Positive minimal time for the control of state-constrained dynamical systems

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Introduction

Consider a (finite- or infinite-dimensional) linear control system

$$\boxed{\dot{y}(t) = Ay(t) + Bu(t)}$$

assumed to be controllable, without any state and control constraints.

For example, in finite dimension: Kalman condition

$$\Rightarrow \ \forall y^0, y^1 \in \mathbb{R}^n \qquad \forall T > 0 \qquad \exists u \in L^\infty(0, T; \mathbb{R}^m) \ \mid \ y(0) = y^0, \ \ y(T) = y^1$$

i.e., minimal controllability time $T_{\mathbb{R}^n}(y^0, y^1, A, B) = 0$

$\hookrightarrow$ *one can steer the system from any point to any other in arbitrarily small time.*

Introduction

Now, let $C \subset \mathbb{R}^n$ with nonempty interior: set of state constraints.

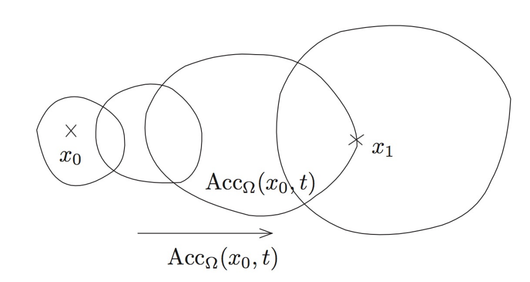
Question:

Given $y^0, y^1 \in C$, is it possible to steer the system from y^0 to y^1 in arbitrarily small time $T > 0$, guaranteeing that $y(t) \in C \quad \forall t \in [0, T]$?

N.B.: no control constraint.

Introduction

Remark: Existence of a positive minimal time is obvious under control constraints, without state constraints:



Introduction

$$\dot{y}_1(t) = y_2(t), \quad \dot{y}_2(t) = u(t),$$

State constraint

$$y_2(t) \geqslant 0$$

$\Rightarrow y_1(t)$ nondecreasing, and then one cannot pass from any point to any other.

Existing results on *controllability under state constraints*:

Krastanov Veliov 1992, Krastanov 2008, Heemels Camlibel 2007,
Le Marigonda 2017 (+ upper estimates for the minimal time)

$\rightarrow$ This is not our objective here. Objective: minimal time under state constraints.

In the sequel we assume that y^0 and y^1 are steady-states.

Introduction

$\bar{y} \in C$ is a *steady-state* if there exists $\bar{u} \in \mathbb{R}^m$ such that $A\bar{y} + B\bar{u} = 0$.

Remark: for $C = [0, +\infty)^n$, any point of C is a steady-state if and only if

$$\mathrm{Im}(B) \cap \mathrm{Cone}^+(a_1, \ldots, a_n) \neq \emptyset,$$

where $a_1, \ldots, a_n \in \mathbb{R}^n$ are the columns of A.

Assumption

$y^0, y^1 \in \mathring{C}$ steady-states.
We assume that there exists a path of steady-states $\tau \mapsto \bar{y}(\tau)$, $0 \leqslant \tau \leqslant 1$, such that $\bar{y}(0) = y^0$ and $\bar{y}(1) = y^1$, and $\bar{y}(\tau) \in \mathring{C} \quad \forall \tau \in [0, 1]$.

(satisfied if $\mathring{C}$ is convex)

Introduction

Under this steady-state connectedness assumption:

$$\exists T>0 \quad \exists u \in L^\infty(0,T;\mathbb{R}^m) \mid y(0)=y^0,\ \ y(T)=y^1, \qquad y(t)\in C \quad \forall t\in[0,T]$$

(argument: iterated use of local controllability along the path of steady-states)

i.e., one can pass from any steady-state to any other one, remaining in $\mathring{C}$, *in time sufficiently large*.

Introduction

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i.e., one can pass from any steady-state to any other one, remaining in $\mathring{C}$, *in time sufficiently large*.

Question:

Can the time T be chosen arbitrarily small?

N.B.: no control constraint.

Answer: NO in general, even for unilateral state constraints!

Introduction

Geometric explanation: $m = 1$, i.e., B vector in $\mathbb{R}^n$

- No constraint on $u \Rightarrow$ "instantaneous" motions along $\mathbb{R}B$ (with $u = \pm M$, $M \gg 1$)
- To move along $\mathbb{R}AB$, take $u = +M, -M, +M, -M$ with $M \gg 1$ (Lie bracket)
- Etc

$\hookrightarrow$ A state constraint $y(t) \in C$ may forbid such motions (in arbitrarily small time).

At this stage:

If C is bounded and if $y^0 \neq y^1$ then $T_C(y^0, y^1; A, B) > 0$.

i.e., *a positive minimal time is required to steer the system from y^0 to y^1 under the state constraint $y(t) \in C$.*

$\rightarrow$ Actually this is also true for unilateral constraints (depending on y^0, y^1)

Analysis under unilateral state constraints

$$\boxed{\dot{y}(t) = Ay(t) + Bu(t)}$$

- $A : n \times n, \quad B : n \times m,$ Kalman condition
- No control constraint: $u(t) \in \mathbb{R}^m$
- State constraint: $y(t) \in C$, assumed to be unilateral and affine

$\rightarrow$ Analysis in several steps.

Analysis under unilateral state constraints

First step

Feedback equivalence and Brunovsky normal form

(A, B) feedback equivalent to $(\tilde{A}, \tilde{B})$ if:

$$\exists T \in \mathrm{GL}_n(\mathbb{R}) \quad \exists V \in \mathrm{GL}_m(\mathbb{R}) \quad \exists F \in \mathcal{M}_{m,n}(\mathbb{R}) \quad | \quad T^{-1}(A+BF)T = \tilde{A} \quad \text{and} \quad T^{-1}BV = \tilde{B}$$

i.e., changes of variables $y = T\tilde{y}$ and $u = Fy + V\tilde{u}$. New control system:

$$\dot{\tilde{y}}(t) = \tilde{A}\tilde{y}(t) + \tilde{B}\tilde{u}(t)$$

satisfying:

- Kalman
- No control constraint: $\tilde{u}(t) \in \mathbb{R}^m$
- State constraint: $\tilde{y}(t) \in T^{-1}C$

Analysis under unilateral state constraints

First step

Feedback equivalence and Brunovsky normal form

Setting $r = \mathrm{rank}(B)$, (A, B) is feedback equivalent to

$$A = \begin{pmatrix} A_{k_1} & 0 & \cdots & 0 \\ 0 & A_{k_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{k_r} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{k_1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & b_{k_2} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{k_r} & 0 & \cdots & 0 \end{pmatrix},$$

with

$$A_{k_i} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad b_{k_i} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Analysis under unilateral state constraints

First step

Feedback equivalence and Brunovsky normal form

Assume that $m = 1$ for simplicity. Then, in the new variables:

$$\dot{y}_1 = y_2, \qquad \dot{y}_2 = y_3, \qquad \cdots \qquad \dot{y}_{n-1} = y_n, \qquad \dot{y}_n = u$$

Unilateral affine state constraint:

$$\langle \alpha, y(t) \rangle = \alpha_1 y_1(t) + \alpha_2 y_2(t) + \cdots + \alpha_n y_n(t) \geqslant \beta$$

for some $\alpha \in \mathbb{R}^n \setminus \{0\}$ and $\beta \in \mathbb{R}$.

Analysis under unilateral state constraints

Second step

Reduction by Goh transformation

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = y_3, \quad \dots \quad \dot{y}_{n-1} = y_n, \quad \dot{y}_n = u$$

$$\alpha_1 y_1(t) + \alpha_2 y_2(t) + \cdots + \alpha_n y_n(t) \geqslant \beta, \qquad u(t) \in \mathbb{R}$$

Set $v(t) = y_n(t)$: new control. Then:

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = y_3, \quad \dots \quad \dot{y}_{n-1} = v$$

$$\alpha_1 y_1(t) + \alpha_2 y_2(t) + \cdots + \alpha_{n-1} y_{n-1}(t) + \alpha_n v(t) \geqslant \beta$$

- if $\alpha_n = 0$, reiterate.
- Do it until the coefficient in v is nonzero.

Analysis under unilateral state constraints

Second step

Reduction by Goh transformation

Then:

$$\dot{y}_1 = y_2, \qquad \dot{y}_2 = y_3, \qquad \dots \qquad \dot{y}_k = v$$

$$\alpha_1 y_1(t) + \alpha_2 y_2(t) + \cdots + \alpha_k y_k(t) + \alpha_{k+1} v(t) \geqslant \beta$$

$\rightarrow$ mixed state-control constraint

and the initial optimal control problem is equivalent to the problem of steering in minimal time the above reduced control system in $\mathbb{R}^k$ from $\pi_k y^0$ to $\pi_k y^1$ under the mixed state-control constraint.

Multi-input case: same procedure on each block.

Analysis under unilateral state constraints

Third step

Change of control

$$\dot y_1 = y_2, \qquad \dot y_2 = y_3, \qquad \dots \qquad \dot y_k = v \qquad i.e., \quad \dot y = A_k y + b_k v$$

$$\underbrace{\alpha_1 y_1(t) + \alpha_2 y_2(t) + \cdots + \alpha_k y_k(t) + \alpha_{k+1} v(t)}_{w(t):\ \text{new control}} \geqslant \beta$$

Then:

$$\dot y = (A_k - b_k \alpha^\top) y + \alpha_{k+1} b_k w$$

$$w(t) \geqslant \beta$$

Multi-input case ($m > 1$):
this third step is performed on one control only, all other $m - 1$ controls being unconstrained.

Analysis under unilateral state constraints

Fourth step

Static equivalence and Brunovsky normal form

(A, B) *equivalent* to $(\tilde{A}, \tilde{B})$ if $\quad \exists T \in \mathrm{GL}_n(\mathbb{R}) \quad | \quad T^{-1}AT = \tilde{A}$ and $T^{-1}B = \tilde{B}$

i.e., change of variable $y = T\tilde{y}$. Normal form:

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{k_1} & * & \cdots & * \\ 0 & \tilde{A}_{k_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \tilde{A}_{k_r} \end{pmatrix}, \qquad \tilde{B}G = \begin{pmatrix} b_{k_1} & 0 & \cdots & 0 \\ 0 & b_{k_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{k_r} \end{pmatrix}$$

for some $G \in \mathcal{M}_{m,k_r}(\mathbb{R})$, with

$$\tilde{A}_{k_i} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n^{k_i} & -a_{n-1}^{k_i} & \cdots & \cdots & -a_1^{k_i} \end{pmatrix}, \qquad b_{k_i} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Analysis under unilateral state constraints

Fourth step

Static equivalence and Brunovsky normal form

We have thus reduced our problem to the minimal time problem:

$$\min t_f, \qquad \dot{y} = Ay + Bu, \qquad y(0) = y^0, \quad y(t_f) = y^1, \qquad u_1(t) \geqslant \beta$$

Replacing the control u_1 with $\beta + u_1$, setting $r = \beta b_1$ (with b_1 = first column of B), we have the minimal time problem

$$\min t_f, \qquad \dot{y} = Ay + Bu + r, \qquad y(0) = y^0, \quad y(t_f) = y^1, \qquad u_1(t) \geqslant 0$$

Since $y(t) = e^{tA} y^0 + \int_0^t e^{(t-s)A}\, ds\, r + \int_0^t e^{(t-s)A} Bu(s)\, ds$, we consider

$$\min t_f, \qquad \dot{y} = Ay + Bu, \qquad y(0) = 0, \quad y(t_f) = y^1, \qquad u_1(t) \geqslant 0$$

Analysis under unilateral state constraints

$$\min t_f, \qquad \dot y = Ay + Bu, \qquad y(0) = 0, \quad y(t_f) = y^1, \qquad u_1(t) \geqslant 0$$

Accessible set in time T:

$$A_0(0,T) = \left\{ y^f \in \mathbb{R}^n \mid \exists u \in L^\infty(0,T;\mathbb{R}^m), \quad \begin{array}{l} \dot y = Ay + Bu, \quad y(0) = 0,\ y(T) = y^f \\ u_1(t) \geqslant 0 \quad \forall t \in [0,T] \end{array} \right\}$$

$\rightarrow$ convex cone with vertex at 0, evolving continuously wrt T

$\hat{B}_1$ = matrix B of which the first column has been removed.

- If $m > 1$ and if $(A, \hat{B}_1)$ satisfies Kalman then $A_0(0,T) = \mathbb{R}^n \quad \forall T > 0$.
- Otherwise, $A_0(0,T)$ is a proper convex cone, isomorphic to the positive quadrant of $\mathbb{R}^n$, for $T > 0$ small enough.

$\Rightarrow$ this explains why $T_C(y^0, y^1, A, B) > 0$ or $= 0$, depending on y^0 and y^1

Analysis under unilateral state constraints

$$\min t_f, \qquad \dot y = Ay + Bu, \qquad y(0)=0, \quad y(t_f)=y^1, \qquad u_1(t)\geqslant 0$$

Accessible set in time T:

$$A_0(0,T) = \left\{ y^f \in \mathbb{R}^n \mid \exists u \in L^\infty(0,T;\mathbb{R}^m), \quad \begin{array}{l} \dot y = Ay+Bu, \quad y(0)=0,\ y(T)=y^f \\ u_1(t)\geqslant 0 \quad \forall t\in[0,T] \end{array} \right\}$$

$\rightarrow$ convex cone with vertex at 0, evolving continuously wrt T

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- If $m>1$ and if $(A,\hat{B}_1)$ satisfies Kalman then $A_0(0,T)=\mathbb{R}^n \quad \forall T>0$.
- Otherwise, $A_0(0,T)$ is a proper convex cone, isomorphic to the positive quadrant of $\mathbb{R}^n$, for $T>0$ small enough.

Only for T small. Indeed take $y_1' = y_2, \quad y_2' = -y_1 + u, \quad u \geqslant 0$,
shoot in time $\tau > 0$ small a point $\neq 0$, then take $u=0$ and follow the circle around 0 (in time $\leqslant 2\pi$). Hence $A_0(0,T)=\mathbb{R}^2 \quad \forall T>2\pi$.

Further results

- Exactly at time $T = T_C(y^0, y^1, A, B)$, there does not exist any (classical) L^∞ control steering the system from y^0 to y^1.
- $\lim_{M\to+\infty} T_C^M(y^0, y^1; A, B) = T_C(y^0, y^1; A, B)$
  (i.e., $\|u\| \leqslant M$, with $M \to +\infty$)

Alternative issues for investigation:

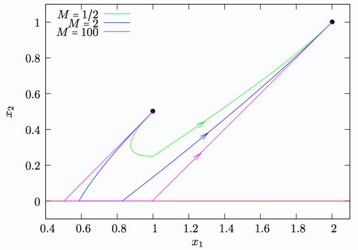
- Impulsive optimal control (sparsity, time support of impulses).
- Regularity of the minimal time.

(many existing results...)

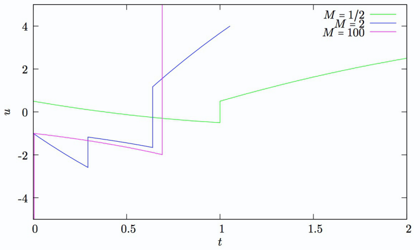
Examples

$$\dot{y}_1 = y_1 + u \qquad y^0 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}, \quad y^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad y_1(t) \geqslant 0, \quad y_2(t) \geqslant 0$$
$$\dot{y}_2 = 2y_2 + u$$

Then: $T_C(y^0, y^1) = \ln(2), \quad T_C(y^1, y^0) = 0.$



State trajectories



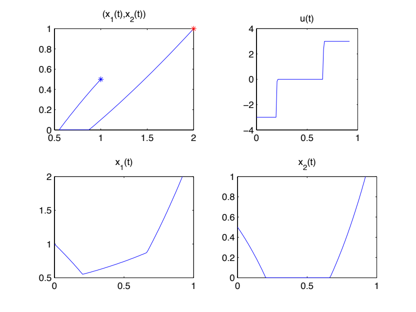
Control

Examples

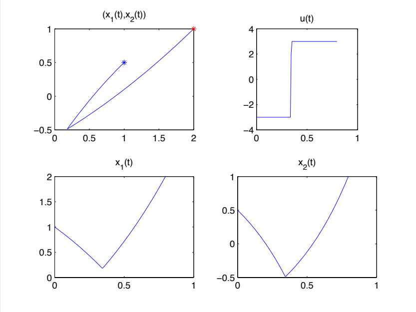
$$\begin{aligned}\dot y_1 &= y_1 + u\\ \dot y_2 &= 2y_2 + u\end{aligned} \qquad y^0 = \begin{pmatrix}1\\1/2\end{pmatrix}, \quad y^1 = \begin{pmatrix}2\\1\end{pmatrix}, \qquad y_1(t) \geqslant 0, \quad y_2(t) \geqslant 0$$

Then: $T_C(y^0, y^1) = \ln(2), \quad T_C(y^1, y^0) = 0.$

Steer the system from y^0 to y^1 under the control constraint $|u| \leqslant 3$



under the state constraints $y_1 \geqslant 0$, $y_2 \geqslant 0$ (turnpike... cf L. Grüne)



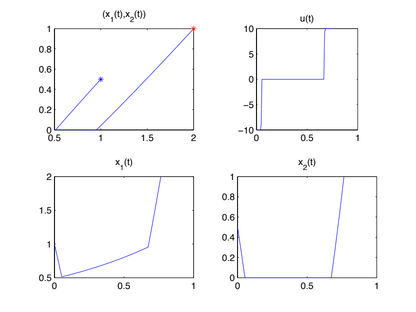
under no state constraint

Examples

$$\begin{array}{l}\dot{y}_1 = y_1 + u \\ \dot{y}_2 = 2y_2 + u\end{array} \qquad y^0 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}, \quad y^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad y_1(t) \geqslant 0, \quad y_2(t) \geqslant 0$$

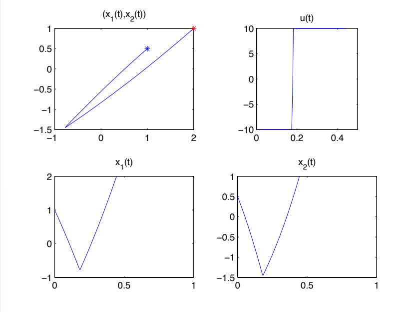
Then: $T_C(y^0, y^1) = \ln(2), \quad T_C(y^1, y^0) = 0.$

Steer the system from y^0 to y^1 under the control constraint $|u| \leqslant 10$



under the state constraints $y_1 \geqslant 0$, $y_2 \geqslant 0$

(turnpike... cf L. Grüne)



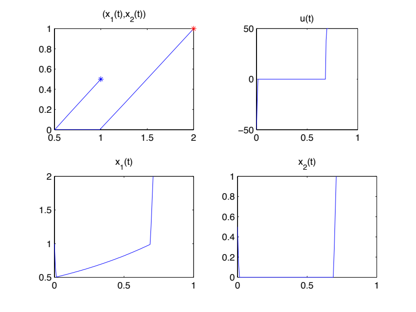
under no state constraint

Examples

$$\begin{cases}\dot y_1 = y_1 + u \\ \dot y_2 = 2y_2 + u\end{cases} \qquad y^0 = \begin{pmatrix}1\\1/2\end{pmatrix}, \quad y^1 = \begin{pmatrix}2\\1\end{pmatrix}, \qquad y_1(t) \geqslant 0, \quad y_2(t) \geqslant 0$$

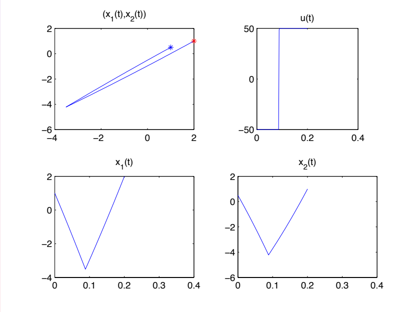
Then: $\quad T_C(y^0, y^1) = \ln(2), \quad T_C(y^1, y^0) = 0.$

Steer the system from y^0 to y^1 under the control constraint $|u| \leqslant 50$



under the state constraints $y_1 \geqslant 0$, $y_2 \geqslant 0$

(turnpike... cf L. Grüne)



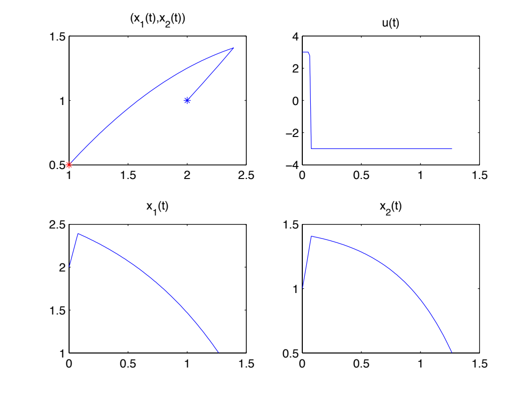
under no state constraint

Examples

$$\begin{aligned}\dot y_1 &= y_1 + u\\ \dot y_2 &= 2y_2 + u\end{aligned} \qquad y^0 = \begin{pmatrix}1\\1/2\end{pmatrix}, \quad y^1 = \begin{pmatrix}2\\1\end{pmatrix}, \qquad y_1(t) \geqslant 0, \quad y_2(t) \geqslant 0$$

Then: $T_C(y^0, y^1) = \ln(2), \quad T_C(y^1, y^0) = 0.$

Steer the system from y^1 to y^0 under the control constraint $|u| \leqslant 3$

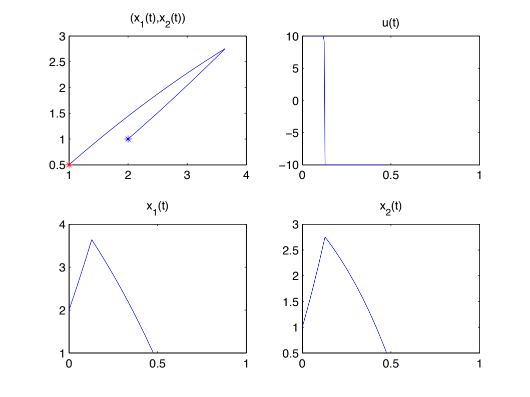


Examples

$$\dot{y}_1 = y_1 + u \quad \dot{y}_2 = 2y_2 + u \qquad y^0 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}, \quad y^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad y_1(t) \geqslant 0, \quad y_2(t) \geqslant 0$$

Then: $T_C(y^0, y^1) = \ln(2), \quad T_C(y^1, y^0) = 0.$

Steer the system from y^1 to y^0 under the control constraint $|u| \leqslant 10$

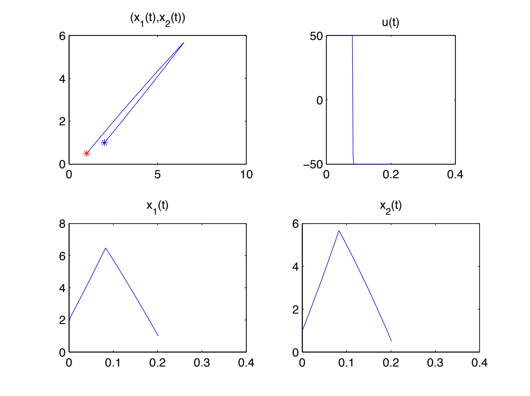


Examples

$$\dot y_1 = y_1 + u \qquad y^0 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}, \quad y^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad y_1(t) \geqslant 0, \quad y_2(t) \geqslant 0$$
$$\dot y_2 = 2y_2 + u$$

Then: $T_C(y^0, y^1) = \ln(2), \quad T_C(y^1, y^0) = 0.$

Steer the system from y^1 to y^0 under the control constraint $|u| \leqslant 50$

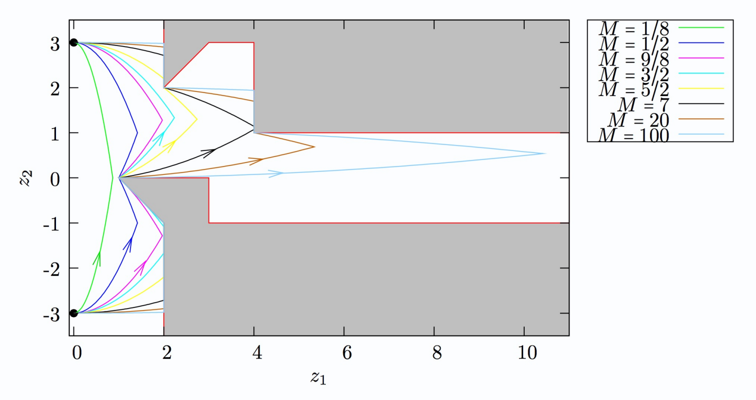


Examples

$$\dot z_1 = v \qquad \dot z_2 = z_1 \qquad z^0 = \begin{pmatrix} 0 \\ -3 \end{pmatrix}, \quad z^1 = \begin{pmatrix} 0 \\ 3 \end{pmatrix},$$

more complicated state constraints

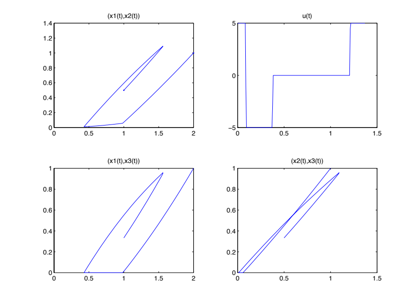
Then: $T_C(z^0, z^1) = \ln(2) + 7/4.$



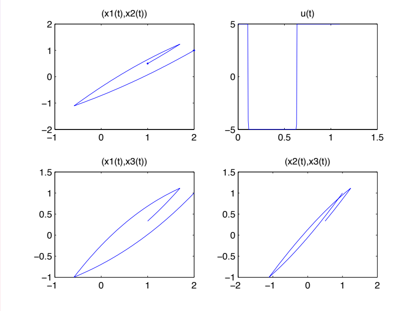
Examples

$$\begin{aligned} \dot{y}_1 &= y_1 + u \\ \dot{y}_2 &= 2y_2 + u \\ \dot{y}_3 &= 3y_3 + u \end{aligned} \qquad z^0 = \begin{pmatrix} 1 \\ 1/2 \\ 1/3 \end{pmatrix}, \quad z^1 = \begin{pmatrix} 2 \\ 1 \\ 2/3 \end{pmatrix}$$

Control constraint $|u| \leqslant 5$:



With state constraints
$y_1(t) \geqslant 0, \quad y_2(t) \geqslant 0, \quad y_3(t) \geqslant 0$



Without state constraints

Generalizations

- Linear control system with nonlinear state constraint $c(y) \geqslant 0$: similar analysis, leading to
$$\dot{y} = Ay - Bc(y) + Bw, \qquad w \geqslant 0$$

- Similar analysis for some classes of control-affine systems
$$\dot{y}(t) = f_0(y(t)) + \sum_{i=1}^{m} u_i(t) f_i(y(t))$$

  with nonlinear Brunovsky normal form.
  (involutive distribution of controlled vector fields, cf Isidori)

  N.B.: zero minimal time if Hörmander condition on the controlled vector fields.

Minimal time for the heat equation

Heat equation

$$\boxed{\partial_t y = \triangle y}$$

- Under homogeneous Dirichlet conditions: nonnegativity is preserved.
- Infinite velocity of propagation $\Rightarrow$ controllability in arbitrarily small time (with internal or boundary control).

Question

What happens under a nonnegativity state constraint?

Minimal time for the heat equation

1D heat equation with Dirichlet boundary control, under nonnegativity state constraint:

$$\begin{array}{llll} \partial_t y = \partial_x^2 y, & 0 < x < 1 & & \\ y(t,0) = u_0(t), & y(t,1) = u_1(t) & & \text{Dirichlet controls} \\ y(0) = y^0 \geqslant 0, & y(T) = y^1 \text{ constant (steady-state)} & & \\ y(t,x) \geqslant 0 & \forall t \in [0,T] & \forall x \in (0,1) & \text{state constraint} \end{array}$$

Minimal time for the heat equation

1D heat equation with Dirichlet boundary control, under nonnegativity state constraint:

$$\begin{array}{lll} \partial_t y = \partial_x^2 y, & 0 < x < 1 & \\ y(t,0) = u_0(t), & y(t,1) = u_1(t) & \text{Dirichlet controls} \\ y(0) = y^0 \geqslant 0, & y(T) = y^1 \text{ constant (steady-state)} & \\ y(t,x) \geqslant 0 \quad \forall t \in [0,T] & \forall x \in (0,1) & \text{state constraint} \end{array}$$

This is equivalent to:

$$\begin{array}{lll} \partial_t y = \partial_x^2 y, & 0 < x < 1 & \\ y(t,0) = u_0(t) \geqslant 0, & y(t,1) = u_1(t) \geqslant 0 & \text{nonnegative Dirichlet controls} \\ y(0) = y^0 \geqslant 0, & y(T) = y^1 \text{ constant (steady-state)} & \end{array}$$

Then, by comparison principle: $\sup_{x\in[0,1]} y(t,x) \geqslant y^0 \exp(-\pi^2 t)$

and thus if $0 < y^1 < y^0$, then $T(y^0,y^1) \geqslant \frac{1}{\pi^2} \ln\left(\frac{y^0}{y^1}\right) > 0$ (positive minimal time).

Less obviously (and more surprisingly), this is actually also true when $y^0 < y^1$.

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$$
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y(t,0) = u_0(t), & y(t,1) = u_1(t) & \text{Dirichlet controls} \\
y(0) = y^0 \geqslant 0, & y(T) = y^1 \text{ constant (steady-state)} & \\
y(t,x) \geqslant 0 & \forall t \in [0,T] \quad \forall x \in (0,1) & \text{state constraint}
\end{array}
$$

Theorem

Given any $y^0 \in L^2(0,1)$ and any $y^1 > 0$ such that $y^0 \neq y^1$:

- Controllability can be achieved in large enough time, with controls in $L^1(0,T)$ or even in $C^\infty([0,T])$. (proof by duality)
- $T(y^0, y^1) > 0$ positive minimal time
  (proof by spectral decomposition; lower estimates on the minimal time)
- Exactly at time $T = T(y^0, y^1)$, controllability can be achieved with nonnegative Radon measures controls. (more regular controls? open question)

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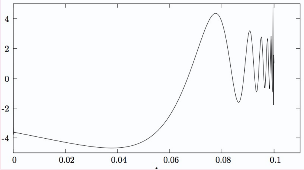
- Simulation with $y^0 = 5$ and $y^1 = 1$
- Simulation with $y^0 = 1$ and $y^1 = 5$:
  - Constraint 50 on the controls
  - Constraint 3000 on the controls



In contrast: control of minimal L^2 norm, without state constraint.

Numerical observation: sparsity of controls and of their support (Dirac impulses).

Minimal time for the heat equation

1D heat equation with Neumann boundary control, under nonnegativity state constraint:

$$\begin{array}{lll} \partial_t y = \partial_x^2 y, & 0 < x < 1 & \\ \partial_x y(t,0) = v_0(t), & \partial_x y(t,1) = v_1(t) & \text{Neumann controls} \\ y(0) = y^0 \geqslant 0, & y(T) = y^1 \text{ constant (steady-state)} & \\ y(t,x) \geqslant 0 & \forall t \in [0,T] \quad \forall x \in (0,1) & \text{state constraint} \end{array}$$

- Similar results.
- Simulation with $y^0 = 5$ and $y^1 = 1$:
  - Constraint 20 on the controls
  - Constraint 3000 on the controls

Numerical observation: nonsaturating control (kind of singular control)
+ "Dirac chattering" at the end.

Minimal time for the heat equation

Generalizations to multi-D heat equations

Positive minimal time in the following cases:

- Dirichlet boundary controls along the whole boundary, under nonnegativity state or control constraints (both are equivalent).

  Controllability exactly in time $T\left(y^0, y^1\right)$ with Radon measure controls.

  Lower estimate for $T(y^0, y^1)$ when the domain is a ball.

- Neumann boundary controls along the whole boundary, under nonnegativity state constraints.

  In contrast, under nonnegativity control constraints: $T(y^0, y^1) = +\infty$ (i.e., controllability fails).

First: extension to the unit ball (and spectral Sturm-Liouville decomposition).

Then, in a general domain: comparison by restriction to an internal ball.

Minimal time for the heat equation

Open questions

- Uniqueness of (Radon measure) controls at the minimal time.
- Regularity of controls at the minimal time: number of switchings, “Dirac chattering”?
- Regularity of the minimal time function.
- Turnpike and sparse structure.
- Convergence of minimal times of discrete finite-dimensional models to infinite dimension.
- More general PDE models.

J. Lohéac, E. Trélat, E. Zuazua,
- *Minimal controllability time for the heat equation under unilateral state or control constraints*, M3AS 2017.
- *Minimal controllability time for finite dimensional systems under state constraints*, ongoing.