Optimal control with several targets: the need of a rate-independent memory

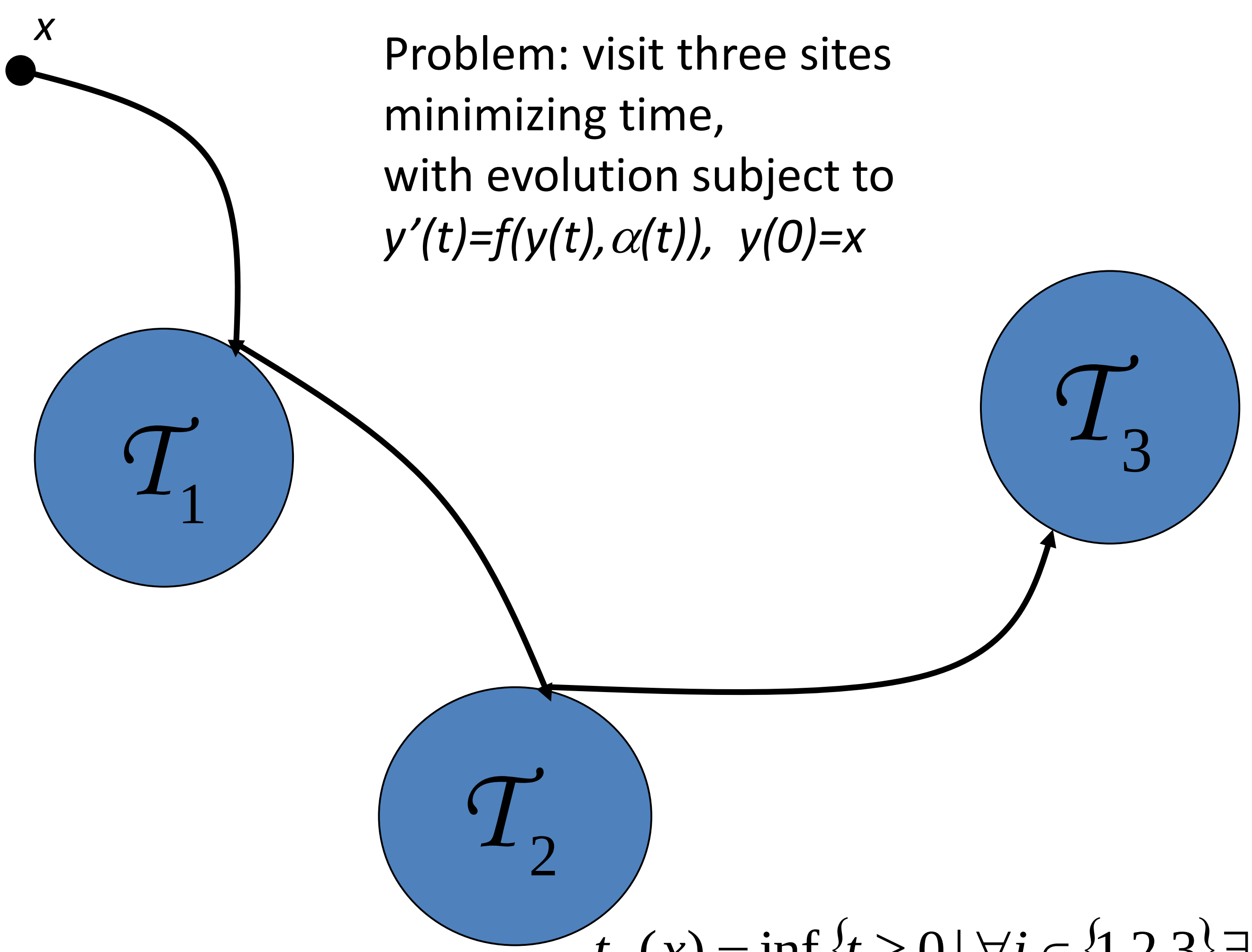
Fabio Bagagiolo
University of Trento – Italy

CoSCDS Padova September 25-29 2017

Contents

- The need of rate-independent memory
  - Continuous memory/hysteresis
    - Dynamic programming with hysteresis
    - Optimal visiting problem
  - Switching memory
    - Mean field game for tourists flow

Optimal visiting problem

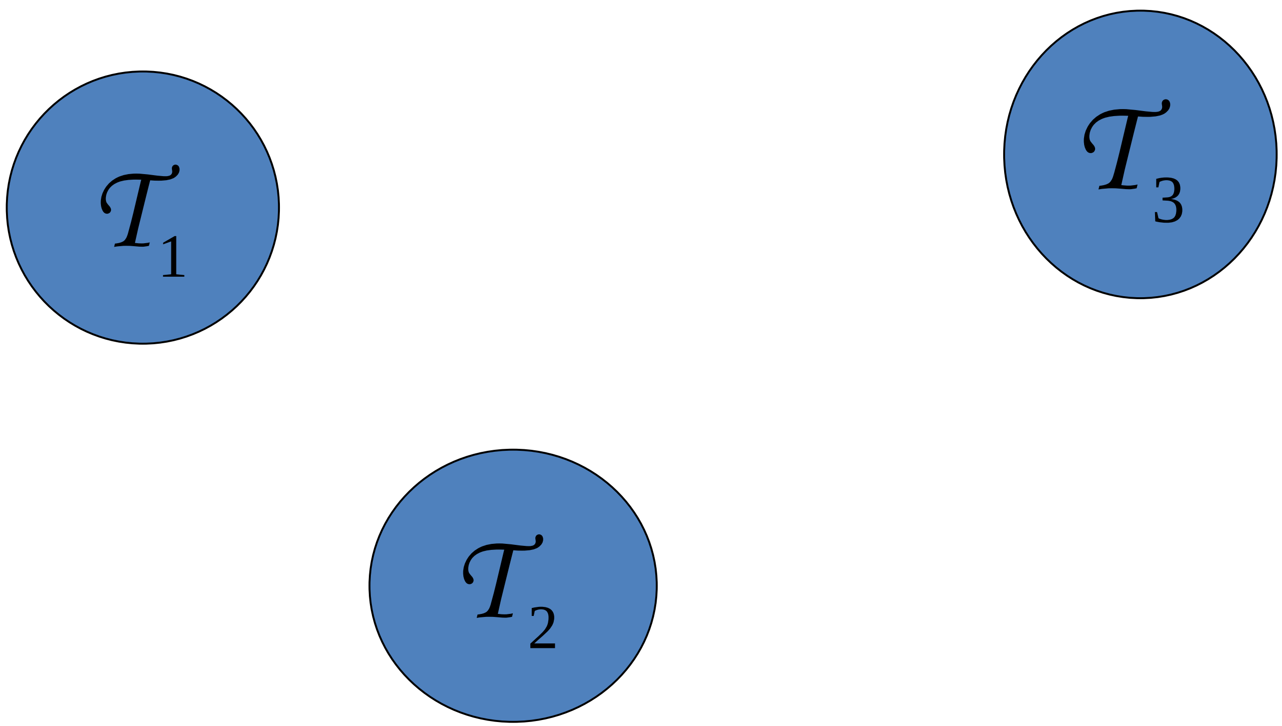


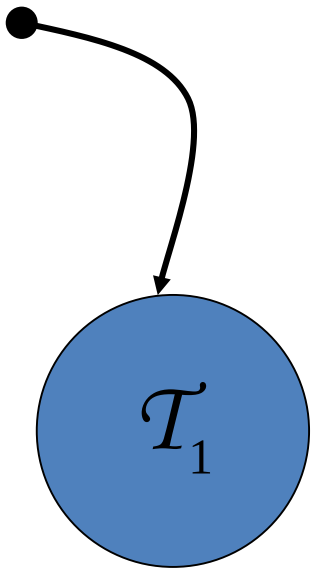
Problem: visit three sites minimizing time, with evolution subject to
$y'(t)=f(y(t),\alpha(t)), \quad y(0)=x$

$$t_\alpha(x) = \inf\{t \geq 0 \mid \forall i \in \{1,2,3\}\ \exists\, 0 \leq t_i \leq t,\ y(t_i) \in \mathcal{T}_i\}$$

$$T(x) = \inf_\alpha t_\alpha(x) \quad \text{optimal visiting function}$$

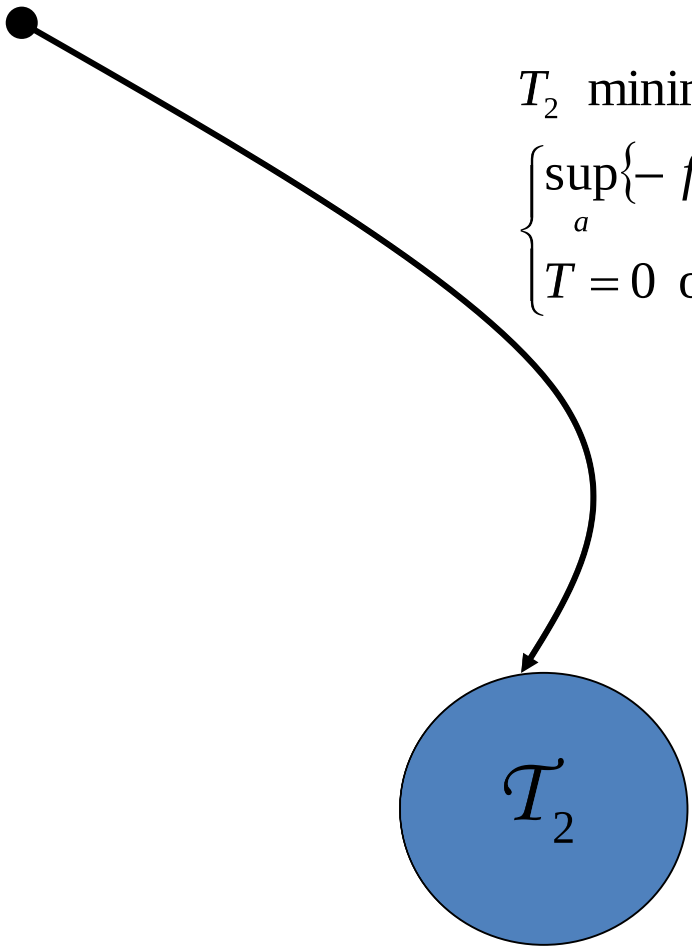
- The problem is obviously reminiscent of the famous Traveling Salesman Problem: minimizing the length of the path for passing through *m* cities.
- It is then characterized by a high computational complexity: many sub-problems must be addressed before solving the initial problem.


$\mathcal{T}_1$
$\mathcal{T}_3$
$\mathcal{T}_2$



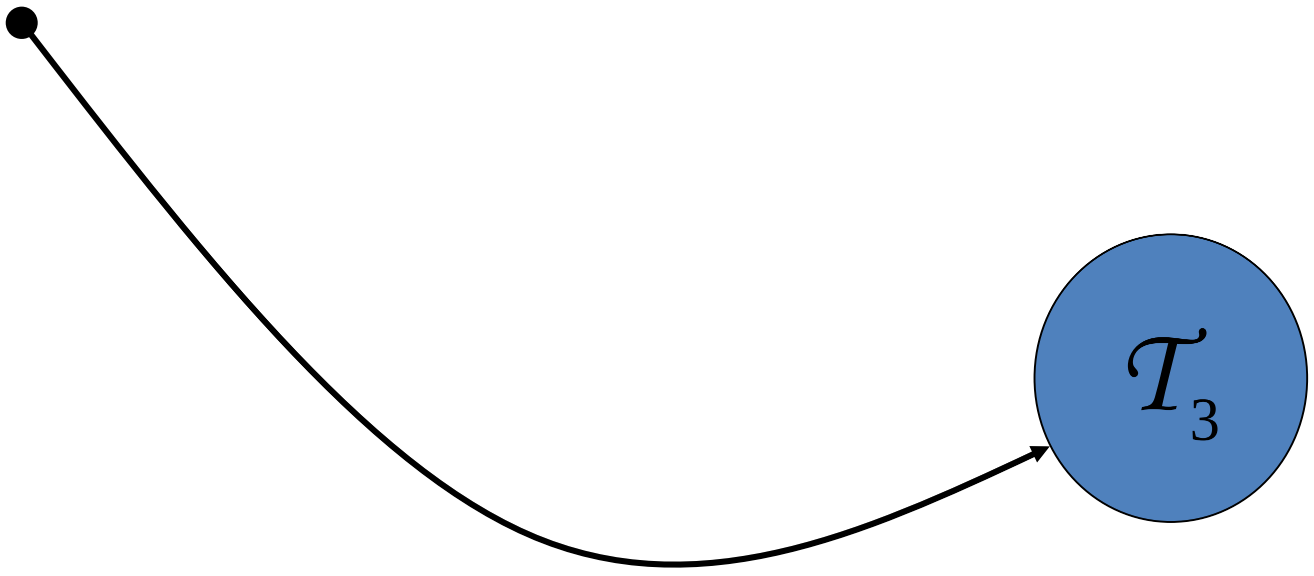
T_1 minimum time function for reaching $\mathcal{T}_1$, which solves

$$\begin{cases} \sup_a \{-f(x,a)\cdot\nabla T(x)\} = 1 \\ T = 0 \text{ on } \partial\mathcal{T}_1 \end{cases}$$



T_2 minimum time function for reaching $\mathcal{T}_2$, which solves

$$\begin{cases} \sup_a \{- f(x,a) \cdot \nabla T(x)\} = 1 \\ T = 0 \text{ on } \partial \mathcal{T}_2 \end{cases}$$

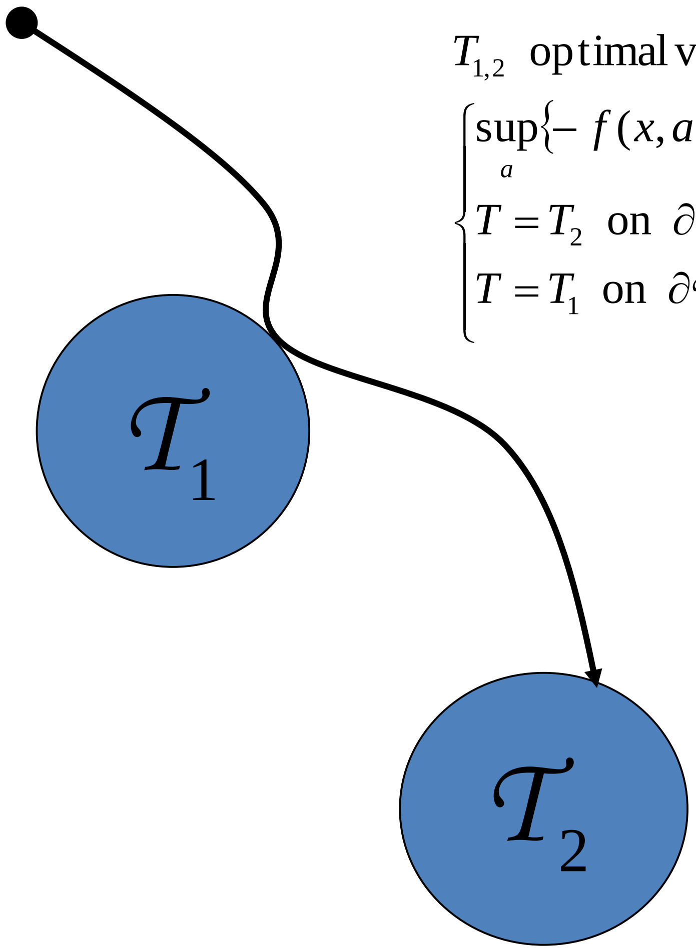


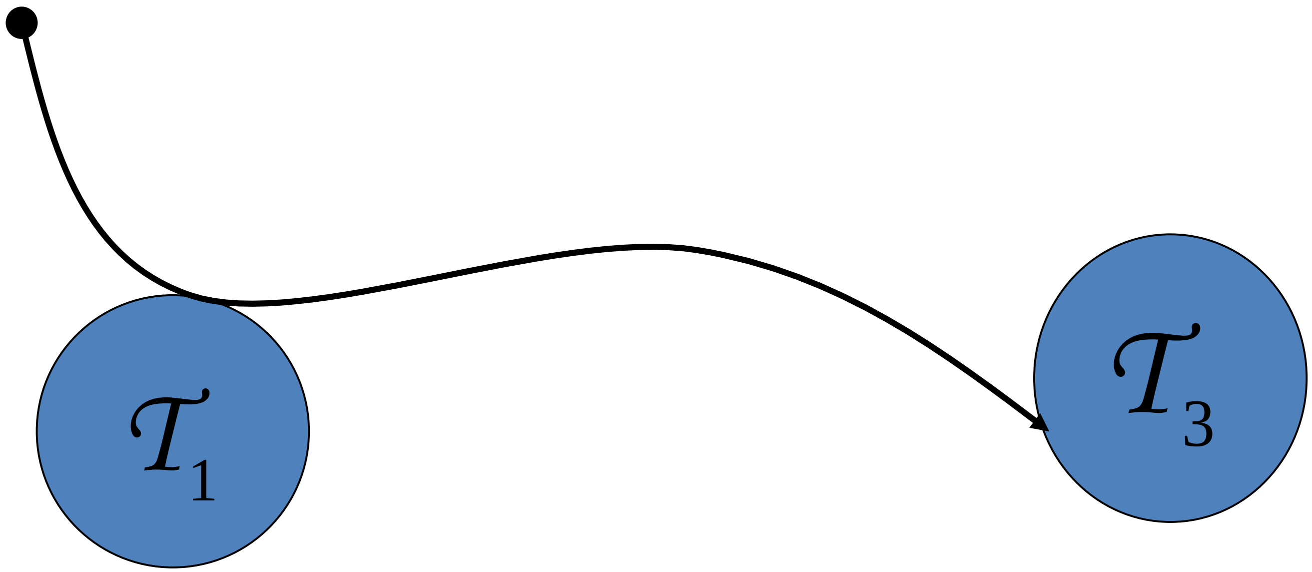
T_3 minimum time function for reaching $\mathcal{T}_3$, which solves

$$\begin{cases} \sup_a \{- f(x,a) \cdot \nabla T(x)\} = 1 \\ T = 0 \text{ on } \partial \mathcal{T}_3 \end{cases}$$

$T_{1,2}$ optimal visting function for reaching $\mathcal{T}_1$ e $\mathcal{T}_2$, which solves

$$\begin{cases} \sup_a \{ -f(x,a) \cdot \nabla T(x) \} = 1 \\ T = T_2 \text{ on } \partial\mathcal{T}_1 \\ T = T_1 \text{ on } \partial\mathcal{T}_2 \end{cases}$$





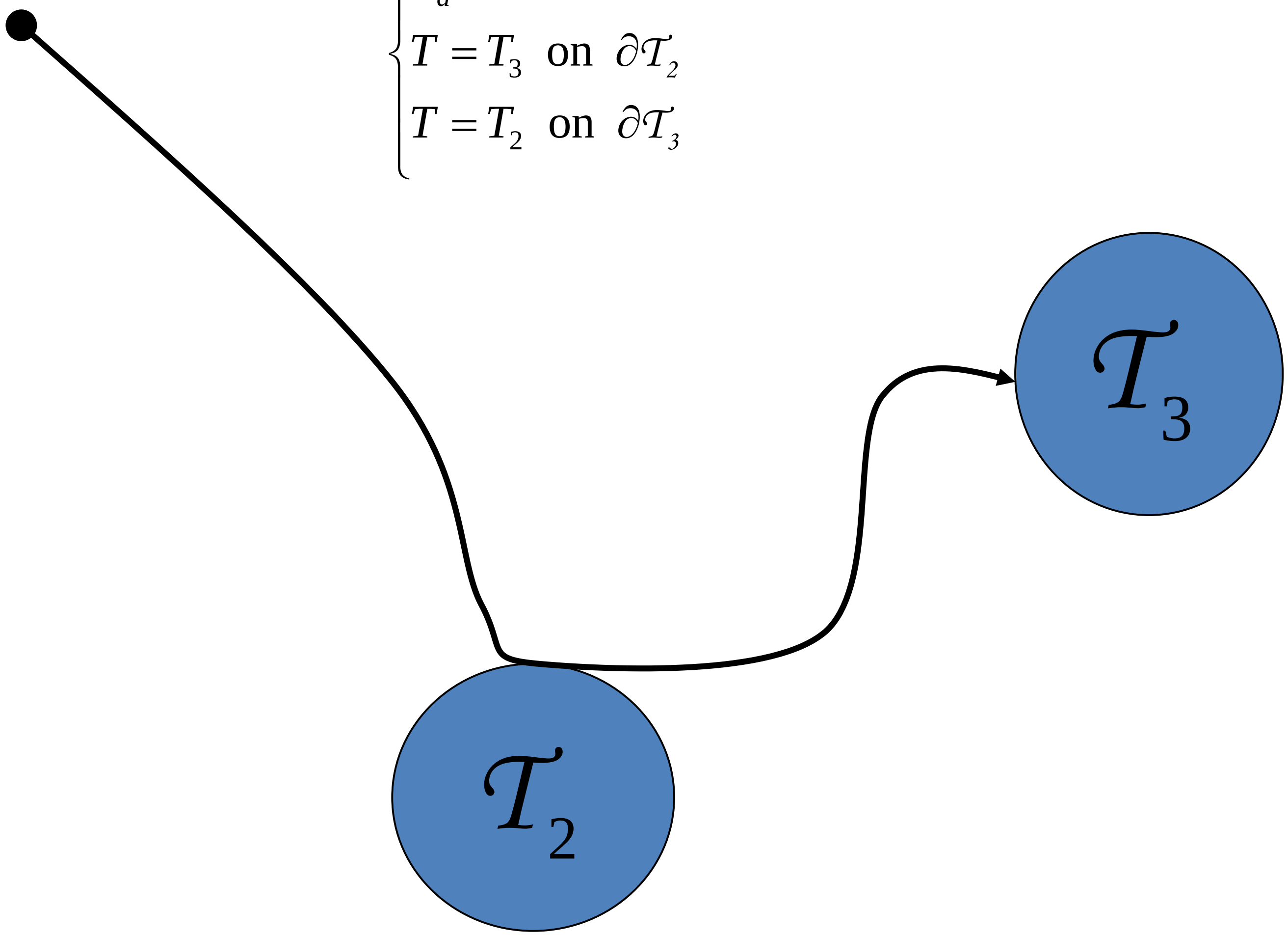
$T_{1,3}$ optimal visiting function for reaching $\mathcal{T}_1$ e $\mathcal{T}_3$, which solves

$$\begin{cases} \sup_a \{ -f(x,a) \cdot \nabla T(x) \} = 1 \\ T = T_3 \text{ on } \partial \mathcal{T}_1 \\ T = T_1 \text{ on } \partial \mathcal{T}_3 \end{cases}$$

$T_{2,3}$ optimal visiting function for reaching $\mathcal{T}_2$ e $\mathcal{T}_3$,

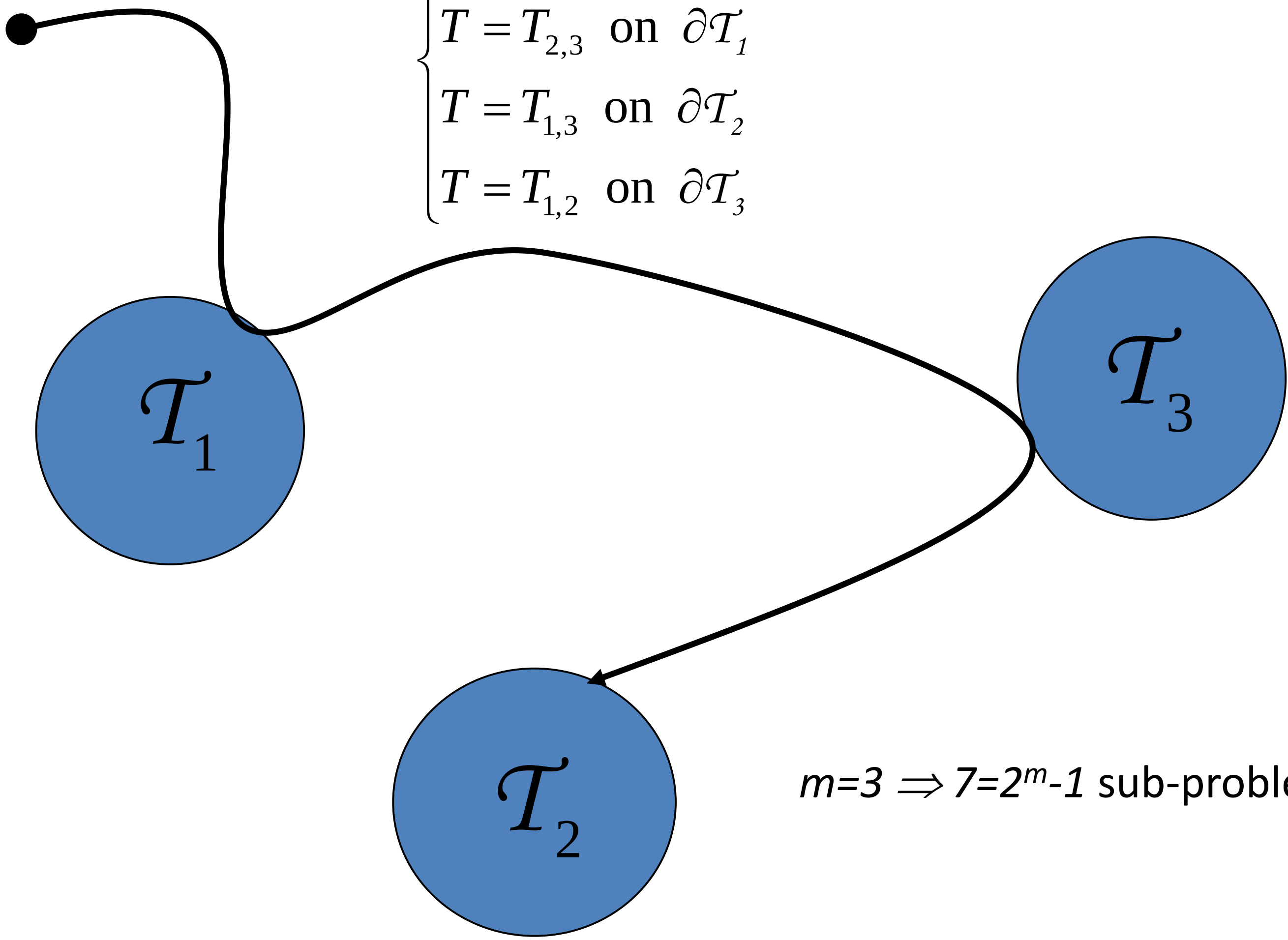
which solves

$$\begin{cases} \sup_a \{-f(x,a)\cdot\nabla T(x)\} = 1 \\ T = T_3 \text{ on } \partial\mathcal{T}_2 \\ T = T_2 \text{ on } \partial\mathcal{T}_3 \end{cases}$$



$T_{1,2,3}$ optimal visting function for reaching $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$, which solves

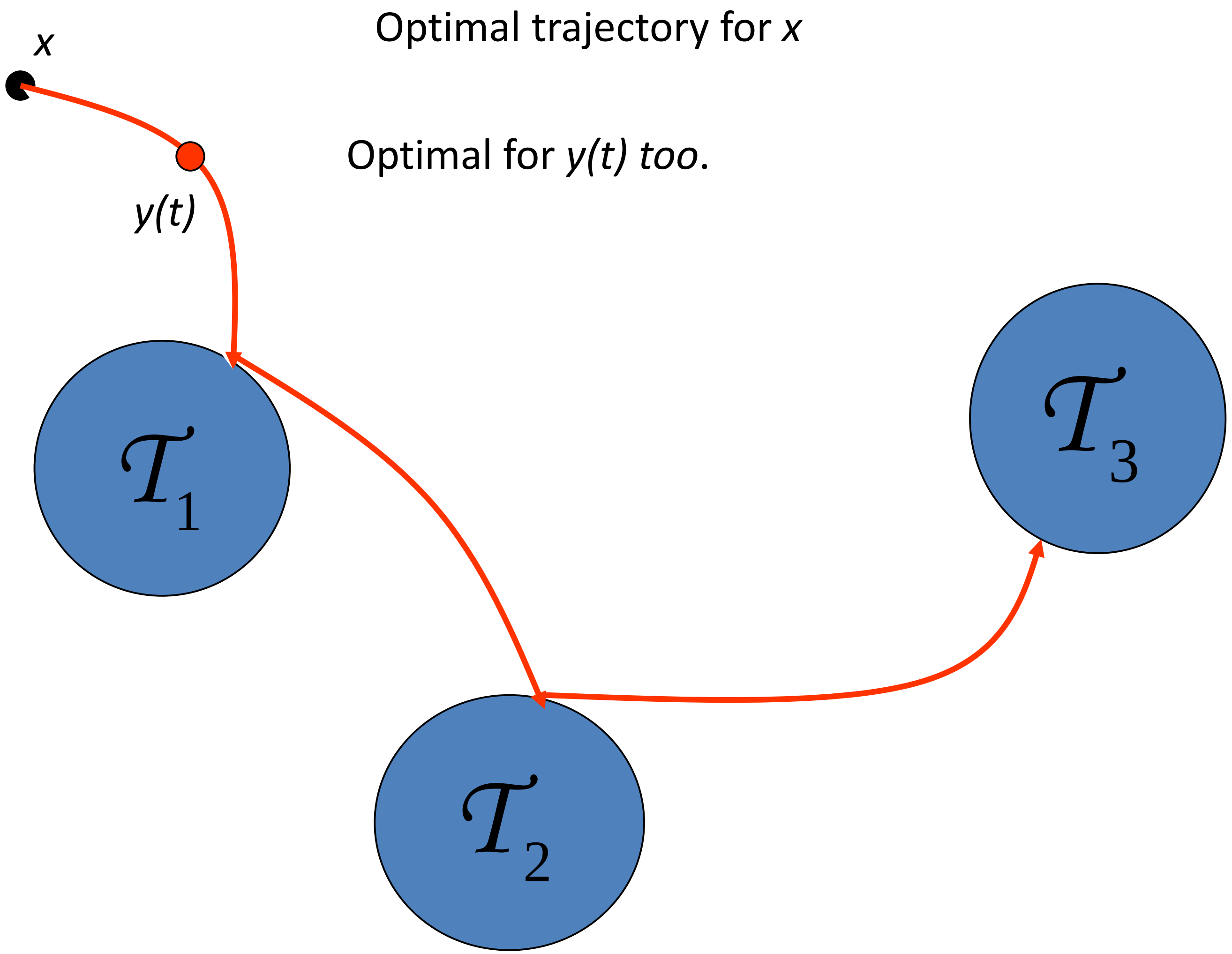
$$\begin{cases} \sup_a \{- f(x,a) \cdot \nabla T(x)\} = 1 \\ T = T_{2,3} \text{ on } \partial \mathcal{T}_1 \\ T = T_{1,3} \text{ on } \partial \mathcal{T}_2 \\ T = T_{1,2} \text{ on } \partial \mathcal{T}_3 \end{cases}$$



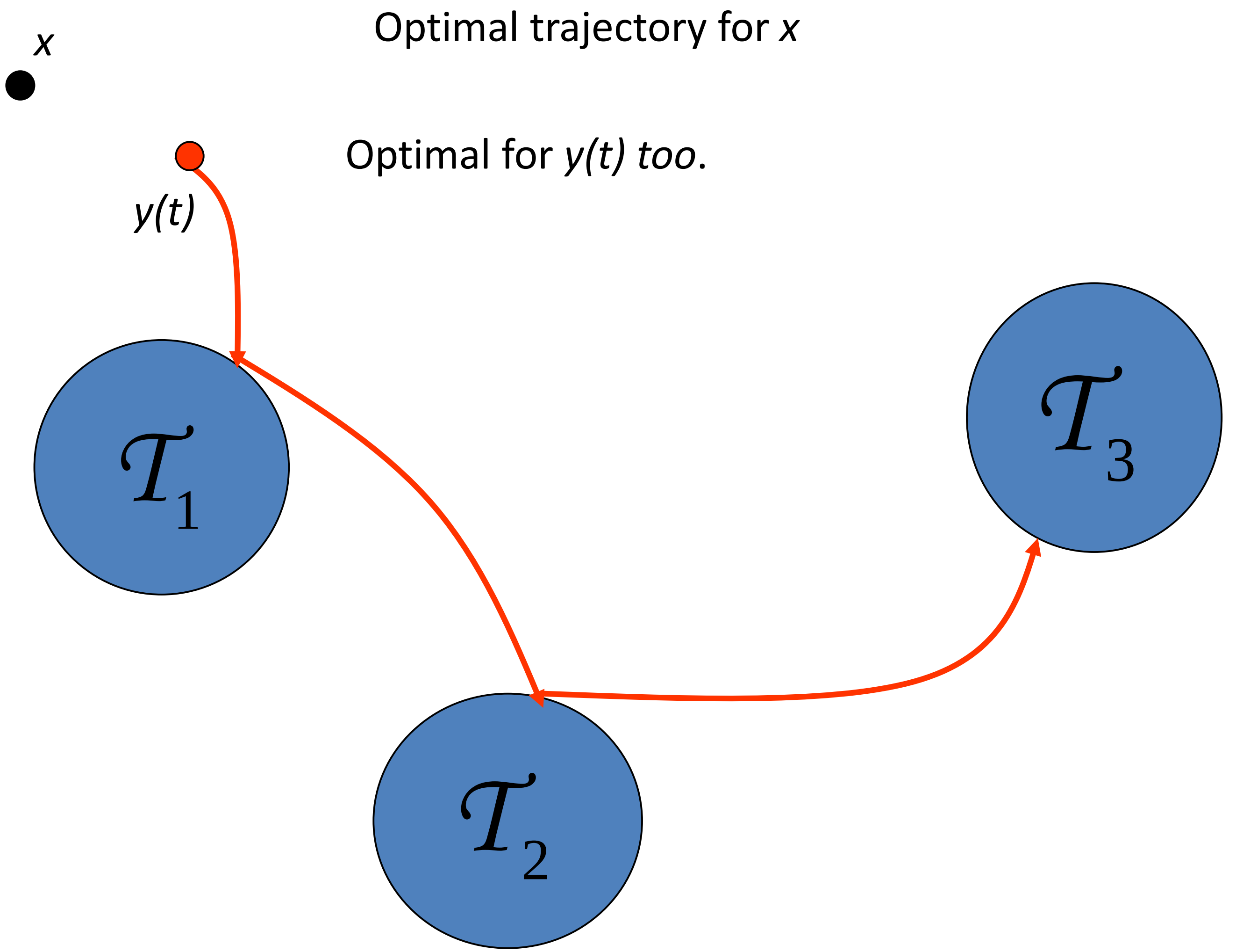
$m=3 \Rightarrow 7=2^m-1$ sub-problems

Goal

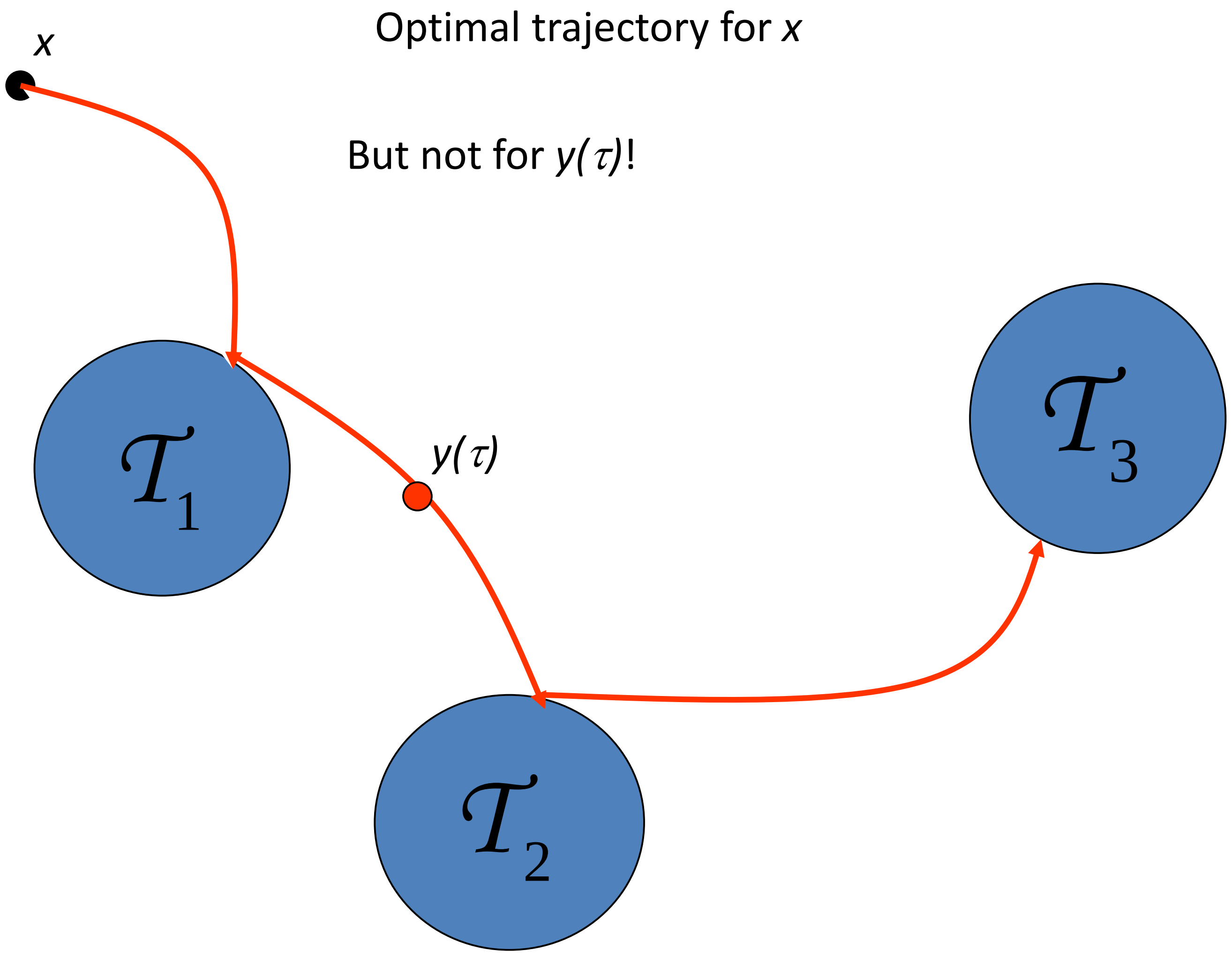
- Use Dynamic Programming for writing a “single” equation uniquely satisfied by the optimal visiting function.
- An immediate problem:
- The Dynamic Programming Principle does not hold.
- “Pieces of optimal trajectories are not optimal”!



Optimal trajectory for *x*
Optimal for *y(t) too*.
x
y(t)
$\mathcal{T}_1$
$\mathcal{T}_2$
$\mathcal{T}_3$

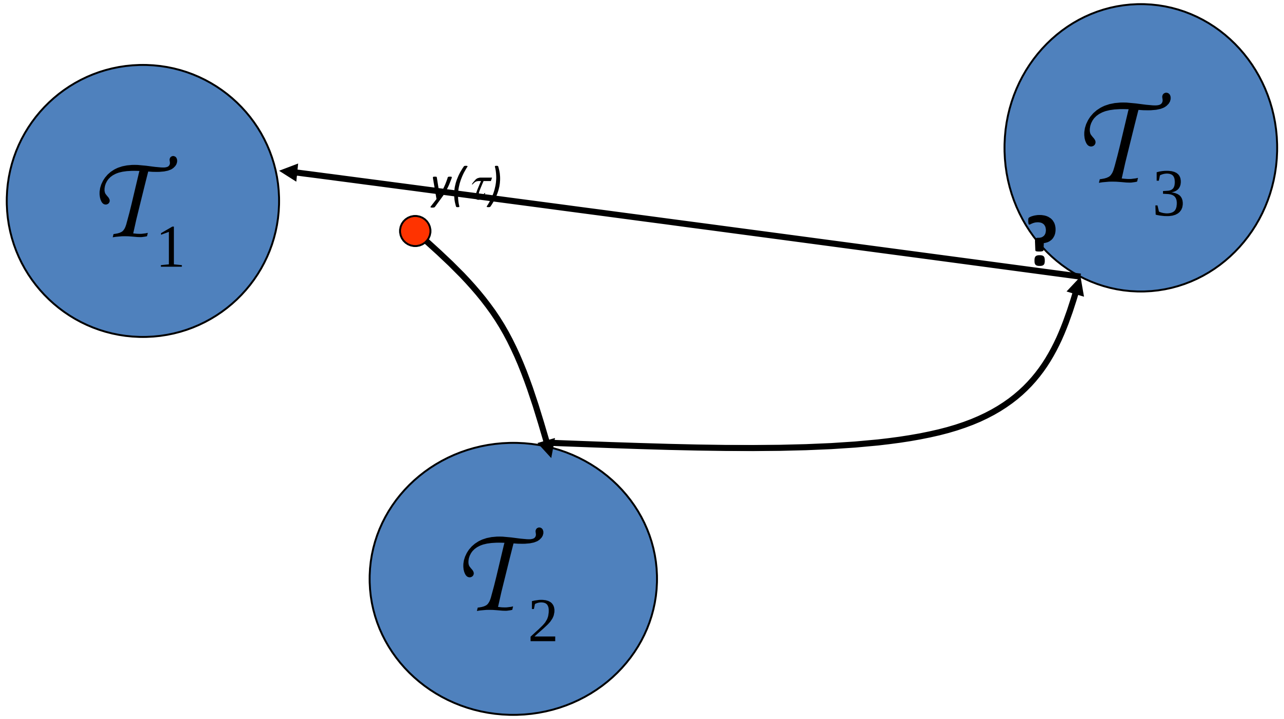


Optimal trajectory for *x*
x
Optimal for *y(t) too*.
y(t)
$\mathcal{T}_1$
$\mathcal{T}_3$
$\mathcal{T}_2$


Optimal trajectory for *x*
But not for *y(τ)*!
x
$y(\tau)$
$\mathcal{T}_1$
$\mathcal{T}_2$
$\mathcal{T}_3$

Optimal trajectory for *x*

But not for *y(τ)*!



Need of memory

- We need a sort of memory!
- We have to keep in mind whether the *i*-th target is already visited or not.
- For every *i*, we need a positive scalar w_i, evolving in time, which is zero if and only if we have already reached the *i*-th target.
- Such memory variables must depend on the sequences of reached values only, and not on the time-scale.
- They must be rate-independent memory variables.
- They exhibit hysteresis.

$$u_i(t) = dist\left(y(t), \mathcal{T}_i\right), \qquad w_i(t) = \min_{\tau \in [0,t]}\left(dist\left(y(\tau), \mathcal{T}_i\right)\right) = \min_{\tau \in [0,t]} u_i(\tau)$$

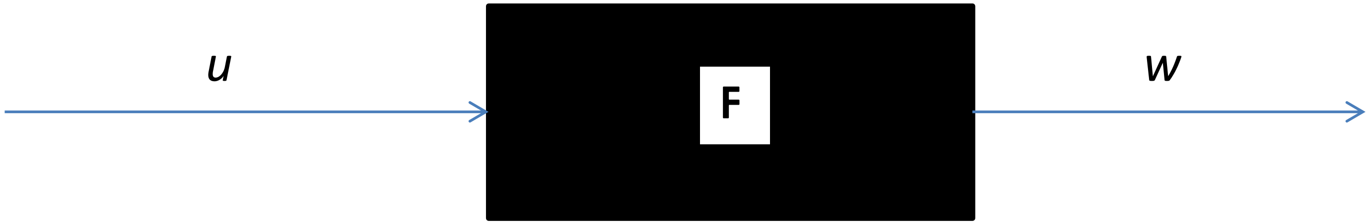
- Bellman ’62 (added variables for TSP and DPP)

Need of memory

- We need a sort of memory!
- We have to keep in mind whether the *i*-th target is already visited or not.
- For every *i*, we need a positive scalar w_i, evolving in time, which is zero if and only if we have already reached the *i*-th target.
- Such memory variables must depend on the sequences of reached values only, and not on the time-scale.
- They must be rate-independent memory variables.
- They exhibit hysteresis.

$$u_i(t) = dist\left(y(t), \mathcal{T}_i\right), \qquad w_i(t) = \min_{\tau \in [0,t]} \left(dist\left(y(\tau), \mathcal{T}_i\right)\right) = \min_{\tau \in [0,t]} u_i(\tau)$$

Hysteresis



$w(t) = \mathbf{F}[u(.)](t)$

In particular, the operator $\mathbf{F}$ is causal (i.e. $\mathbf{F}[u](t)$ depends only on $u_{|[0,t]}$)

it is not linear,

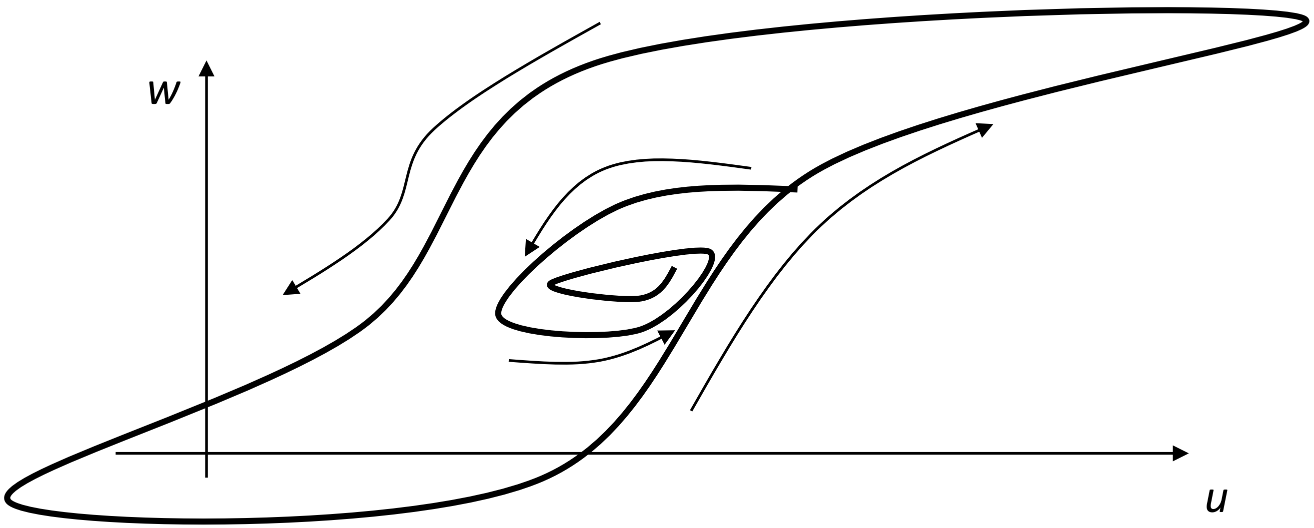
it is not differentiable,

$w(\cdot)$ may be discontinuous,

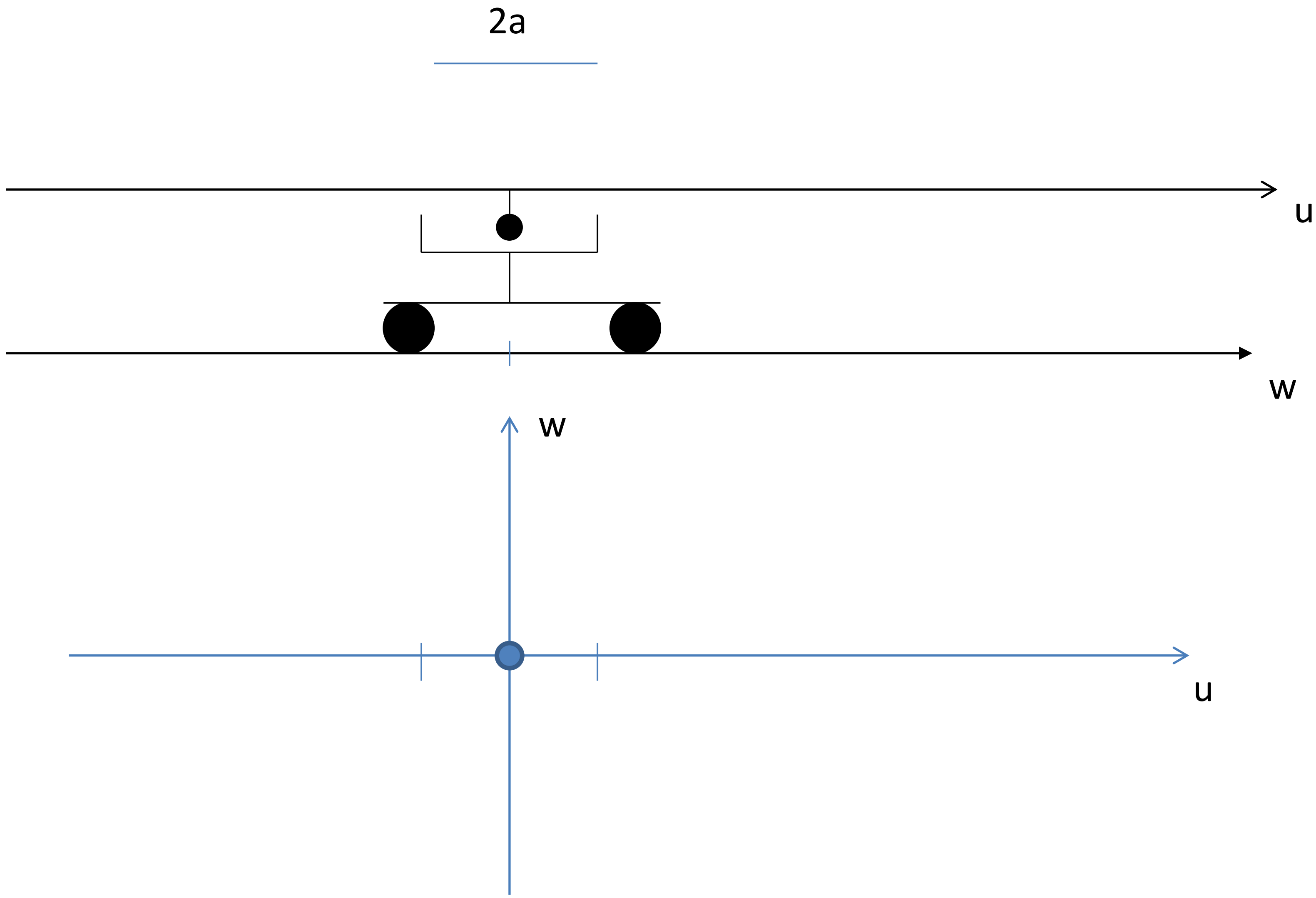
$\mathbf{F}[u \circ \varphi] = \mathbf{F}[u] \circ \varphi$, for any (positive) time-scaling φ (rate-independence),

and tipically the relationship $u \mapsto w = \mathbf{F}[u(\cdot)]$ is not a "differential" relationship

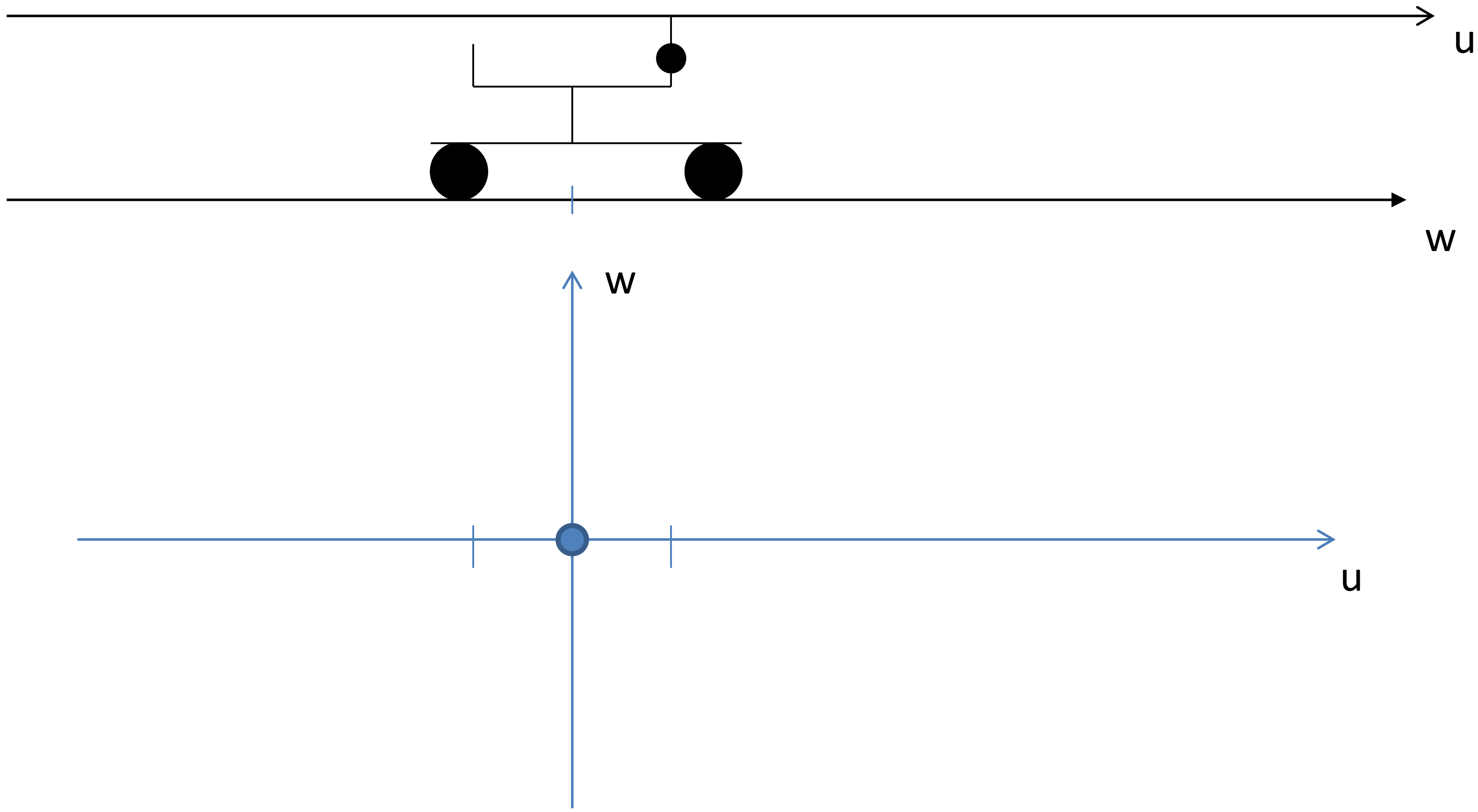
Hysteresis cycle



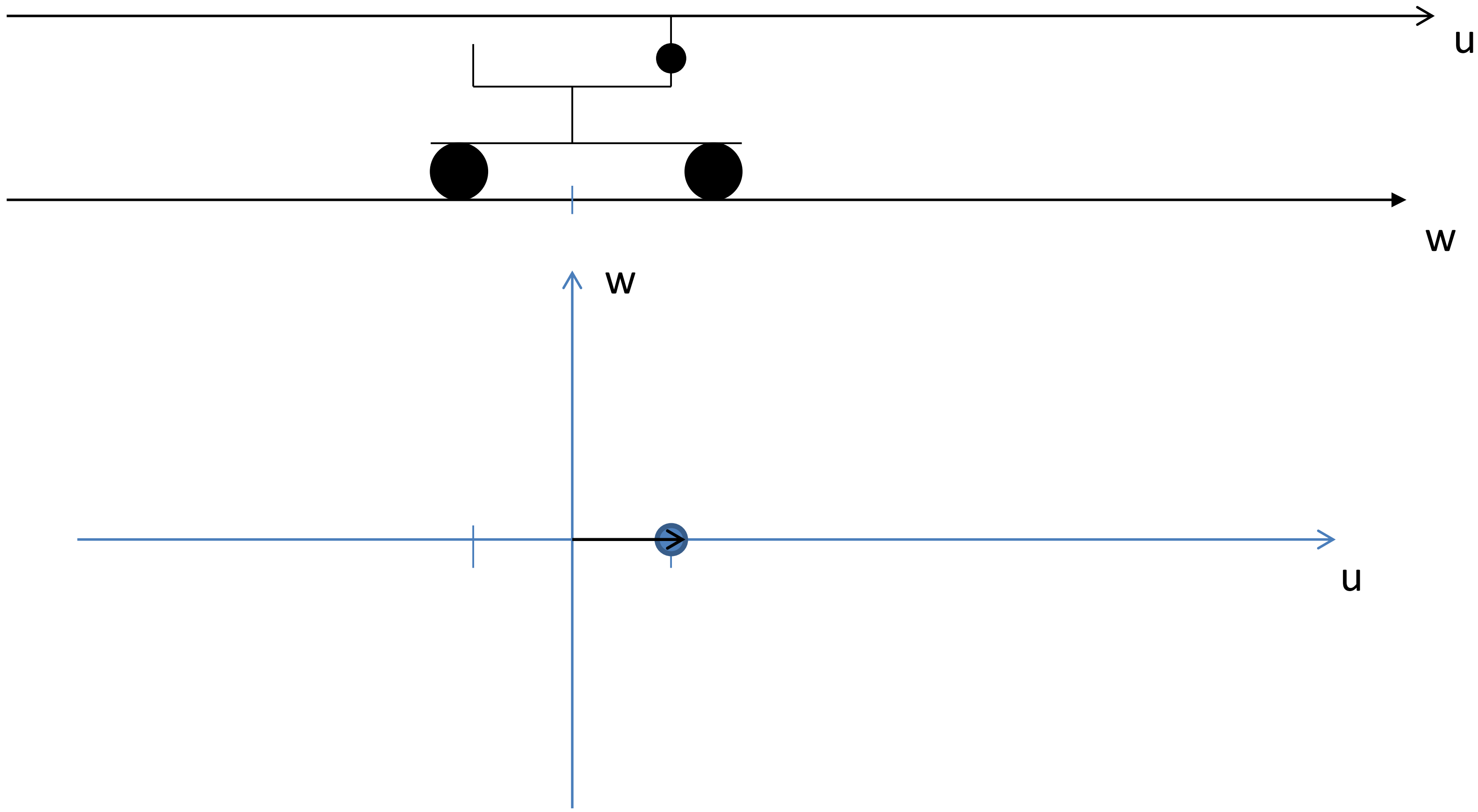
The Play operator



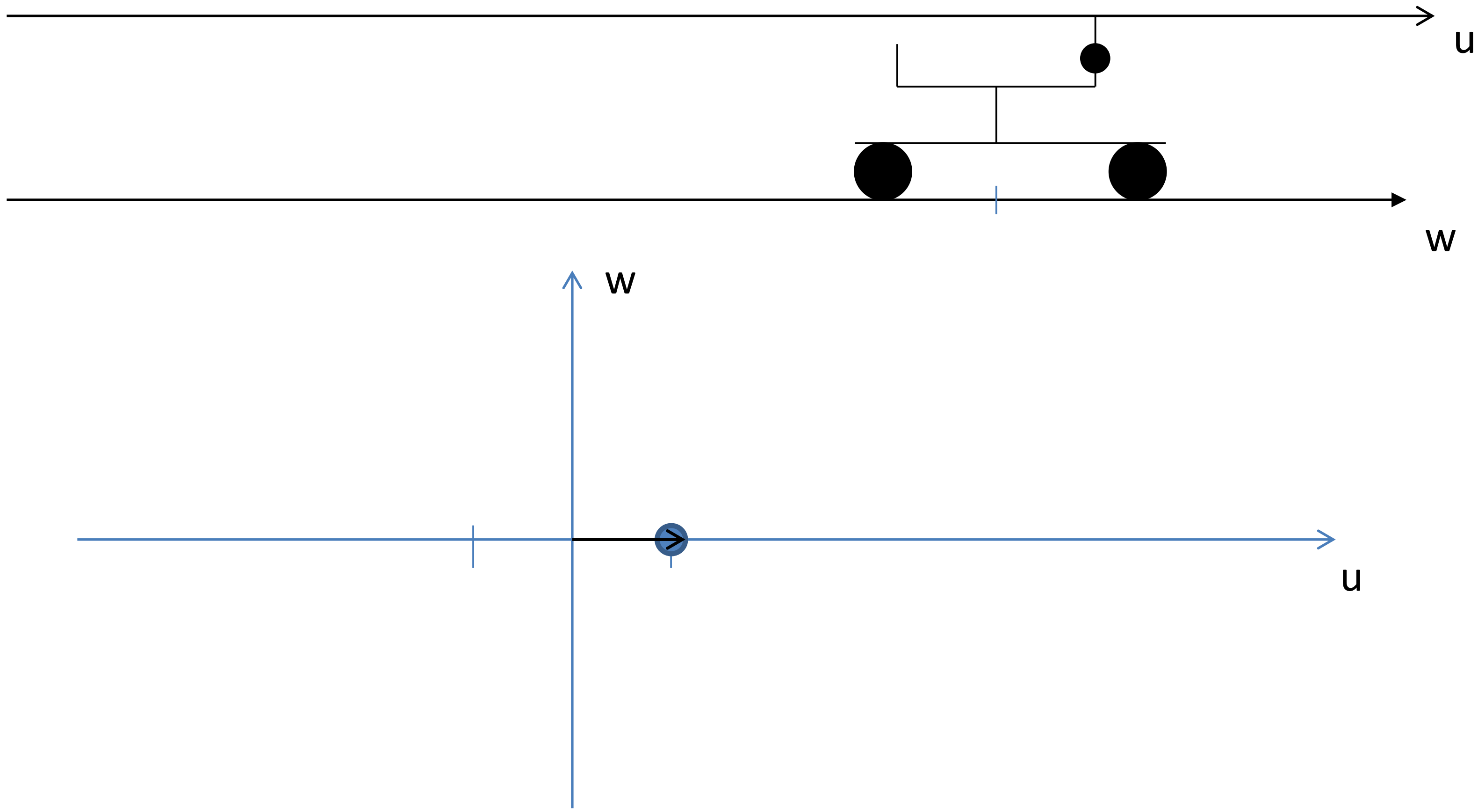
The Play operator



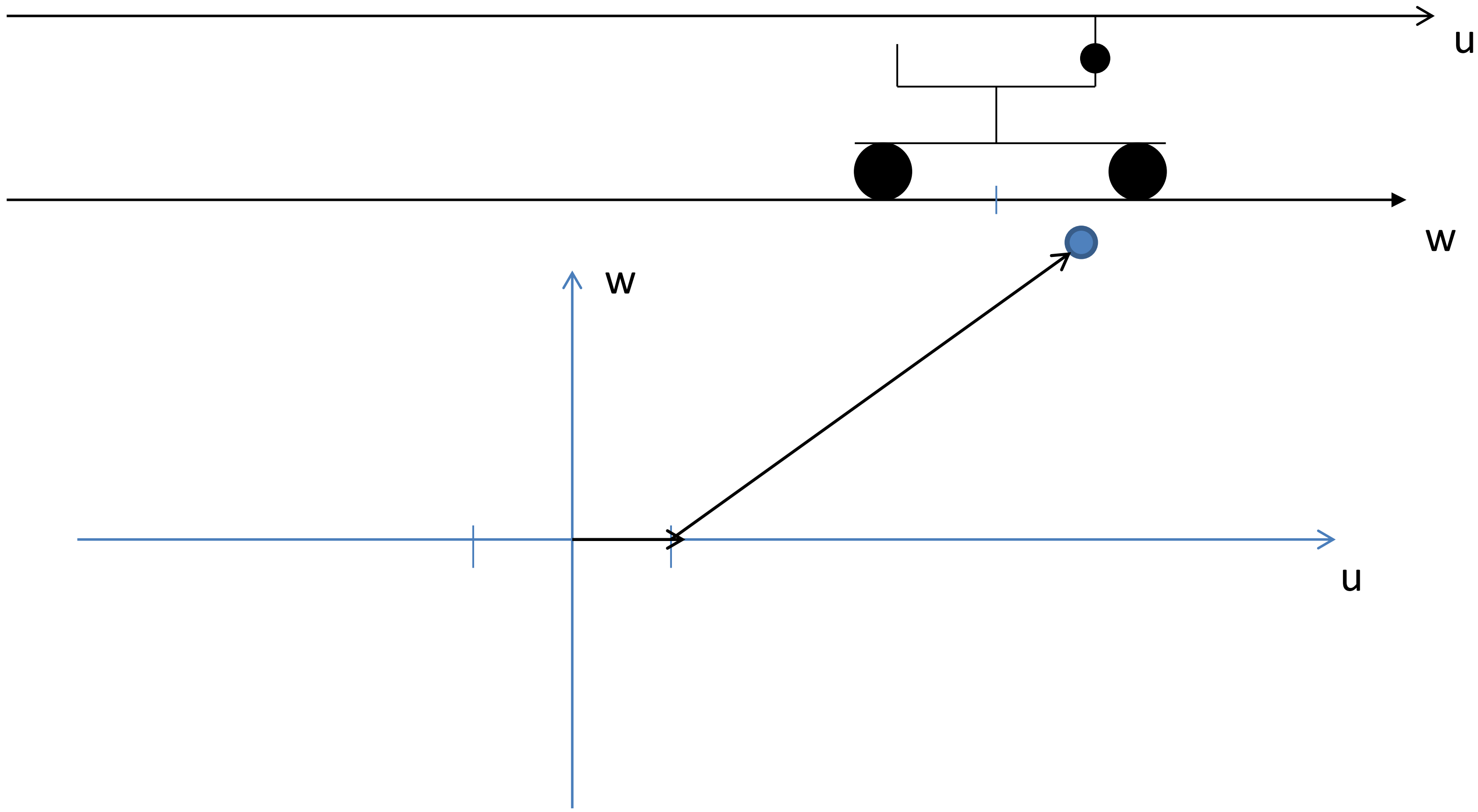
The Play operator



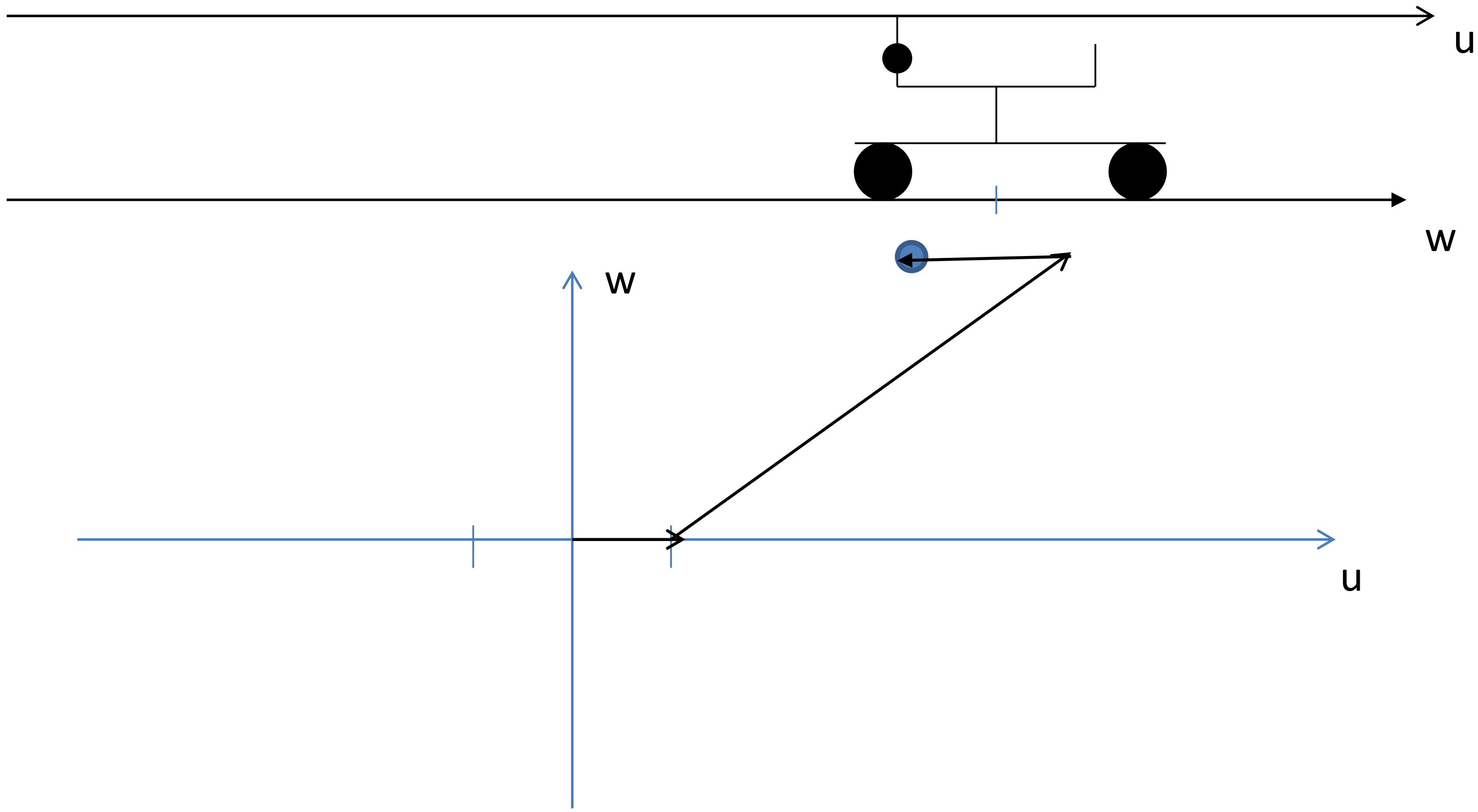
The Play operator



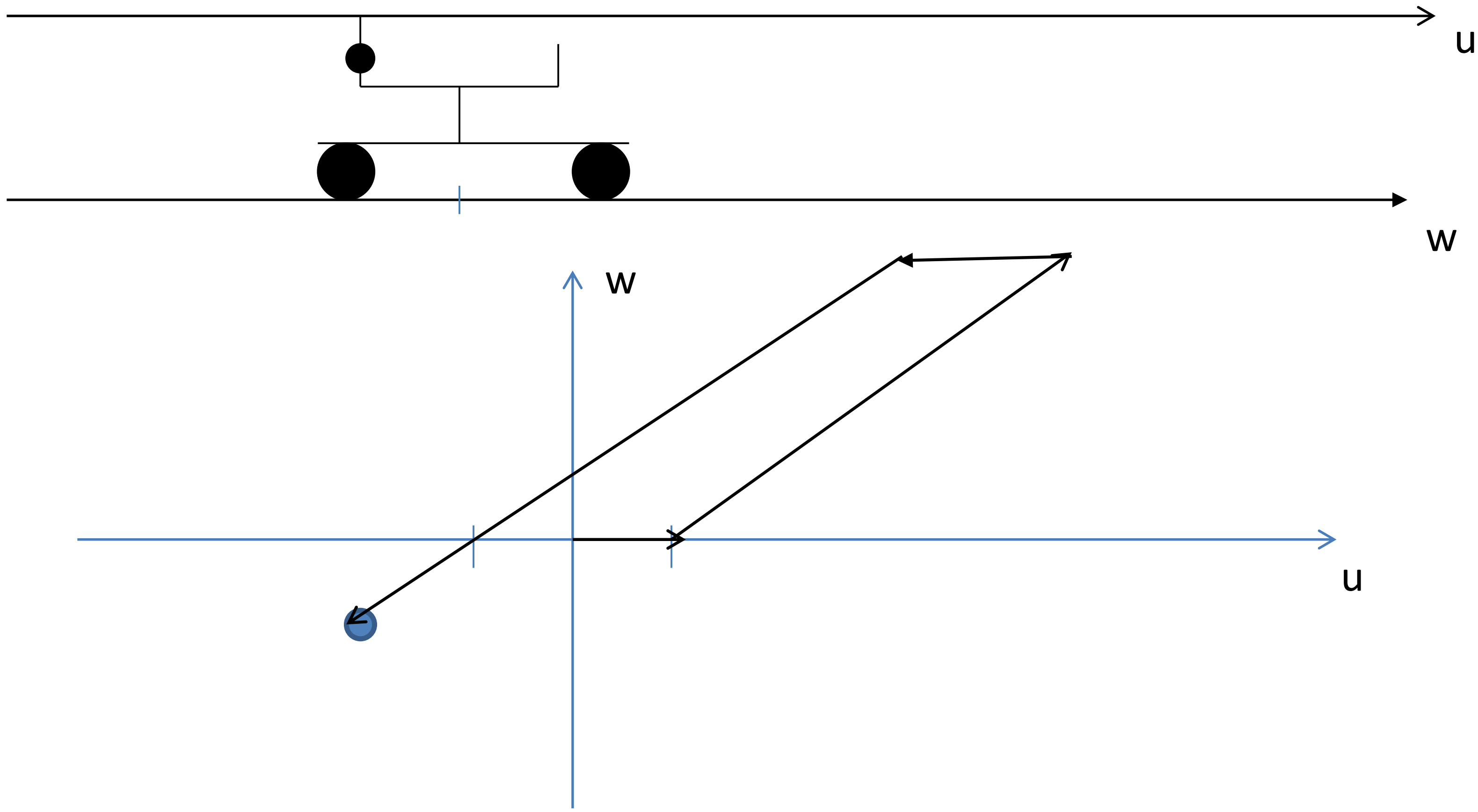
The Play operator



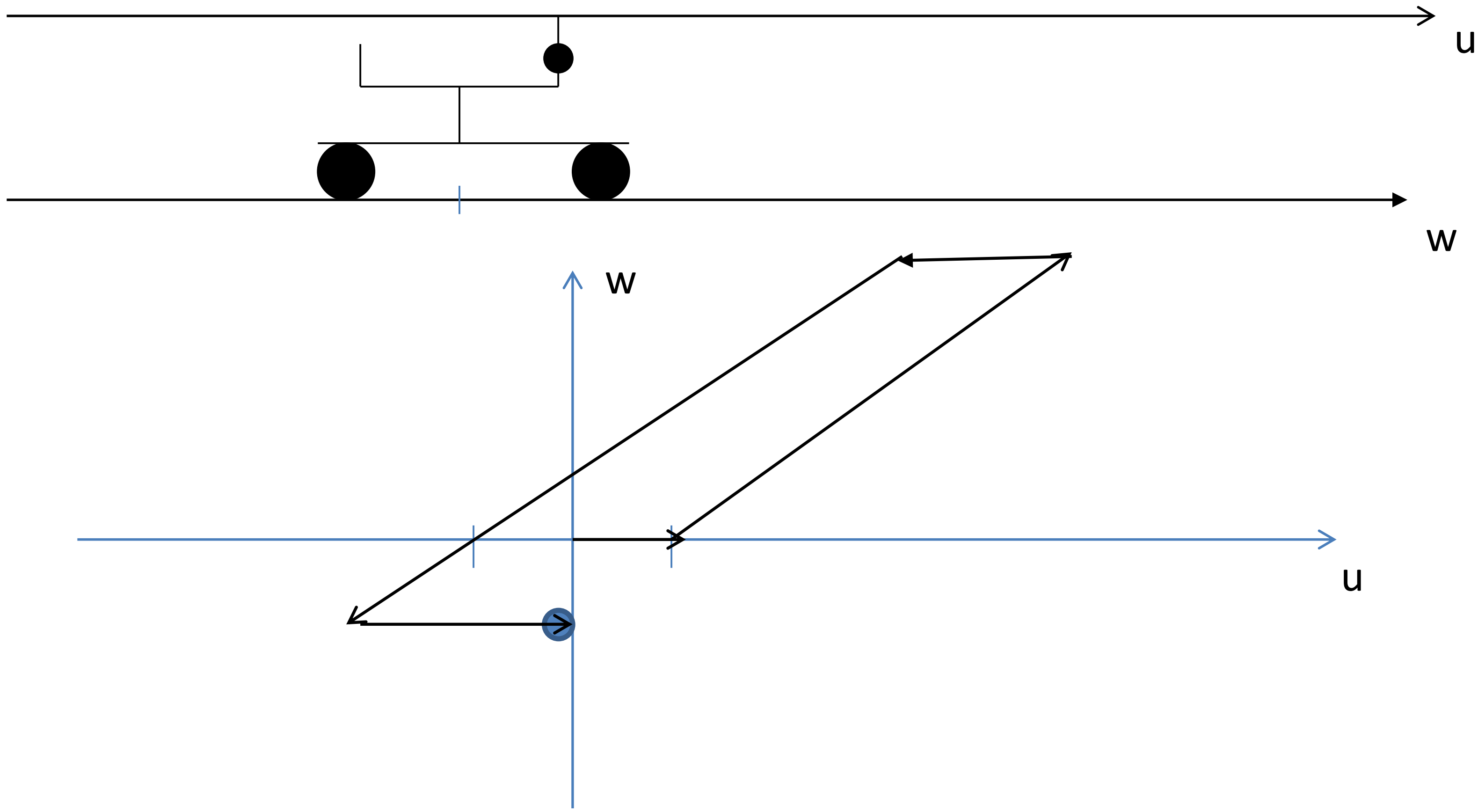
The Play operator



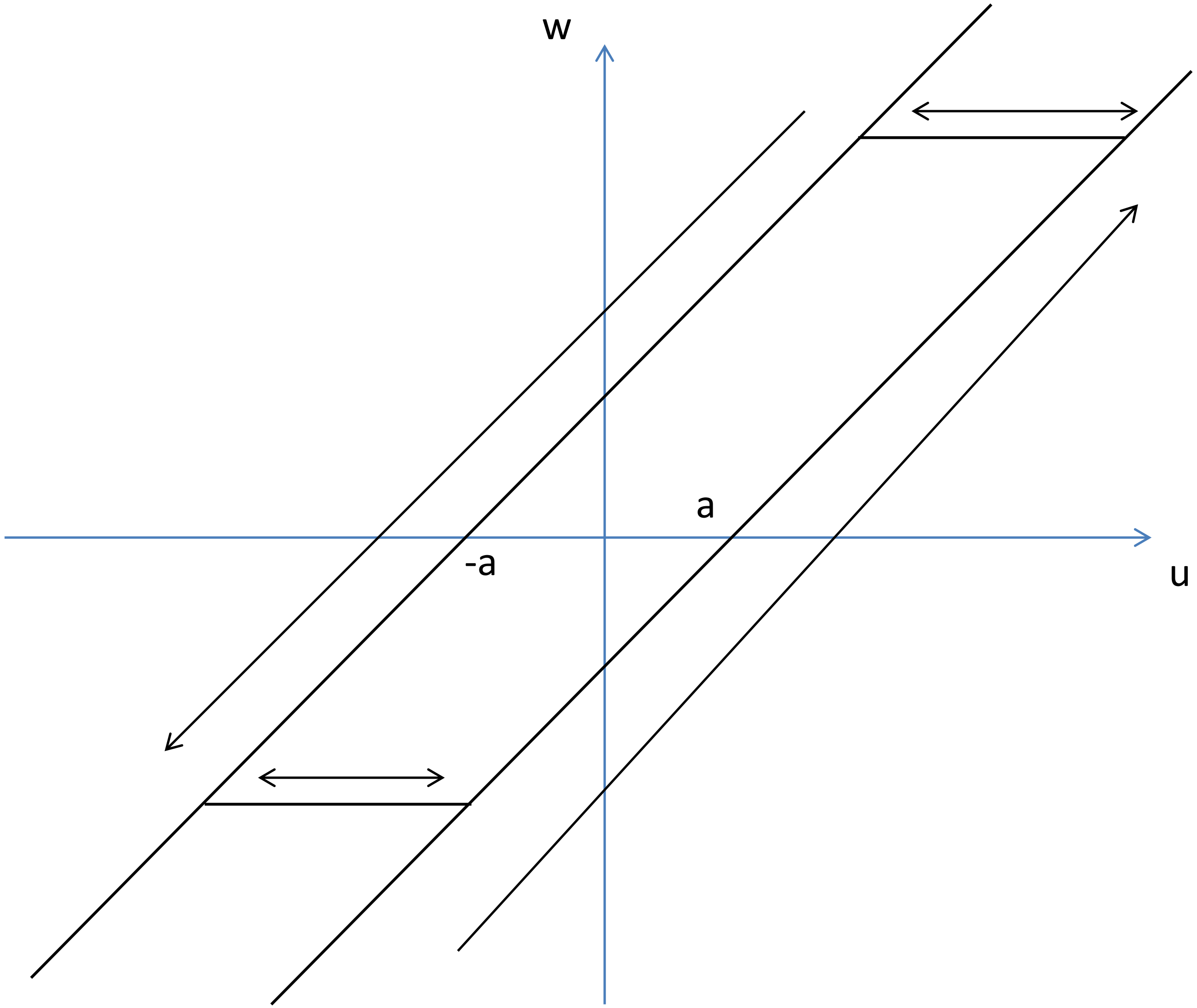
The Play operator



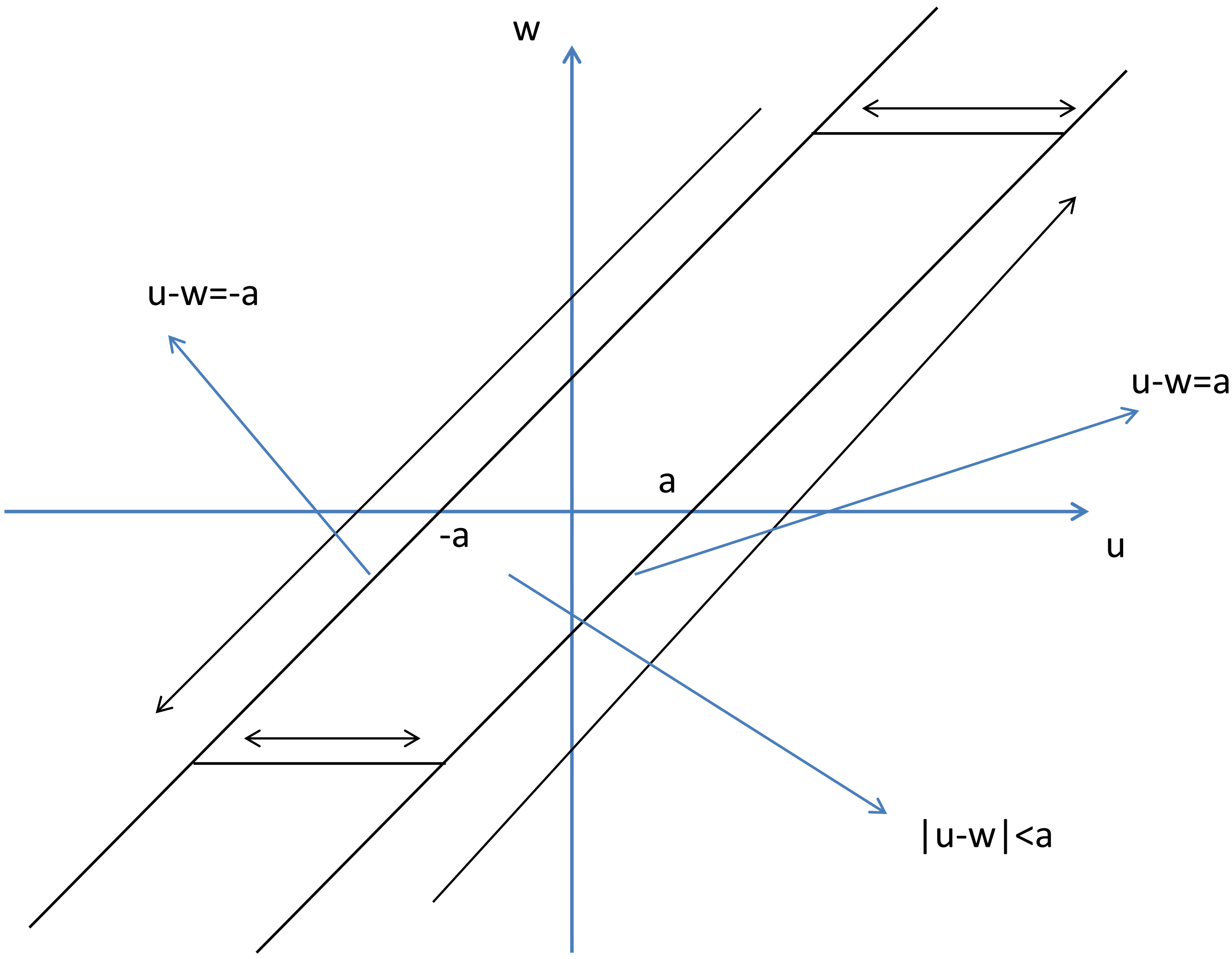
The Play operator



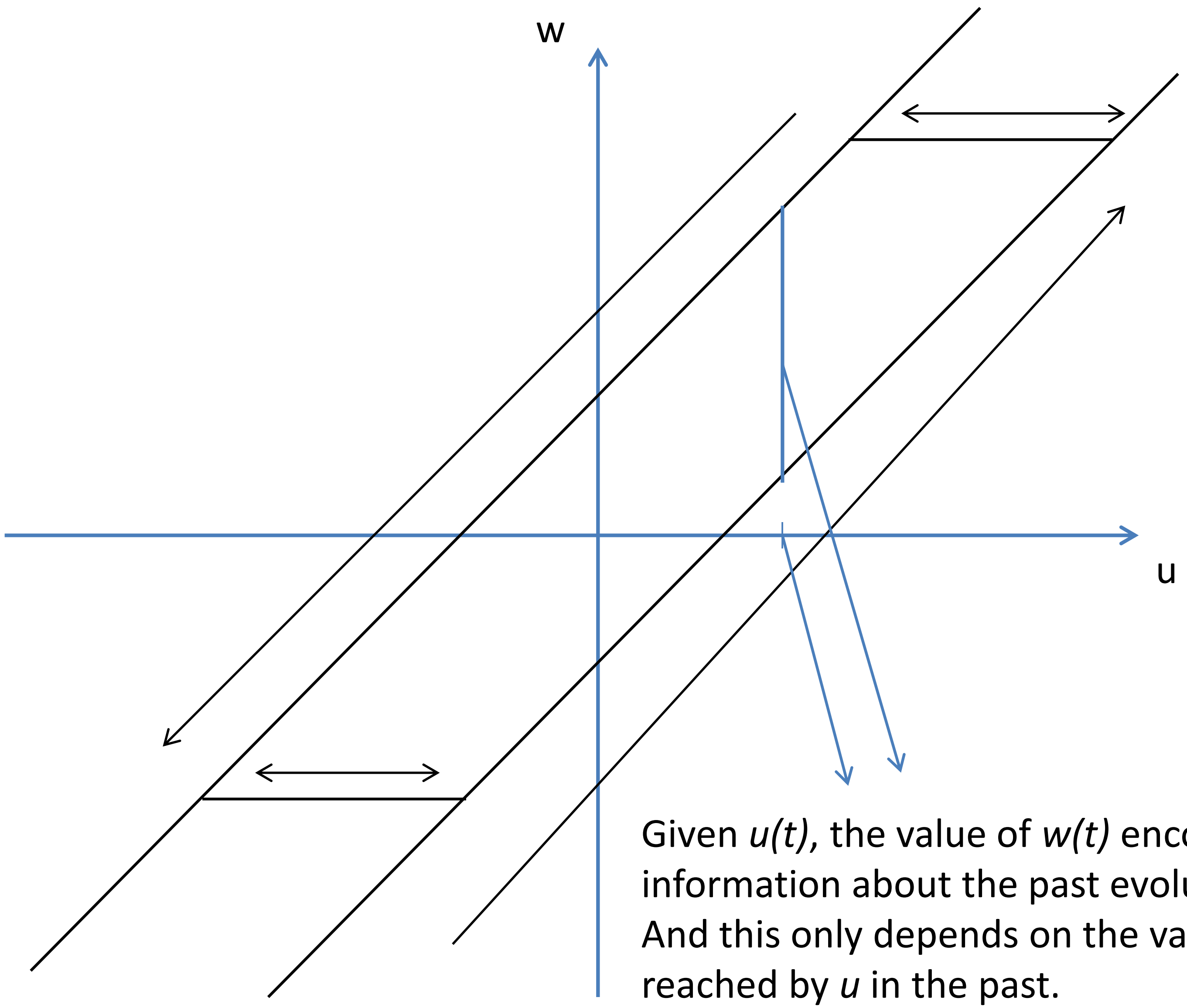
The Play operator



The Play operator



The Play operator



Given *u(t)*, the value of *w(t)* encodes the information about the past evolution of *u.* And this only depends on the values reached by *u* in the past.

The Play operator
(a one-dimensional sweeping process)

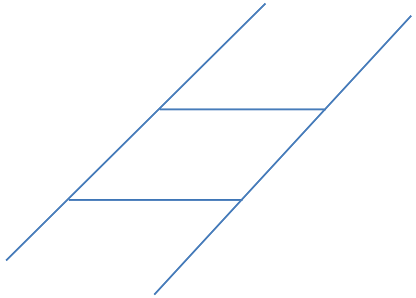
Variational inequality

$$\begin{cases} w'(t)\big(u(t)-w(t)-v\big)\geq 0 \quad \forall\,|v|\leq a \quad \text{for almost every } t \\ |u(t)-w(t)|\leq a \quad \forall t \end{cases}$$

$$w'(t)\in \partial I_{[-a,a]}(u(t)-w(t)) \quad \text{a.e.}\, t$$

Discontinuous ODE

$$w'(t)=\chi_r(u(t),w(t))\big(u'(t)\big)^{+}-\chi_l(u(t),w(t))\big(u'(t)\big)^{-} \quad \text{for almost every } t$$



$$\chi_r(u,w)=\begin{cases}1 & \text{if } u-w=a \\ 0 & \text{otherwise}\end{cases}$$

$$\big(u'(t)\big)^{+}=\max\{u'(t),0\}$$

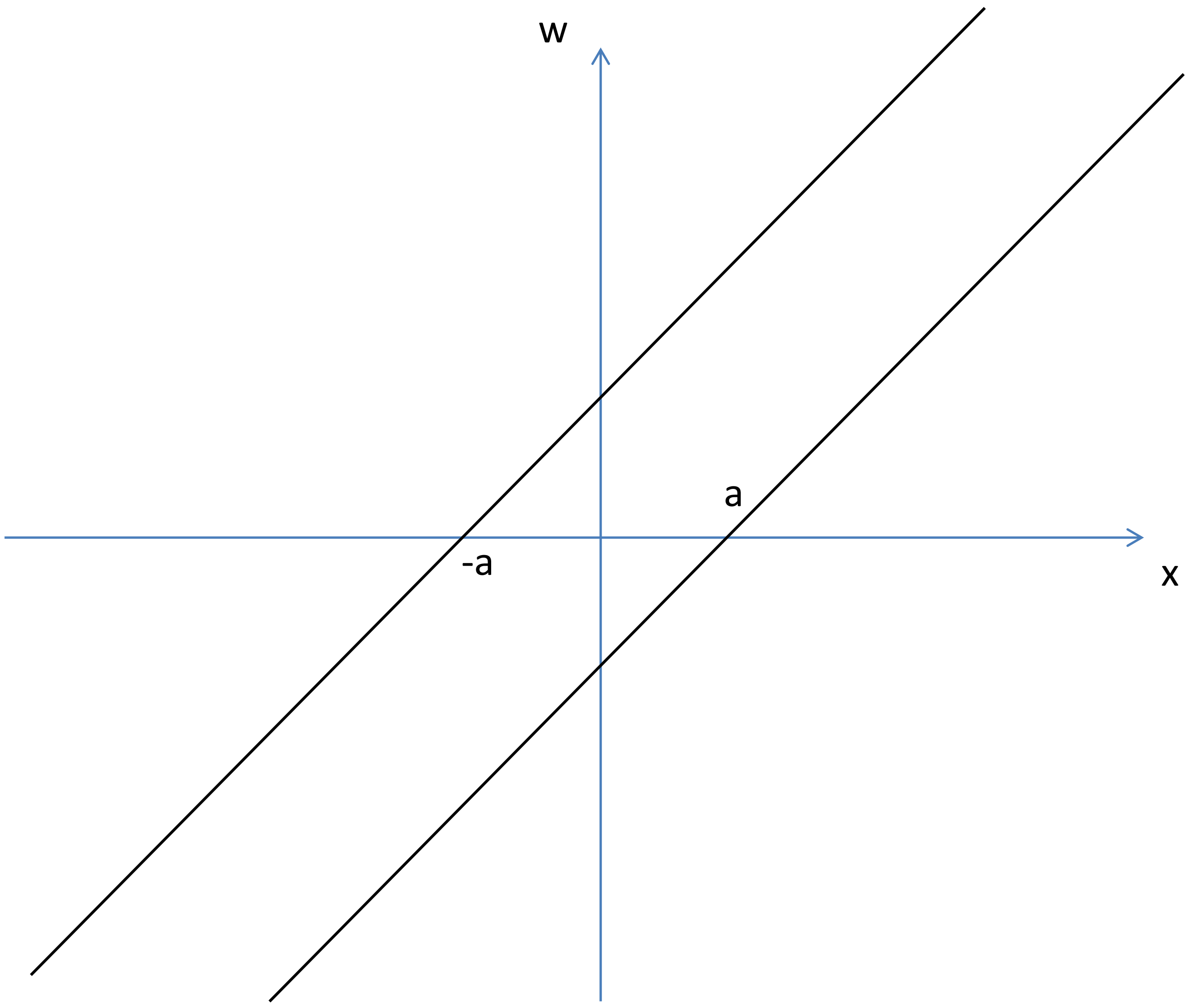
$$\big(u'(t)\big)^{-}=\max\{-u'(t),0\}$$

The Play operator (a one-dimensional sweeping process)

- Reflecting (absorbing) boundary
- Skorokhod problem
- Sweeping process

Optimal control with hysteresis

$$\begin{cases} y' = f(y, w, \alpha) \\ w = \wp[y] \\ (y(0), w(0)) = (x, w^0) \in \overline{\Omega_a} \subset \mathbf{R}^2 \end{cases}$$


w
a
-a
x

Optimal control with hysteresis

$$\begin{cases} y' = f(y, w, \alpha) \\ w = \wp[y] \\ (y(0), w(0)) = (x, w^0) \in \overline{\Omega_a} \subset \mathbf{R}^2 \end{cases}$$

$$J(x, w^0, \alpha) = \int_0^{+\infty} e^{-\lambda t} \ell(y(t), w(t), \alpha(t)) dt$$

$$V(x, w^0) = \inf_{\alpha} J(x, w^0, \alpha)$$

Optimal control with hysteresis

$$\lambda V + H(x, w, V_x, V_w) = 0 \text{ in } \overline{\Omega_a}$$

Discontinuous ODE

$$w'(t) = \chi_r(u(t), w(t))\big(u'(t)\big)^+ - \chi_l(u(t), w(t))\big(u'(t)\big)^- \quad \text{for almost every } t$$

$$H(x, w, p, q) = \sup_a \begin{Bmatrix} -pf(x, w, a) \\ -q\big(\chi_r(x, w) f^+(x, w, a) - \chi_l(x, w) f^-(x, w, a)\big) \\ -\ell(x, w, a) \end{Bmatrix}$$

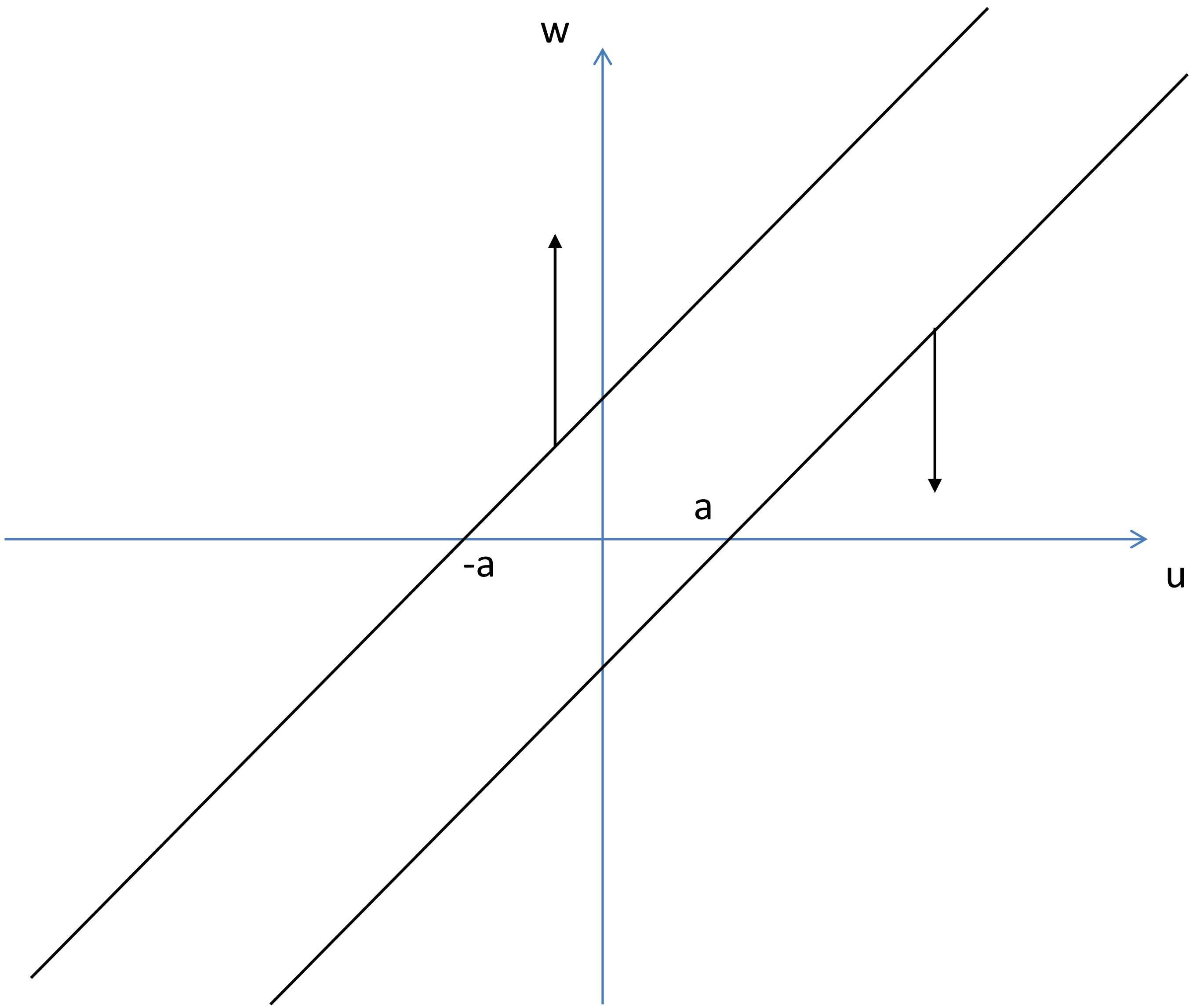
Optimal control with hysteresis

$$\lambda V + H^*(x, w, V_x, V_w) \geq 0 \quad \text{in } \overline{\Omega_a}$$

$$\lambda V + H_*(x, w, V_x, V_w) \leq 0 \quad \text{in } \overline{\Omega_a}$$

$$H^*(x, w, p, q) = \sup_a \left\{ \begin{array}{l} -pf(x, w, a) \\ +q^- \chi_r(x, w) f^+(x, w, a) + q^+ \chi_l(x, w) f^-(x, w, a) \\ -\ell(x, w, a) \end{array} \right\}$$

$$\begin{cases} \lambda V + H(x, w, V_x, 0) = 0 \text{ in } \Omega_a \\ \text{Neumann boundary conditions on } \partial\Omega_a \end{cases}$$



w
a
-a
u

Characterization

- The value function *V* is the unique continuous viscosity solution of the Hamilton-Jacobi problem.
- F. B.: Dynamic Programming for some optimal control problems with hysteresis, NODEA, 2002

Back to optimal visiting

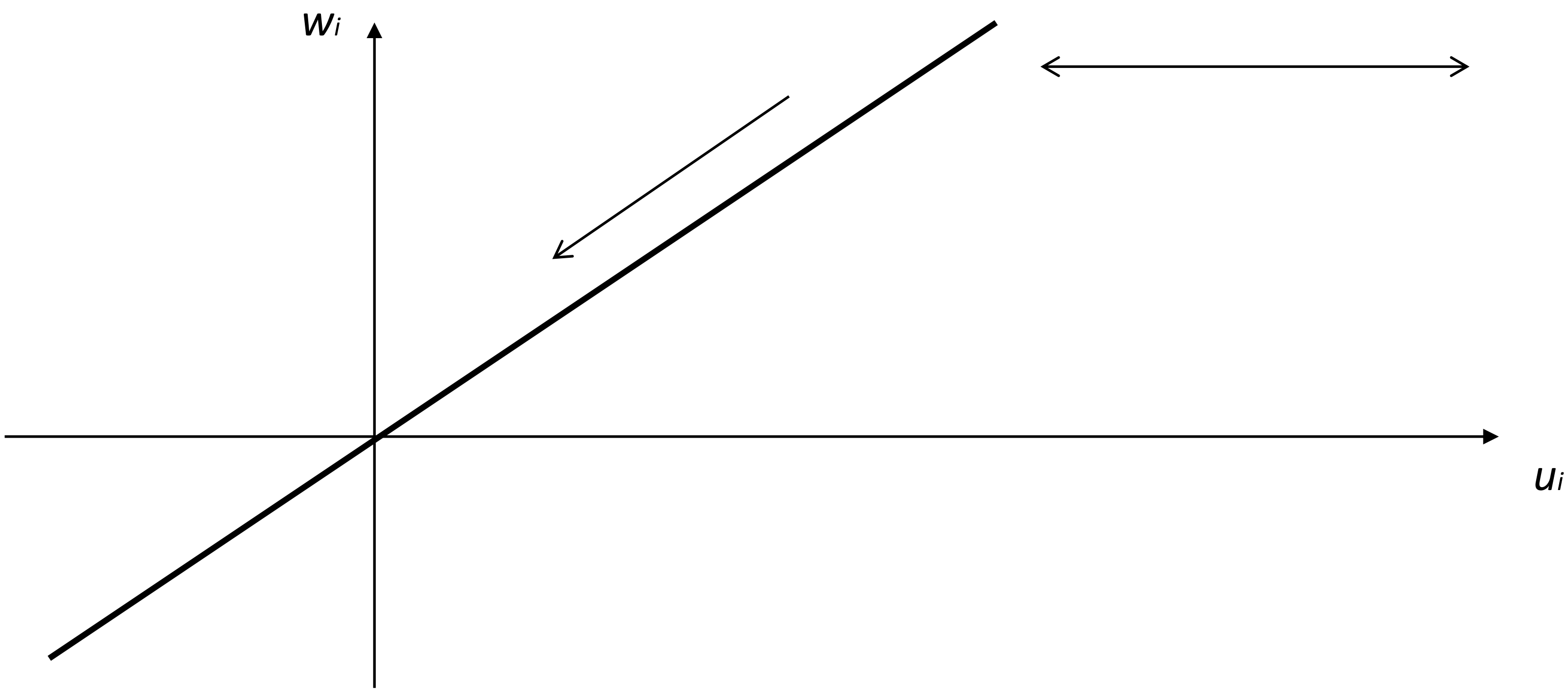
Suppose that we have m targets to visit, $\mathcal{T}_i \subset \mathbf{R}^n$.

$$u_i(t) = dist\left(y(t), \mathcal{T}_i\right), \qquad w_i(t) = \min_{\tau\in[0,t]}\left(dist\left(y(\tau), \mathcal{T}_i\right)\right) = \min_{\tau\in[0,t]} u_i(\tau)$$

The new state variable is $(x, w) = (x, w_1, \ldots, w_m) \in \mathbf{R}^n \times \mathbf{R}^m$

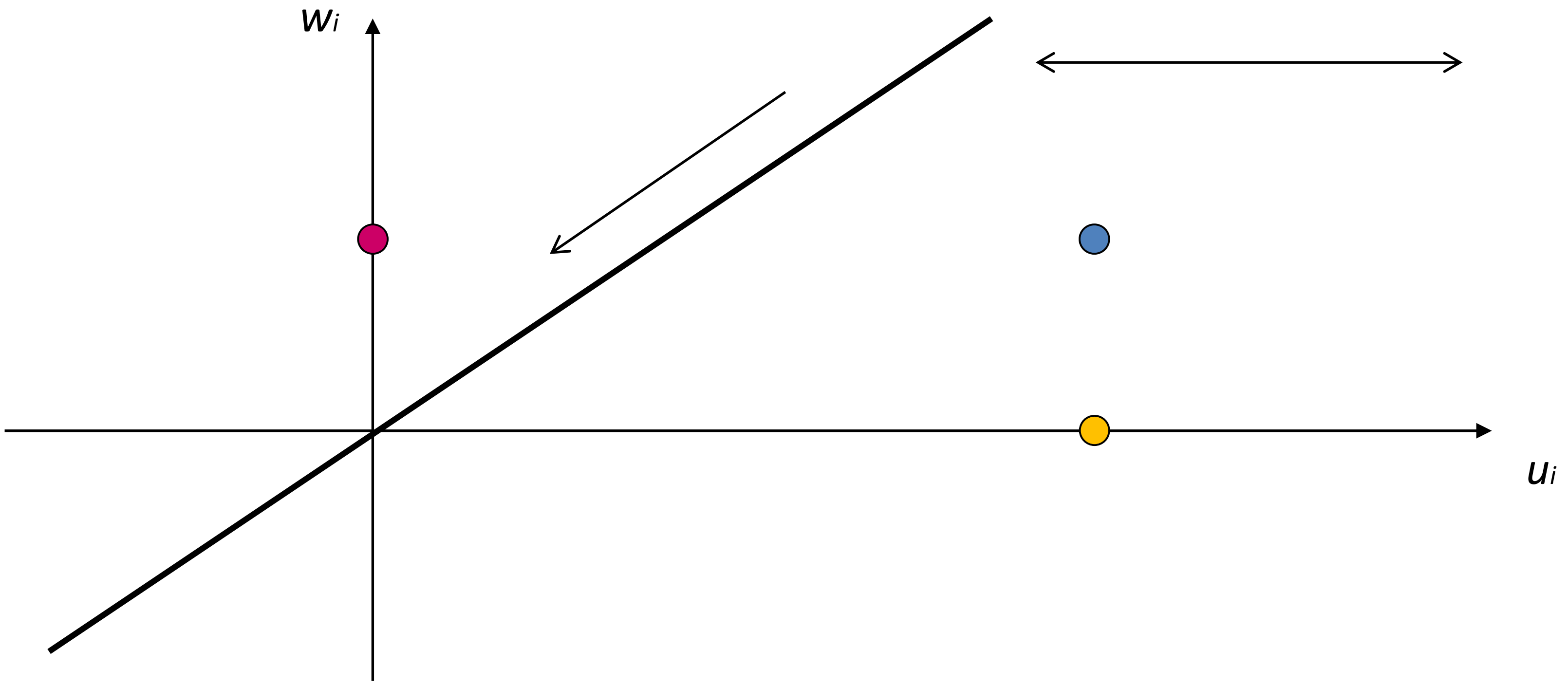
Back to optimal visiting

$$u_i(t) = dist\left(y(t), \mathcal{T}_i\right), \qquad w_i(t) = \min_{\tau\in[0,t]}\left(dist\left(y(\tau), \mathcal{T}_i\right)\right) = \min_{\tau\in[0,t]} u_i(\tau)$$



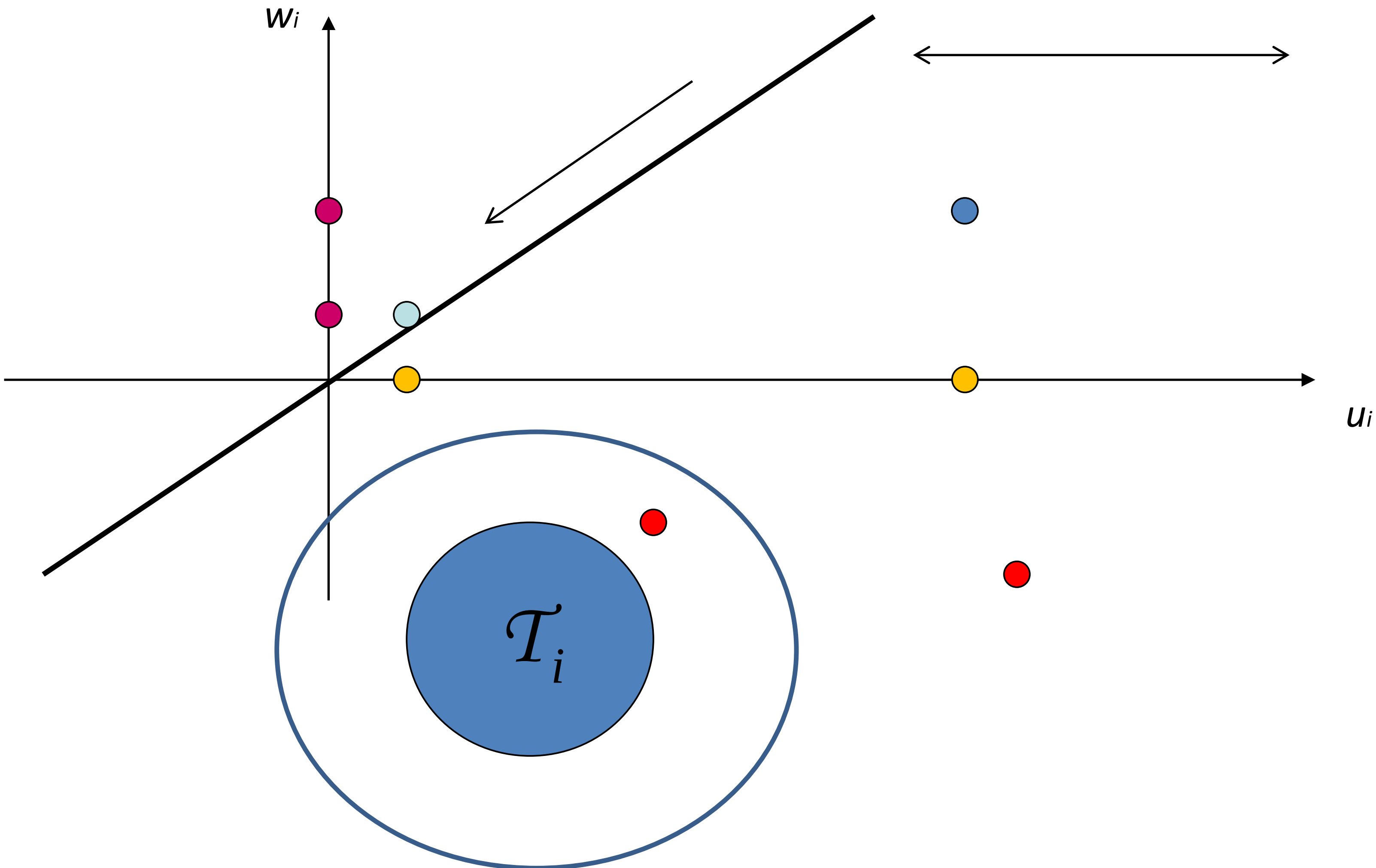
Back to optimal visiting

$$u_i(t) = dist\left(y(t), \mathcal{T}_i\right), \qquad w_i(t) = \min_{\tau \in [0,t]} \left(dist\left(y(\tau), \mathcal{T}_i\right)\right) = \min_{\tau \in [0,t]} u_i(\tau)$$



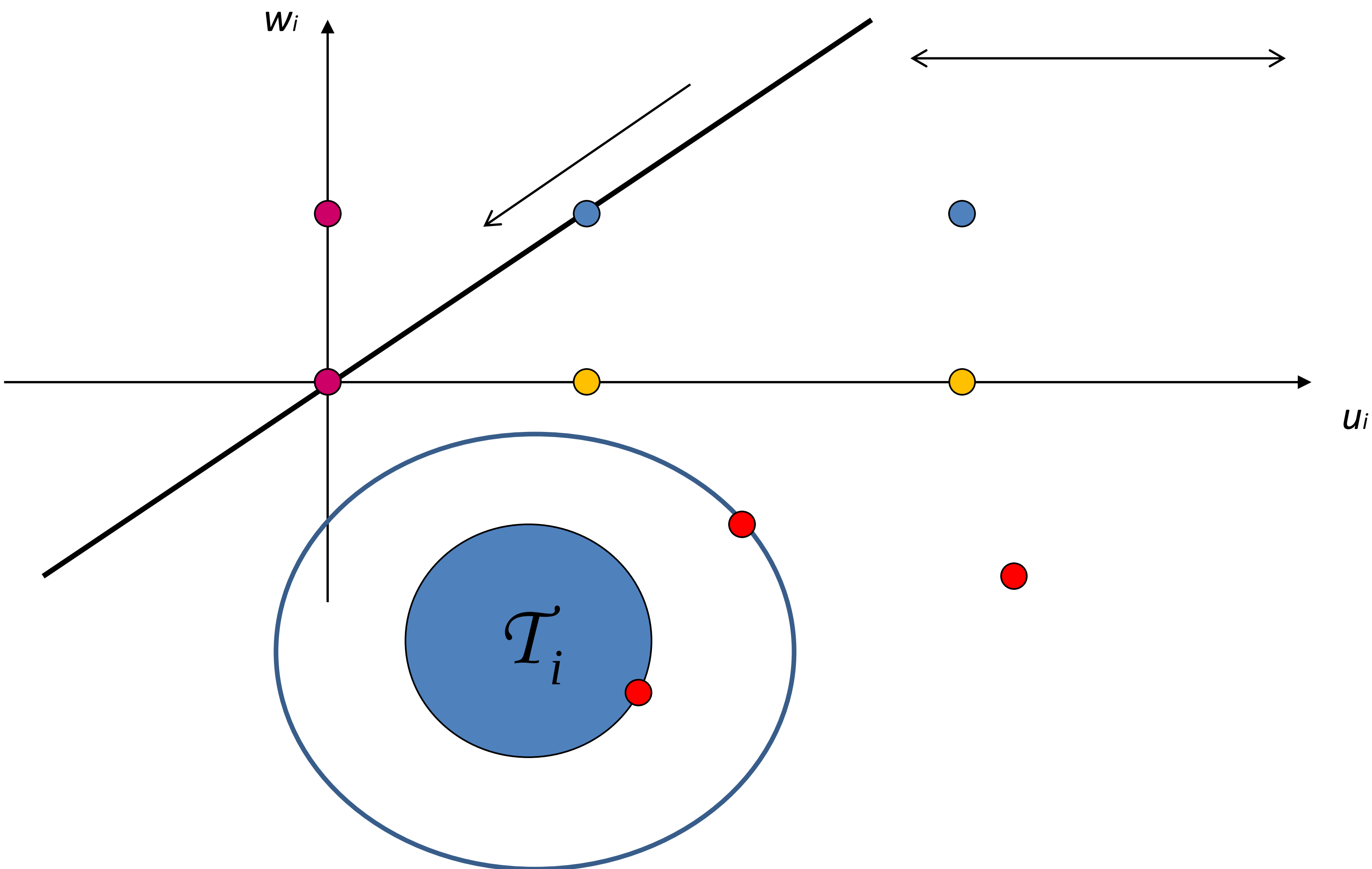
Back to optimal visiting

$$u_i(t) = dist\left(y(t), \mathcal{T}_i\right), \qquad w_i(t) = \min_{\tau \in [0,t]} \left(dist\left(y(\tau), \mathcal{T}_i\right)\right) = \min_{\tau \in [0,t]} u_i(\tau)$$



Back to optimal visiting

$$u_i(t) = dist\left(y(t), \mathcal{T}_i\right), \qquad w_i(t) = \min_{\tau\in[0,t]}\left(dist\left(y(\tau), \mathcal{T}_i\right)\right) = \min_{\tau\in[0,t]} u_i(\tau)$$



The optimal visiting problem

The distance function is not regular in $\mathbf{R}^n$.

$\forall j = 1,\ldots m,$ let $g_j : \mathbf{R}^n \to \mathbf{R}$ be such that

$g_j \in C^1(\mathbf{R}^n, \mathbf{R}) \cap Lip(\mathbf{R}^n, \mathbf{R}),\ g_j \geq 0,\ g_j(x) = 0 \Leftrightarrow x \in \mathcal{T}_j,$

We consider the controlled system with hysteresis

in the new extended variable $(y, w) \in \mathbf{R}^{n+m}$

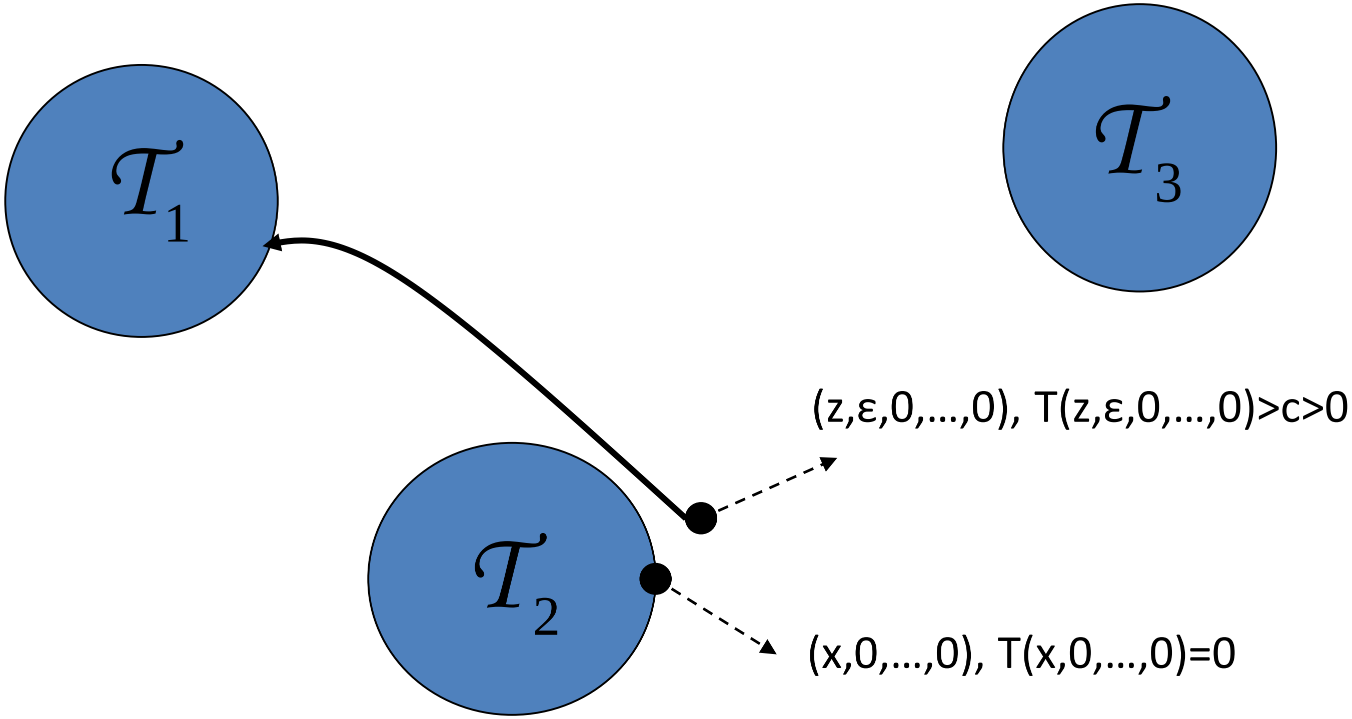
$$\begin{cases} y'(t) = f(y(t), \alpha(t)) \\ w_j(t) = \mathcal{SP}[g_j \circ y](t), \quad j = 1,\ldots,m \\ y(0) = x,\ w_1(0) = w_1^0, \ldots, w_m(t) = w_m^0 \end{cases}$$

$T(x, w_1, \ldots, w_m)$ minimum time to reach

$$\mathcal{T} = \left\{ (x, w_1, \ldots, w_m) \in \mathbf{R}^{\mathrm{n+m}} \middle| w_j = 0 \forall j = 1, \ldots, m \right\}$$

DPP holds but *T is not continuous*

- *T* is not continuous on the boundaries of the targets!



A Mayer problem

We then consider the value function for a Mayer problem

$$V(x, w_1^0, \ldots, w_m^0, t) = \inf_{\alpha} \left(w_1(t) + \cdots + w_m(t) \right)$$

$$T(x) = \inf \left\{ t \geq 0 \middle| V(x, w^0, t) = 0 \right\}$$

Knowing V, we can get informations on the optimal visiting function T.

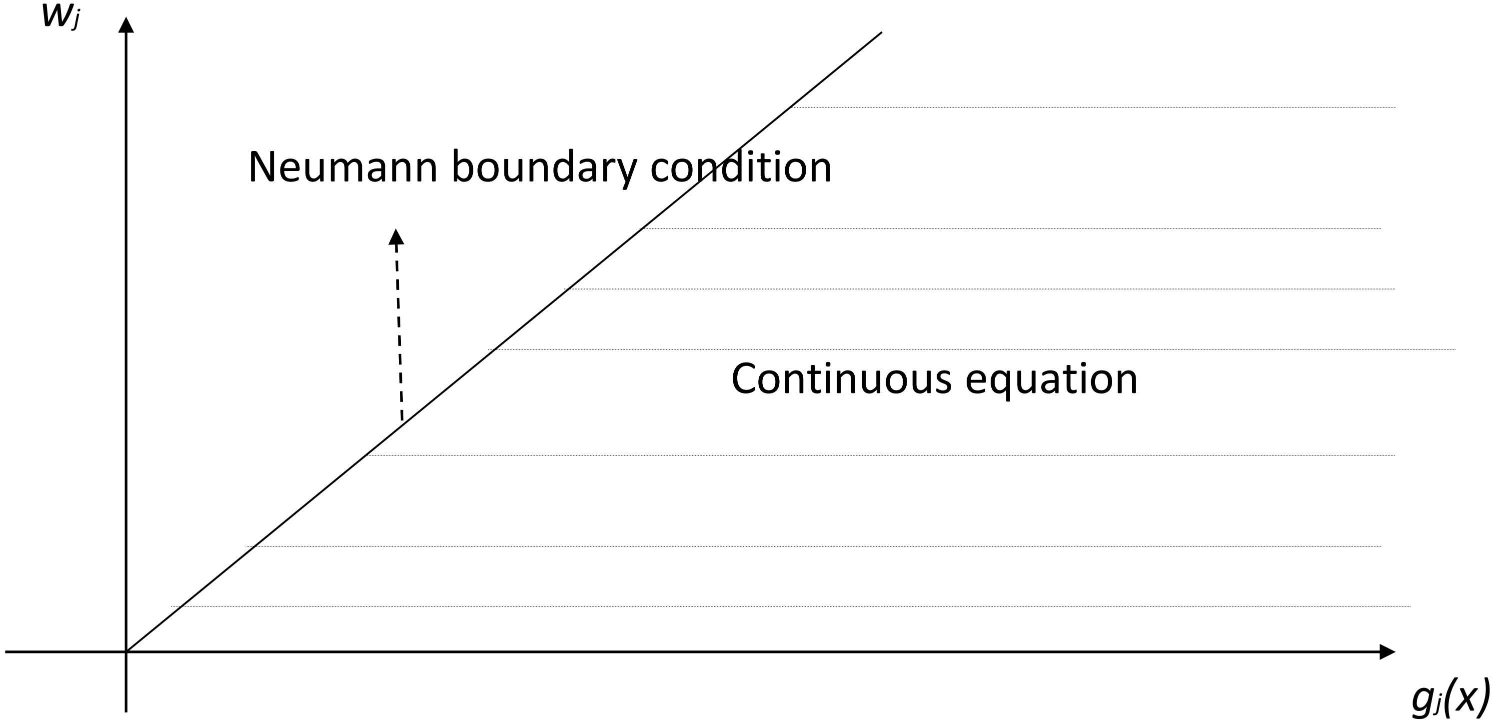
A Mayer problem

- The value function *V* is continuous.
- DPP holds.
- *V* is the unique continuous viscosity solution of

$$\begin{cases} V_t(x,w,t) + \sup_a \left\{ - f(x,a)\cdot\nabla_x V(x,w,t) + \sum_{j=1}^{m} \dfrac{\partial V(x,w,t)}{\partial w_j} \chi(g_j(x),w_j)\big(\nabla g_j(x)\cdot f(x,a)\big)^- \right\} = 0 \\ V(x,w,0) = w_1 + w_2 + \cdots + w_m \end{cases}$$

F. B.-M. Benetton: About an optimal visting problem, Appl.Math.Optim, 2012

‘splitting the equation on $(g_j(x), w_j)$-planes’



Switching memory

- Up to now the added memory variables were continuous in time.
- This is good, of course.
- However, we can also consider switching memory variables.
- Every memory variable w_i is a time dependent 'label' taking value 0 and 1
- '1' means: target T_i not reached yet
- '0' means: target T_i already reached

Control of tourists flow

- The problem I am going to present takes inspiration from the problem of governing the flow of tourists inside the historical center of a heritage art city.
- F. B.-R. Pesenti: Non-memoryless pedestrian flow in a crowded environment with target sets, to appear on Annals of ISDG vol 15
- Here I present a possible model for flow of excursionists (daily tourists that arrive in the city in the morning and go away in the evening).

Control of excursionists flow

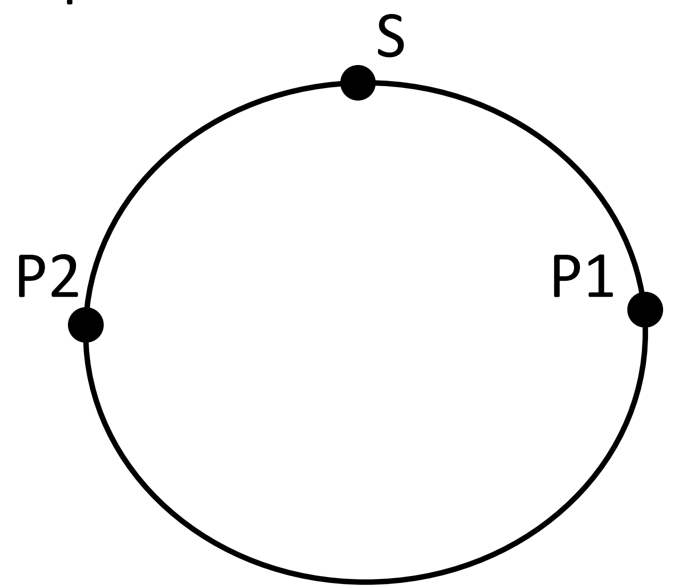
- First of all excursionists have only two main attractions they want to visit.
- The two attractions are not necessarily of the same interest: a main attraction P1 and a minor attraction P2.
- The excursionists arrive at the train station during a fixed interval of time.
- They may decide to first visit attraction P1 and then attraction P2 or vice-versa. This choice may, for example, depend on the crowdedness and on the expected waiting time.
- They have to return back to the station at the fixed time T.

Some features of the model

- (Memory) Excursionists may occupy at the same instant the same place in the path but they may have different purposes: someone has already visited P1 only, someone else P2 only, someone both, someone else nothing.
- At the initial time they all have the same purposes.
- During the day they split into several "populations" with different purposes.
- And possibly they eventually recover into the same population.
- Excursionists in the same point at the same instant may have "different past histories".

The model

- We describe the path of excursionists inside the city as a circular graph with three identified points:

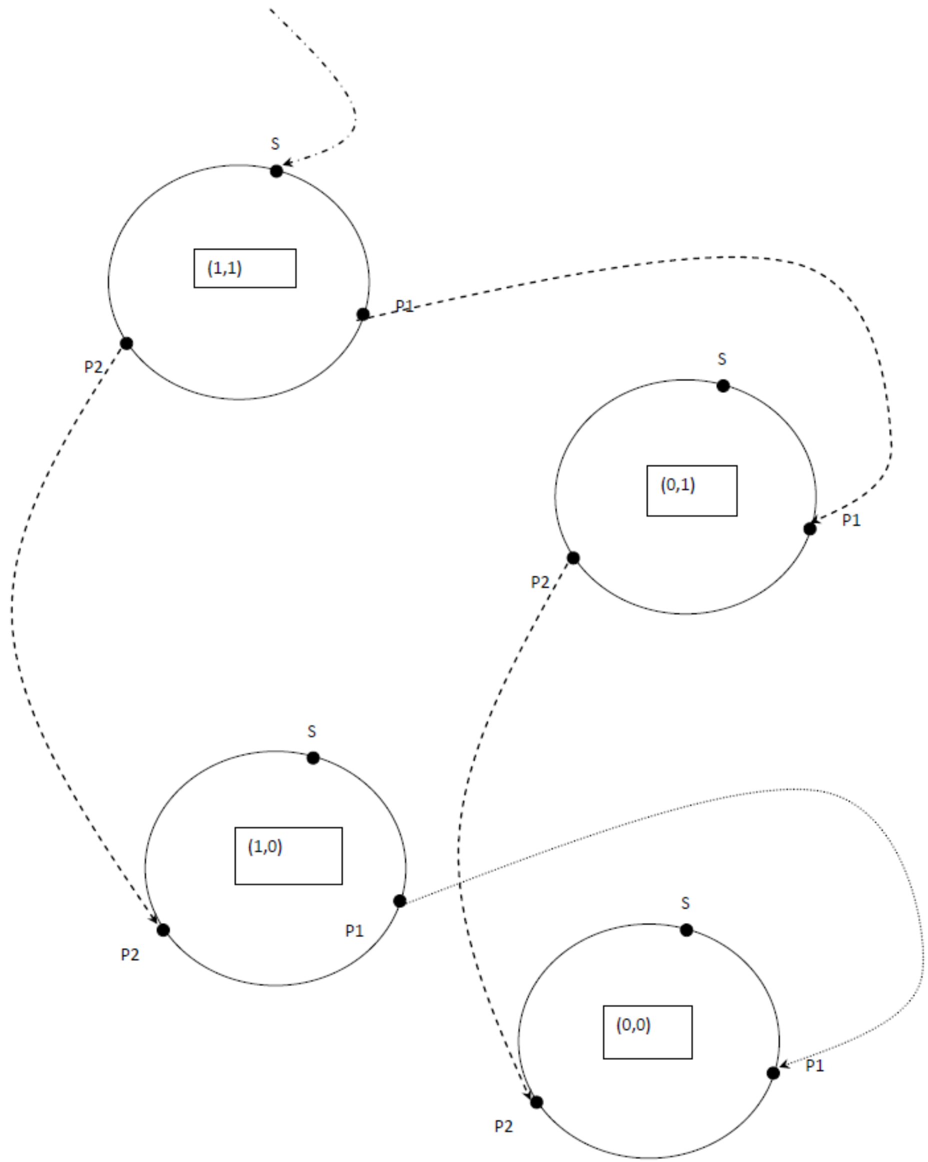


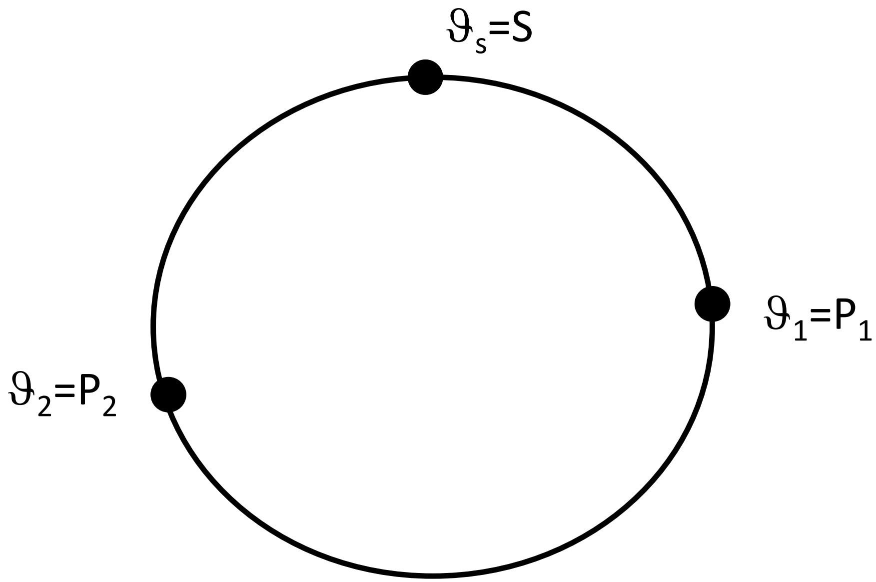
- S the station
- P1 the attraction 1
- P2 the attraction 2

- The position of an excursionist is given by the parameter $\vartheta \in [0, 2\pi]$

Memory

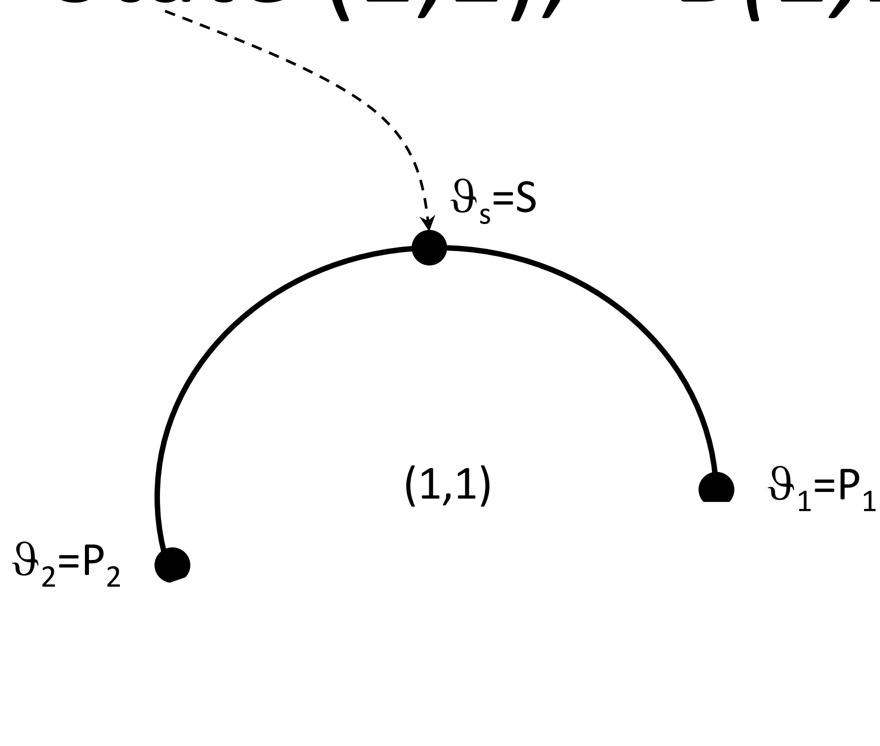
- To the state ϑ we add two more parameters w_1 and w_2 which may take values 0 or 1.
- $w_1=1$ means P1 is not visited yet.
- $w_1 =0$ means P1 is already visited
- $w_2=1$ means P2 is not visited yet
- $w_2=0$ means P2 is already visited.
- We have then four states/modes : $(\vartheta,1,1)$, $(\vartheta,0,1)$, $(\vartheta,1,0)$, $(\vartheta,0,0)$.


S
(1,1)
P1
P2
S
(0,1)
P1
P2
S
(1,0)
P1
P2
S
(0,0)
P1
P2



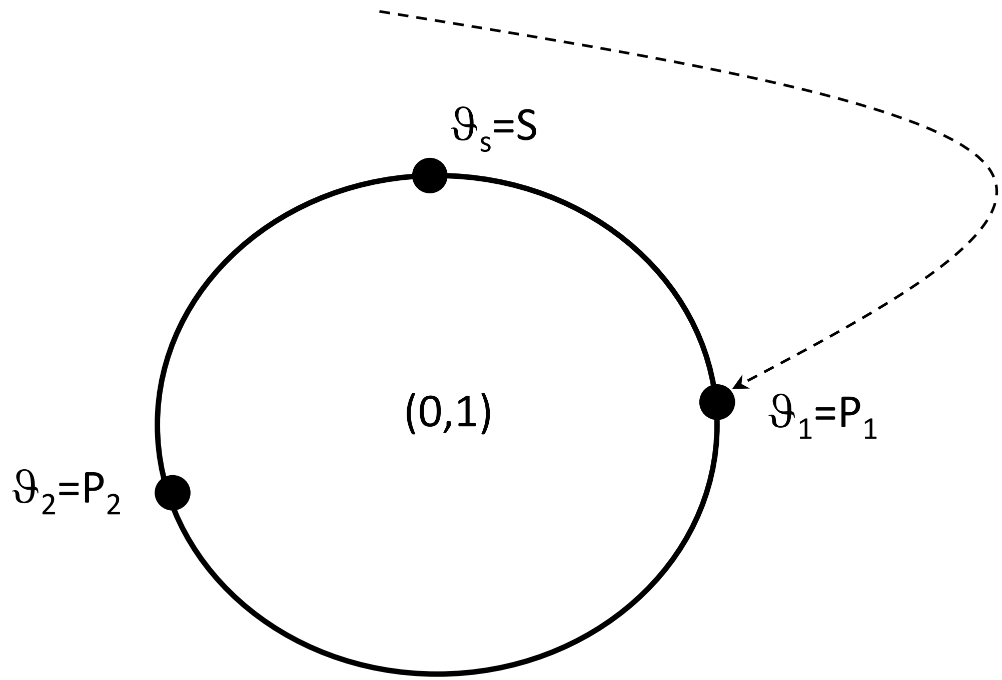
$$0 \leq \vartheta_1 < \vartheta_S < \vartheta_2 < 2\pi$$

State (1,1), *B(1,1)*



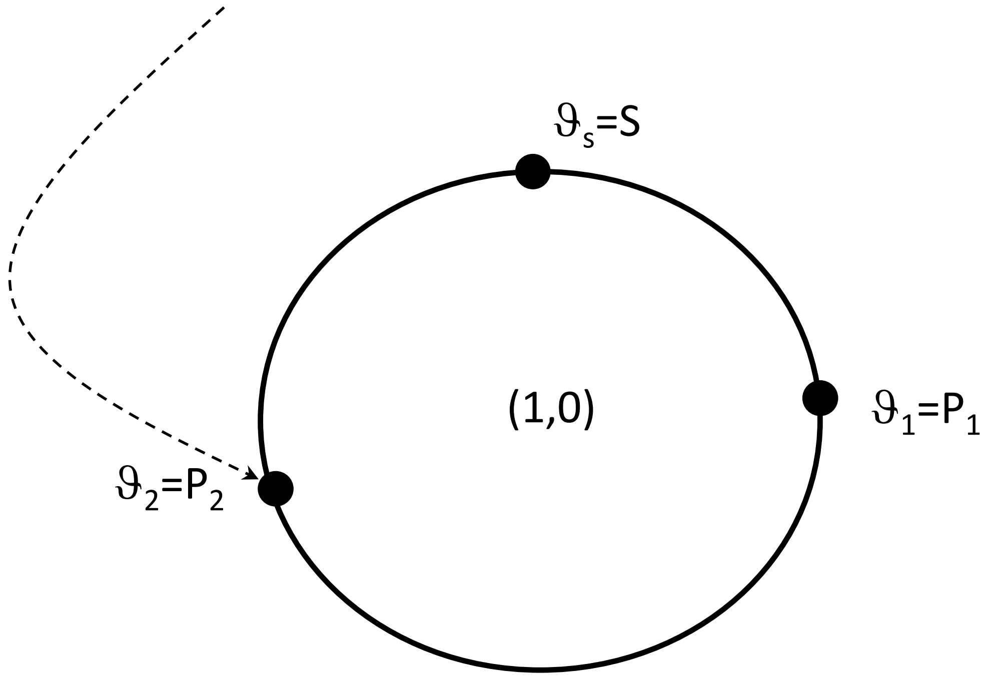
$$\vartheta \in [\vartheta_1, \vartheta_2]$$

State (0,1),   *B(0,1)*



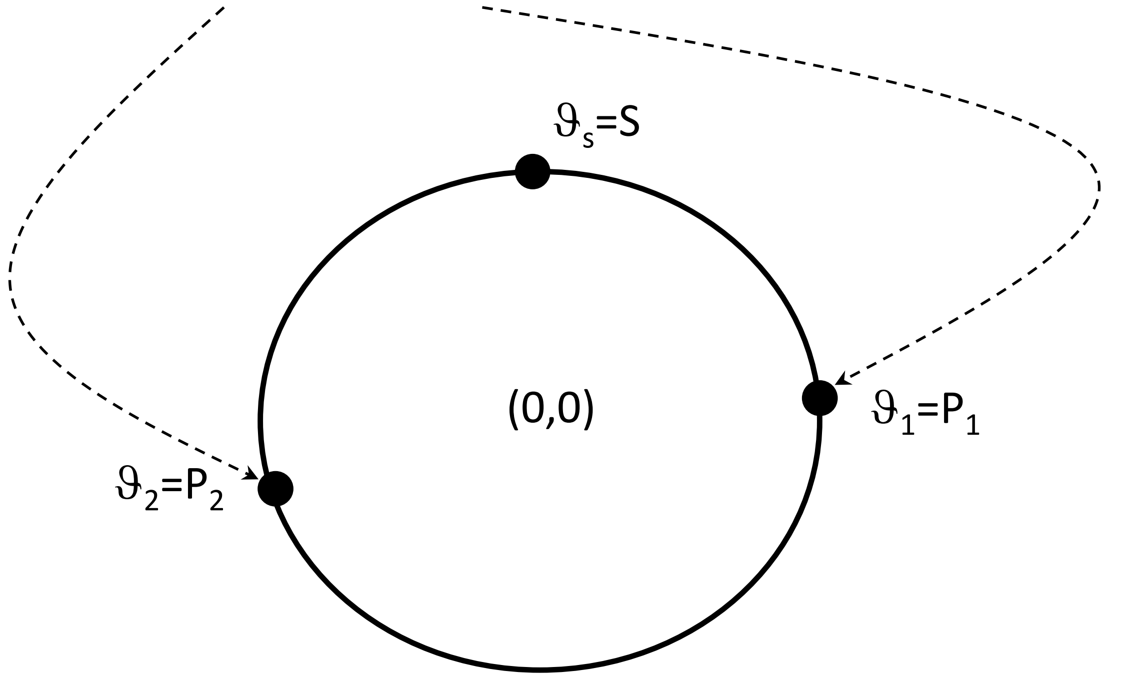
$\vartheta \in [\vartheta_2 - 2\pi, \vartheta_2]$

State (1,0), *B(1,0)*



$\vartheta \in [\vartheta_1, \vartheta_1 + 2\pi]$

State (0,0), *B(0,0)*



$\vartheta \in [0,2\pi]$

State space

- The state space is then

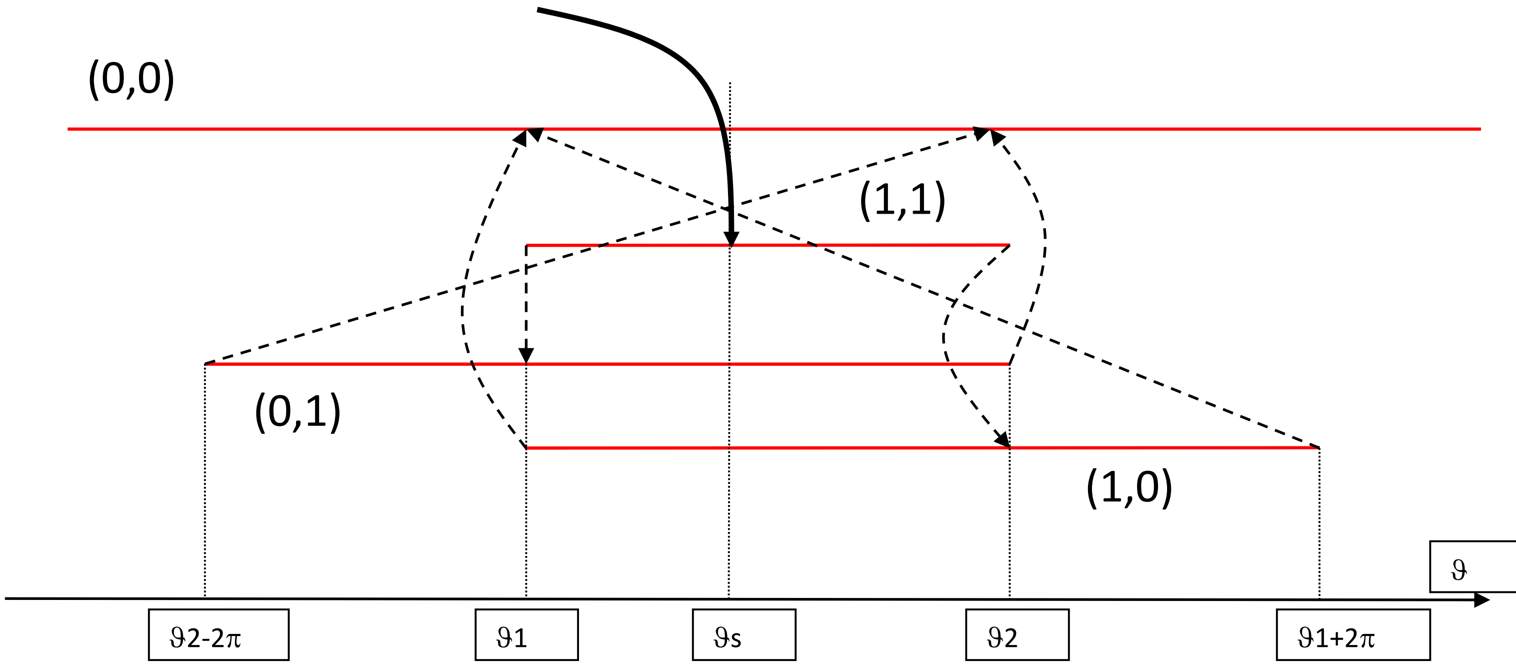
$$\big(B(1,1)\times(1,1) \cup B(0,1)\times(0,1)\cup$$

$$B(1,0)\times(1,0) \cup B(0,0)\times(0,0)\big)$$

$$\times[0,T]=$$

$$B \times[0,T]$$

Switching representation in line



The mean field game model

$$\vartheta'(s) = u(s), \vartheta(t) = \vartheta$$

$$M = (m^{1,1}, m^{0,1}, m^{1,0}, m^{0,0}) : B \times [0,T] \to [0,+\infty[,$$

$$(\vartheta, w_1, w_2, t) \mapsto m^{w_1, w_2}(\vartheta, t)$$

$$J(\vartheta, w_1, w_2, t, u, M) = \int_t^T \left(\frac{u^2(s)}{2} + F^{w_1(s), w_2(s)}(M[s]) \right) ds$$

$$+ c_1 w_1(T) + c_2 w_2(T) + c_3 (\vartheta(T) - \vartheta_s)^2$$

Switching

- From B(1,1) we may switch on B(0,1) and on B(1,0)
- From B(0,1) we may switch on B(0,0)
- From B(1,0) we may switch on B(0,0)
- From B(0,0) we do not switch away

Exit time interpretation

- Given *M*
- In every one of the first three branches we may interpret the optimal control problem as a finite horizon/exit time optimal control problem
- The exit cost is given by the value function on the point where we switch on.
- On the fourth branch B(0,0), the problem is just a finite horizon problem with all given data.

HJB problem

$$\begin{cases} -V_t(\theta,1,1,t) + \frac{1}{2}|V_\theta(\theta,1,1,t)|^2 = F^{(1,1)}(\mathcal{M}(t)) & \text{in }]\theta_1,\theta_2[\times]0,T[\\ V(\theta_1,1,1,t) = V(\theta_1,0,1,t) & \text{in }]0,T] \\ V(\theta_2,1,1,t) = V(\theta_2,1,0,t) & \text{in }]0,T] \\ V(\theta,1,1,T) = c_1 + c_2 + c_3(\theta-\theta_s)^2 & \text{in }]\theta_1,\theta_2[\end{cases}$$

→ V(.,1,1.)

V(.,0,1,.) ←

$$\begin{cases} -V_t(\theta,0,1,t) + \frac{1}{2}|V_\theta(\theta,0,1,t)|^2 = F^{(0,1)}(\mathcal{M}(t)) & \text{in }]\theta_2-2\pi,\theta_2[\times]0,T[\\ V(\theta_2-2\pi,0,1,t) = V(\theta_2,0,0,t) & \text{in }]0,T] \\ V(\theta_2,0,1,t) = V(\theta_2,0,0,t) & \text{in }]0,T] \\ V(\theta,0,1,T) = c_2 + c_3(\theta-\theta_s)^2 & \text{in }]\theta_2-2\pi,\theta_2[\end{cases}$$

$$\begin{cases} -V_t(\theta,1,0,t) + \frac{1}{2}|V_\theta(\theta,1,0,t)|^2 = F^{(1,0)}(\mathcal{M}(t)) & \text{in }]\theta_1,\theta_1+2\pi[\times]0,T[\\ V(\theta_1,1,0,t) = V(\theta_1,0,0,t) & \text{in }]0,T] \\ V(\theta_1+2\pi,1,0,t) = V(\theta_1,0,0,t) & \text{in }]0,T] \\ V(\theta,1,0,T) = c_1 + c_3(\theta-\theta_s)^2 & \text{in }]\theta_1,\theta_1+2\pi[\end{cases}$$

→ V(.1.0,.)

V(.,0,0,.) ←

$$\begin{cases} -V_t(\theta,0,0,t) + \frac{1}{2}|V_\theta(\theta,0,0,t)|^2 = F^{(0,0)}(\mathcal{M}(t)) & \text{in } [0,2\pi]\times]0,T[\\ V(\theta,0,0,T) = c_3(\theta-\theta_s)^2 & \text{in } [0,2\pi] \end{cases}$$

The transport equation

- If it optimally behaves, then every excursionist moves with the optimal feedback

$$u^*(\vartheta, w_1, w_2, t) = -V_\vartheta(\vartheta, w_1, w_2, t)$$

- Due to our simple model (the simple controlled dynamics, the non-dependence of $F^{w1,w2}$ on ϑ, the one-dimensionality,…)
- (We also suppose that the initial distribution m_0 is everywhere zero in all branches).
- The feedback optimal control has some good properties

The transport equation

- No excursionist will return back on its path when inside the same branch (that is not an optimal behavior).
- To stop is not an optimal behavior (apart the case that we are at the station and that we stop there until *T*.)
- When arrived on a switching point, the best choice is to immediately switch.
- These facts simplify a little bit the transport equation.

The transport equation

$m^{1,1}$ ←
$$\begin{cases} (m^{1,1})_t(\theta,t) + [u^*(\theta,1,1,t)m^{1,1}(\theta,t)]_\theta = 0 \\ m^{1,1}(\theta_s,t) = g(t) \end{cases}$$

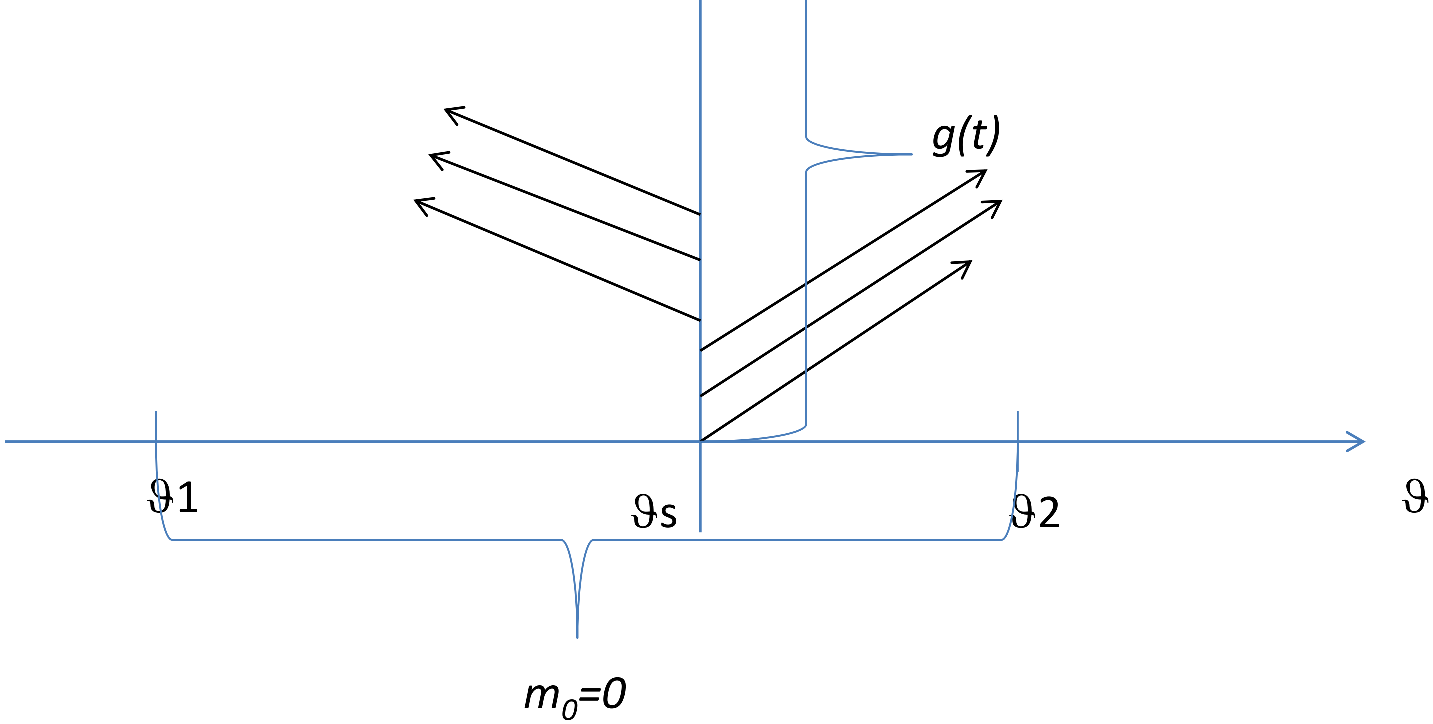
$m^{1,0}$ ←
$$\begin{cases} (m^{1,0})_t(\theta,t) + [u^*(\theta,1,0,t)m^{1,0}(\theta,t)]_\theta = 0 \\ m^{1,0}(\theta_2,t) = m^{1,1}(\theta_2,t) \end{cases}$$

$m^{0,1}$ ←
$$\begin{cases} (m^{0,1})_t(\theta,t) + [u^*(\theta,0,1,t)m^{0,1}(\theta,t)]_\theta = 0 \\ m^{0,1}(\theta_1,t) = m^{1,1}(\theta_1,t) \end{cases}$$

$m^{0,0}$ ←
$$\begin{cases} (m^{0,0})_t(\theta,t) + [u^*(\theta,0,0,t)m^{0,0}(\theta,t)]_\theta = 0 \\ m^{0,0}(\theta_1,t) = m^{1,0}(\theta_1,t) + m^{1,0}(\theta_1+2\pi,t) \\ m^{0,0}(\theta_2,t) = m^{0,1}(\theta_2-2\pi,t) + m^{0,1}(\theta_2,t) \end{cases}$$

“characteristics”

- We can treat the datum *g(t)* as a boundary condition in ϑs at the right as well as at the left



Equilibrium Mean Field

$$M \rightarrow V \rightarrow u^* = -V_\vartheta \rightarrow \tilde{M}$$

- This function (after some relaxation/convexification) is usc (closed graph) and convex/compact as function from *C([0,T];P(B))* into itself which is convex.
- Hence, there exists a fixed point (an equilibrium).

Mathematical Hysteresis Models

- M.A. Krasnoselskii - A. Pokrowskii: *Systems with Hysteresis*, Springer, 1989 (translation from russian)
- I.D. Mayergoyz: *Mathematical Models of Hysteresis*, Springer, 1991
- A. Visintin: *Differential Models of Hysteresis*, Springer, 1994
- P. Krejci: *Hysteresis, Convexity and Dissipation in Hyperbolic Equations*, Gakkotosho, 1996.
- M. Brokate – J. Sprekels: *Hysteresis and Phase Transitions*, Springer 1996

Viscosity Solutions for Hamilton-Jacobi-Bellman equations

- M. Bardi – I. Capuzzo Dolcetta: *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuser, 1997

- The theory goes back to P.L. Lions and to Crandall-Evans-Lions in early ‘80s.

*F. B. – M. Benetton: About an optimal visiting problem,*Applied Mathematics and Optimization, 2012

F. B. – K. Danieli: Infinite horizon optimal control problems with multiple thermostatic hybrid dynamics, Nonlinear Analysis: Hybrid Systems, 2012

F. B.: Optimal control of finite horizon type for a multidimensional delayed switching system, Discrete Contin. Dyn. Syst. Ser. B, 2005

F. B.: Viscosity solutions for an optimal control problem with Preisach hysteresis nonlinearities, ESAIM: Control, Optimisation, and Calculus of Variations, 2004

F. B. : Dynamic programming for some optimal control problems with hysteresis, NoDEA Nonlinear Differ. Equ. Appl., 2002

F. B.: An infinite horizon optimal control problem for some switching systems, Discrete Contin. Dyn. Syst. Ser. B, 2001.

Other DPP for some kind of hysteresis and sweeping process

- Belbas-Mayergoyz, *Dynamic programming for systems with hysteresis*, Physica B, 2001
- O. S. Serea, *On reflecting boundary problem for optimal control,* SIAM J. Control Optim, 2003
- Gudovich-Quincampoix: *Optimal control with hysteresis nonlinearity and multidimensional play operator*, SIAM J. Control Optim., 2011
- C. Castaing, M. D. P. Monteiro Marques, and P. Raynaud de Fitte, *Some problems in optimal control governed by the sweeping process*, J. Nonlinear Convex Anal., 2014,
- Colombo-Palladino: *The minimum time function for the controlled Moreau's sweeping process*, SIAM J. Control Otpim., 2016

Mean field games

- Lasry-Lions, 2006
- Huang-Caines-Malhame’ 2006
- Cardaliaguet (notes) 2012
- Camilli-Carlini-Marchi 2015 (on networks)