Galerkin-Like method for generalized perturbed sweeping process with nonregular sets

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[1] Joint work with Abderrahim Jourani

Outline

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Framework

Let H be a separable Hilbert space. In this talk we are interested in the following differential inclusion

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0, \end{cases} \tag{$\mathcal{P}$}$$

where $F\colon [T_0, T] \times H \rightrightarrows H$ is a set-valued map with nonempty closed and convex values.

Framework: Standing Hypotheses ($\mathcal{H}^F$)

We assume the following *Standing Hypotheses*:

($\mathcal{H}_1^F$) For each $x \in H$, $F(\cdot, x)$ is measurable.

($\mathcal{H}_2^F$) For a.e. $t \in [T_0, T]$, $F(t, \cdot)$ is upper semicontinuous from H into H_w.

($\mathcal{H}_3^F$) There exist $c, d \in L^1(T_0, T)$ such that

$$d(0, F(t, x)) := \sup\{\|w\| : w \in F(t, x)\} \leq c(t)\|x\| + d(t)$$

for all $x \in H$ and a.e. $t \in [T_0, T]$.

Existence for $(\mathcal{P})$?

Under $(\mathcal{H}^F)$, let us consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0. \end{cases} \tag{$\mathcal{P}$}$$

There is a solution for $(\mathcal{P})$?

Theorem (Hájek-Johanis, 2010)

Let X be a separable infinite dimensional Banach space. Then there is a continuous mapping $f : X \to X$ such that the differential equation $\dot{x}(t) = f(x(t))$ has no solutions.

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The Galerkin-Like method

Approximation

For every $n \in \mathbb{N}$ let us consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(t, P_n(x(t))) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = P_n(x_0), \end{cases} \tag{$\mathcal{P}_n$}$$

where P_n is the projector from H into $\text{span}\{e_1, \ldots, e_n\}$ and $(e_n)_{n\in\mathbb{B}}$ is an orthonormal basis of H.

The Galerkin-Like method

Existence for the approximation method

For every $n \in \mathbb{N}$ let us consider:

$$\begin{cases} \dot{x}(t) \in F(t, P_n(x(t))) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = P_n(x_0), \end{cases} \qquad (\mathcal{P}_n)$$

Proposition (Jourani-Vilches, 2017)

Assume that $(\mathcal{H}^F)$ holds. Then, for each $n \in \mathbb{N}$ there exists at least one solution $x_n \in \mathrm{AC}([T_0, T]; H)$ of $(\mathcal{P}_n)$.

Moreover, the sequence $(x_n)_n$ is uniformly bounded in $\mathrm{AC}([T_0, T]; H)$.

The Galerkin-Like method

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The Galerkin-Like method

Convergence of the approximation method

Theorem (Jourani-Vilches, 2017)

Assume that $(\mathcal{H}^F)$ *holds. Assume that the sequence* $(P_n(x_n(t)))_n$ *is relatively compact for all* $t \in [T_0, T]$. *Then there exists a subsequence* $(x_{n_k})_k$ *of* $(x_n)_n$ *converging strongly pointwise to a solution* $x \in \mathrm{AC}\left([T_0, T]; H\right)$ *of*

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & a.e.\ t \in [T_0, T], \\ x(T_0) = x_0. \end{cases}$$

Generalized perturbed sweeping process (GPSP)

The *generalized perturbed sweeping process* (GPSP):

$$\begin{cases} -\dot{u}(t) = Bv(t) & \text{a.e. } t \in [T_0, T]; \\ -\dot{v}(t) \in N\left(C(t,u(t),v(t)); v(t)\right) + F(t,u(t),v(t)) + Au(t) & \text{a.e. } t \in [T_0, T]; \\ u(T_0) = u_0, v(T_0) = v_0 \in C(T_0,u_0,v_0), \end{cases}$$

where $A\colon U \to V$ and $B\colon V \to U$ are two bounded linear operators.

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Assumptions on the moving sets

$(\mathcal{H}^C)$ $C\colon [T_0, T] \times U \times V \rightrightarrows V$ has nonempty closed values.

- There exist $\zeta \in \mathrm{AC}([T_0, T]; \mathbb{R})$, $L_1 \geq 0$ and $L_2 \in [0, 1[$ such that for all $s, t \in [T_0, T]$ and all $x, y \in U$ and $u, v \in V$

$$\mathrm{Hauss}(C(t, x, u), C(s, y, v)) \leq |\zeta(t) - \zeta(s)| + L_1\|x - y\| + L_2\|u - v\|.$$

- For every $t \in [T_0, T]$, every $r > 0$ and every pair of bounded sets A, B, the set $C(t, A, B) \cap r\mathbb{B}$ is relatively compact.

Geometry of the moving sets

Convex	$\Rightarrow$	Uniformly Prox-regular	$\Rightarrow$	Uniformly Subsmooth	$\Rightarrow$	Positively α-far

Reduction of GPSP

Reduction of the GPSP to an unconstrained differential inclusion

$$\begin{cases} -\dot{u}(t) = Bv(t) & \text{a.e. } t \in [T_0, T], \\ -\dot{v}(t) \in N\left(C(t, u(t), v(t)); v(t)\right) & \\ \qquad\qquad + F(t, u(t), v(t)) + Au(t) & \text{a.e. } t \in [T_0, T], \\ u(T_0) = u_0, v(T_0) = v_0 \in C(T_0, u_0, v_0). & \end{cases}$$

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Reduction of GPSP

Reduction of the GPSP to an unconstrained differential inclusion

$$\begin{cases} -\dot{u}(t) = Bv(t) & \text{a.e. } t \in [T_0, T]; \\ -\dot{v}(t) \in m(t, u(t), v(t))\partial d_{C(t,u(t),v(t))}(v(t)) & \\ \qquad\qquad + F(t, u(t), v(t)) + Au(t) & \text{a.e. } t \in [T_0, T], \\ u(T_0) = u_0, v(T_0) = v_0 \in C(T_0, u_0, v_0), & \end{cases} \qquad (\mathcal{P}_{\text{Red}})$$

where $m(t, u, v)$ is a positive function.

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Reduction of GPSP

Reduction of the GPSP to an unconstrained differential inclusion

By using the inclusion:

$$\partial d_S(x) \subseteq N(S;x) \cap \mathbb{B} \quad x \in S.$$

If we can prove that

$$v(t) \in C(t, u(t), v(t)) \text{ for all } t \in [T_0, T].$$

Then, any solution of $(\mathcal{P}_{\text{Red}})$ is a solution of GPSP.

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First main result

Theorem (Jourani-Vilches, 2017)

Assume that the following assumptions hold true:

1. $(\mathcal{H}^F)$ *and* $(\mathcal{H}^C)$ *hold.*
2. *The family* $(C(t,u,v))_{\{(t,u,v)\in[T_0,T]\times H\times H\}}$ *is equi-uniformly subsmooth.*

Then, there exists at least one solution of the GPSP:

$$\begin{cases} -\dot{u}(t) = Bv(t) & a.e.\ t\in[T_0,T],\\ -\dot{v}(t) \in N\left(C(t,u(t),v(t));v(t)\right) + F(t,u(t),v(t)) + Au(t) & a.e.\ t\in[T_0,T],\\ u(T_0)=u_0, v(T_0)=v_0 \in C(T_0,u_0,v_0), \end{cases}$$

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Second main result

Theorem (Jourani-Vilches, 2017)

Assume that the following assumptions hold true:

1. $(\mathcal{H}^F)$ *and* $(\mathcal{H}^C)$ *hold.*
2. *The family* $(C(t))_{\{t\in[T_0,T]\}}$ *is positively* α*-far.*

Then, there exists at least one solution of the GPSP:

$$\begin{cases} -\dot{u}(t) = Bv(t) & \text{a.e. } t \in [T_0, T]; \\ -\dot{v}(t) \in N\left(C(t); v(t)\right) + F(t, u(t), v(t)) + Au(t) & \text{a.e. } t \in [T_0, T]; \\ u(T_0) = u_0, v(T_0) = v_0 \in C(T_0), \end{cases}$$

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Moreau's perturbed sweeping process

Corollary (Jourani & Vilches, 2017)

Assume that the following assumptions hold true:

- $(\mathcal{H}^F)$ *and* $(\mathcal{H}^C)$ *hold.*
- *The family* $(C(t))_{t\in[T_0,T]}$ *is positively* α*-far.*

Then, there exists at least one solution of

$$\begin{cases} -\dot{v}(t) \in N\left(C(t); v(t)\right) + F(t, v(t)) & a.e.\ t \in [T_0, T]; \\ v(T_0) = v_0 \in C(T_0). \end{cases}$$

State-dependent sweeping process

Corollary (Jourani & Vilches, 2017)

Assume that the following assumptions hold true:

- $(\mathcal{H}^F)$ *and* $(\mathcal{H}^C)$ *hold.*
- *The family* $\{C(t,v)\colon (t,v) \in [T_0,T] \times H\}$ *is equi-uniformly subsmooth.*

Then, there exists at least one solution of

$$\begin{cases} -\dot{v}(t) \in N\left(C(t,v(t)); v(t)\right) + F(t,v(t)) & a.e.\ t \in [T_0,T]; \\ v(T_0) = v_0 \in C(T_0,v_0). \end{cases}$$

Second-order sweeping process

Corollary (Jourani & Vilches, 2017)

Assume that the following assumptions hold true:

- $(\mathcal{H}^F)$ *and* $(\mathcal{H}^C)$ *hold.*
- *The family* $\{C(t,u,v)\colon (t,u,v) \in [T_0,T]\times H \times H\}$ *is equi-uniformly subsmooth.*

Then, there exists at least one solution of

$$\begin{cases} -\ddot{u}(t) \in N\left(C(t,u(t),\dot{u}(t));\dot{u}(t)\right) + F(t,u(t),\dot{u}(t)) & a.e.\ t \in [T_0,T]; \\ u(T_0) = u_0, \dot{u}(T_0) = v_0 \in C(T_0,u_0,v_0). \end{cases}$$

Sweeping processes with nonlocal initial conditions

We have applied the Galerkin-like method to the perturbed sweeping process with nonlocal initial conditions:

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); x(t)\right) + F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = Mx \in C(T_0), \end{cases}$$

where $M\colon C\left([T_0, T]; H\right) \to H$ is a nonlocal operator, e.g.,

- $Mx = \pm x(T)$ (periodic and anti-periodic initial conditions);
- $Mx = \frac{1}{T-T_0}\int_{T_0}^{T} x(s)ds$ (mean value initial conditions);
- $Mx = \sum_{k=1}^{k_0} \alpha_i x(t_i)$ with $\alpha_i \in \mathbb{R}$ and $\sum_{i=1}^{k_0} |\alpha_i| \leq 1$, where $T_0 < t_1 < \cdots < t_{k_0} \leq T$ (multi-point initial condition).

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[1] P. Hájek and M. Johanis On Peano's theorem in Banach spaces *J. Differential equations*, 249:3342-3351, 2010.

[2] A. Jourani and E. Vilches Galerkin-Like Method and Generalized Perturbed Sweeping Process with Nonregular Sets *SIAM J. Control Optim.*, 55(4):2412-2436, 2017.

[3] A. Jourani and E. Vilches Constrained Differential Inclusions with Nonlocal Initial Conditions *Submitted.*

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