}$$

Above, $l : R^d \times R^d$, $f : [0, 1] \times R^d \times R^m \to R^d$,
$h : R^d \times R^m \to R^s$, $s \geq m$, $\alpha + m - 1 \geq s$, are given mappings and $x : [0, 1] \to R^d$ is the state variable, $u : [0, 1] \to R^m$ is the control unknown.

An implicit parametrization approach in the control for ODE's

As general assumptions, we shall require l continous, f locally Lipschitzian in (x, u) and measurable in t, h of class C^1 and there is a point $(x^0, u^0) \in R^d \times R^m$ such that

$$h(x^0, u^0) = 0 \text{ and } \nabla h(x^0, u^0) \text{ of maximal rank.} \tag{45}$$

Under hypothesis (45), one can obtain a constructive parametric description of the admissible manifold for (44), denoted by $A \subset R^d \times R^m$. This is not the admissibility set for the control problem (42) - (44). However any admissible state-control trajectory should satisfy

$$(x(t), u(t)) \in A, t \in [0, T]. \tag{46}$$

Here and in the sequel we shall assume that the admissible controls $u(t) \in W^{1,2}(0, T; R^m)$, consequently (46) makes sense by regularity properties for (43).

An implicit parametrization approach in the control for ODE's

We also recall that there are regularity results for the optimal pairs, for instance in the classical books of Clarke, Fleming and Rishel that allow to restrict the search for admissible pairs by such regularity conditions.
In the standard terminology for DAE system, relations (43), (44) are semi-explicit of index one. Taking into account (44), (46) and differentiating once, we get:

$$\nabla_x h(x(t), u(t)) f(t, x(t), u(t)) + \nabla_u h(x(t), u(t), u'(t)) = 0. \quad (47)$$

If $\nabla_u h(\cdot, \cdot)$ is invertible on A, then (47) may be put in an explicit form, as an ODE for the control vector $u(t) \in R^m$.

An implicit parametrization approach in the control for ODE's

The important observation is that any point in A provides a consistent initial condition for the differential system (43), (47). This system gives a characterization of the admissible state-control trajectories. It is elementary to show:

Proposition

Any trajectory of (43), (47), starting from a point in A, remains in A and is in $W^{1,2}((0,s);R^d \times R^m)$.

Here, $(0, s)$ is the local existence interval (depending on the initial condition). In control theory, taking into account the form of the cost functional (42), we also require the existence of global solutions, in $[0, 1]$, that has to be checked in each applications.

An implicit parametrization approach in the control for ODE's

We notice that the set of discretization points in A generated by (39) - (41), denoted by $\underset{n\in N}{\cup} A_n$, is dense in A when the discretization of $I_1 \times I_2 \times \ldots \times I_{d-l}$ is finer and finer.
We have, for the terminal set $T = \{x(1); [x(0), u(0)] \in A\}$:

Proposition

Under global existence for the system (43), (47), the discretized terminal set $\underset{n\in N}{\cup} T_n = \{x(1); [x(0), u(0)] \in \underset{n\in N}{\cup} A_n\}$ *is dense in* T.

This is a consequence of continuity results with respect to initial conditions, since f is locally Lipschitz in (x, u) and similar conditions are imposed on $\nabla h(\cdot, \cdot)$.

An implicit parametrization approach in the control for ODE's

We add now to the problem (42) - (44) more constraints:

$$q_r(x(t), u(t)) \leq 0, r = \overline{1, Q}, t \in [0, 1], \tag{48}$$

$$(x(t), u(t)) \in C(t), t \in [0, 1], \tag{49}$$

where $C(t)$ is some closed nonvoid subset, for any $t \in [0, 1]$ and assume that the admissible set for (43), (44), (48), (49) is not empty. The following algorithm is taken into account:

An implicit parametrization approach in the control for ODE's

Algorithm 4.7

1) Fix $n = 1$ and choose some discretization of $I_1 \times I_2 \times \ldots \times I_{d-l}$, a tolerance parameter δ etc.
2) Compute A_n via (39) - (41), the corresponding discretization of A.
3) Compute via (43), (47) the trajectories $[x(t), u(t)]$, with initial conditions in A_n. They automatically satisfy (44).
4) Check the conditions (48), (49) for all the trajectories defined in STEP 3 (in the discretization points). This gives the set of admissible discrete trajectories $\mathcal{O}_n$.
5) Compute $l(x(0), x(1))$ for all $[x, u] \in \mathcal{O}_n$ and find the optimal solutions (which may be not unique) and the optimal cost L_n.
6) If $|L_n - L_{n-1}| < \delta$, then STOP!
Otherwise $n := n + 1$ and GO TO STEP 2.

An implicit parametrization approach in the control for ODE's

We take $d = s = m = 1$ in (42) - (44), with hypothesis (45) and other conditions mentioned above. Then, the constraint (44) gives a curve in R^2, with coordinates (y, v). Its parametric representation can be obtained by the simplest Hamiltonian system:

$$\begin{aligned} \dot{y}(s) &= -\frac{\partial h}{\partial u}(y(s), v(s)), s \in I \\ \dot{v}(s) &= -\frac{\partial h}{\partial x}(y(s), v(s)), s \in I \end{aligned} \tag{50}$$

with initial condition

$$y(0) = x^0, \quad v(0) = u^0. \tag{51}$$

Any admissible trajectory $(x(t), u(t))$, $t \in [0, 1]$, should lie on the curve defined by (50), (51) and is, in fact, completely determined by its initial conditions.

An implicit parametrization approach in the control for ODE's

The admissibility equation (47) for u has the form

$$u'(t(s)) = -\frac{\frac{\partial h}{\partial x}(x(t,s),u(t,s))}{\frac{\partial h}{\partial u}(x(t,s),u(t,s))} f(t, x(t,s), u(t,s)), t \in [0,1] \quad (52)$$
$$x(0,s) = y(s), \quad u(0,s) = v(s)$$

(with obvious notations for the derivatives in s, respectively t) and should be solved together with (43).
If $l(a,b) = (a-4)^2 + (b-15)^2$ and
$f(t,x,u) = x - 5u + 10t + 2$, $h(x,u) = x - 5u - 4$, then
$x(t) = 5t^2 + 6t + 4$, $u(t) = t^2 + 1.2t$ give an optimal pair.

An implicit parametrization approach in the control for ODE's

We fix now $d = 2$, $s = m = 1$ in (42) - (44).
The iterated Hamiltonian system has the form:

$$y_1'(s) = -h_{y_2}(y_1(s), y_2(s), v(s)), s \in I_1,$$
$$y_2'(s) = h_{y_1}(y_1(s), y_2(s), v(s)), s \in I_2, \quad (53)$$
$$v'(s) = 0.$$

$$y_1(0) = x_1^0, \quad y_2(0) = x_2^0, \quad v(0) = u^0; \quad (54)$$

$$\dot{z}_1(\xi, s) = -h_u(z_1(\xi, s), z_2(\xi, s), w(\xi, s)), s \in I_2,$$
$$\dot{z}_2(\xi, s) = 0, s \in I_2, \quad (55)$$
$$\dot{w}(\xi, s) = h_{x_1}(z_1(\xi, s), z_2(\xi, s), w(\xi, s)), s \in I_2,$$

$$z_1(0, s) = y_1(s), \quad z_2(0, s) = y_2(s), \quad w(0, s) = v(s) \quad (56)$$

An implicit parametrization approach in the control for ODE's

The admissibility equation for the control $u(t,\xi,s)$ is

$$\frac{d}{dt}u(t,\xi,s) = \frac{-\sum_{i=1}^{2} f_i(t,x_1(t,\xi,s),x_2(t,\xi,s),u(t,\xi,s))\cdot h_{x_i}(x_1(t,\xi,s),x_2(t,\xi,s),u(t,\xi,s))}{h_u(x_1(t,\xi,s),x_2(t,\xi,s),u(t,\xi,s))} \tag{57}$$

with initial conditions

$$x_1(0,\xi,s) = z_1(\xi,s),\ x_2(0,\xi,s) = z_2(\xi,s),\ u(0,\xi,s) = w(\xi,s) \tag{58}$$

An implicit parametrization approach in the control for ODE's

and has to be solved together with the two equations from (43), under assumption $h_u(x_1^0, x_2^0, u^0) \neq 0$. The admissible trajectory $(x_1(t,\xi,s), x_2(t,\xi,s), u(t,\xi,s))$, $t \in [0,1]$ lies on the admissible surface parametrized via (53) - (56), passing through the initial point (58). If

$$f_1(t, x_1, x_2, u) = x_1 - 2.5x_2 - 10.25t + u + 4.5,$$

$$f_2(t, x_1, x_2, u) = 2x_1 + 3x_2 - \frac{16}{9}u^2 + 5.5t - 15.5,$$

$$h(x_1, x_2, u) = x_1 + 2x_2 - u^2 + u - 6,$$

$$l(x_1(0), x_2(0), x_1(1), x_2(1)) = (x_1(0) - 4)^2 + (x_2(0) - 1)^2 +$$

$$+(x_1(1) - 15)^2 + (x_2(1) + 1.5)^2$$

then an optimal triple is $(5t^2 + 6t + 4,\ 2t^2 - 4.5t + 1,\ 3t)$, passing through $(4, 1, 0)$ at $t = 0$.

An implicit parametrization approach in the control for ODE's

Example

In Fig. 1 we represent the manifold of admissible conditions (the curve is the solution of (53), (54), while Fig. 2 also includes the above optimal triple, the second three dimensional curve situated on this manifold.

An implicit parametrization approach in the control for ODE's

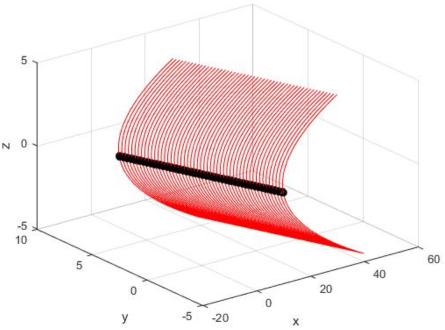

Figure: Manifold of admissible initial conditions

An implicit parametrization approach in the control for ODE's

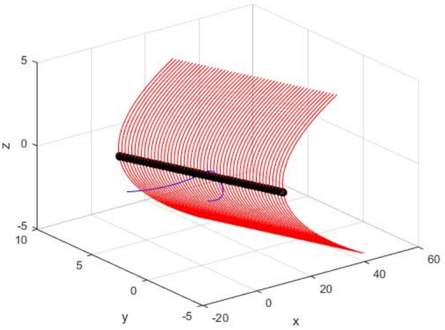

Figure: The optimal trajectory

THANK YOU!