Male optimal and unique stable marriages with partially ordered preferences

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Abstract. The stable marriage problem has a wide variety of practical applications, including matching resident doctors to hospitals, and students to schools. In the classical stable marriage problem, both men and women express a strict order over the members of the other sex. Here we consider a more realistic case, where both men and women can express their preferences via partial orders, i.e., by allowing ties and incomparability. This may be useful, for example, when preferences are elicited via compact preference representations like soft constraint or CP-nets that produce partial orders, as well as when preferences are obtained via multi-criteria reasoning. We study male optimality and uniqueness of stable marriages in this setting. Male optimality gives priority to one gender over the other, while uniqueness means that the solution is optimal, since it is as good as possible for all the participating agents. Uniqueness of solution is also a barrier against manipulation. We give an algorithm to find stable marriages that are male optimal. Moreover, we give sufficient conditions on the preferences (that are also necessary in some special case), that occur often in real-life scenarios, which guarantee the uniqueness of a stable marriage.

1 Introduction

The stable marriage problem (SM) [8] is a well-known collaboration problem. Given n men and n women, where each expresses a strict ordering over the members of the opposite sex, the problem is to match the men to the women so that there are no two people of opposite sex who would both rather be matched with each other than their current partners. In [6] Gale and Shapley proved that it is always possible to find a matching that makes all marriages stable, and provided a quadratic time algorithm which can be used to find one of two extreme stable marriages, the so-called *male optimal* or *female optimal* solutions. The Gale-Shapley algorithm has been used in many real-life scenarios, such as in matching hospitals to resident doctors, medical students to hospitals [9], sailors to ships, primary school students to secondary schools, as well as in market trading.

In the classical stable marriage problem, both men and women express a strict order over the members of the other sex. We consider a potentially more realistic case, where men and women express their preferences via *partial orders*, i.e., given a pair of men (resp., women), the women (resp., the men) can strictly order the elements of the pair, they may say that these elements are in a tie, or that they are incomparable. This is useful in practical applications when a person may not wish (or be able) to choose between alternatives, thus allowing ties in the preference list (or more generally, allowing each preference list to be a partial order) [10]. For example, in the context of centralized matching scheme, some participating hospitals with many applicants have found the task of producing a strictly ordered preference list difficult, and have expressed a desire to use ties [11]. Ties also naturally occur when assigning students to schools, since many students are indistinguishable from the point of view of a given school. Another situation where partial orders are useful is when preferences are elicited with a compact preference representation formalism like soft constraints [1] or CP-nets [2] that give partial orders. Another context where partial orders naturally occur is when preferences are obtained via multi-criteria reasoning.

We study male optimality and uniqueness of solution in this more general context. *Male optimality* can be a useful property since it allows us to give priority to one gender over the other. For example, in matching residents to hospitals in the US, priority is given to the residents. We present an algorithm, based on an extended version of the Gale-Shapely algorithm, to find a male optimal solution in stable marriage problems with partially ordered preferences (SMPs). This algorithm is sound but not complete: it may fail to find a male-optimal solution even when one exists. We conjecture, however, that the incompleteness is rare. We also give a sufficient condition on the preference profile that guarantees to find a male optimal solution, and we show how to find it.

Uniqueness is another interesting concept. For instance, it guarantees that the solution is optimal since it is as good as possible for all the participating agents. Uniqueness is also a barrier against manipulation. This is important as Roth [12] has proved that *all* stable marriage procedures can be manipulated. Uniqueness has previously been investigated in stable marriage problems where only strict orders are allowed [5]. A sufficient condition on the preferences was identified that ensures uniqueness. It was shown that this class of preferences is broad and of particular interest in many real-life scenarios [4]. Properties of preference orderings that satisfy this conditions are vertical heterogeneity and horizontal heterogeneity. Vertical heterogeneity [5] implies that all the agents of the same sex have identical preferences over the mates of the opposite sex, i.e., there is a common ordering over the mates. This is the standard assumption of identical preferences with different endowments [4]. The endowments in the stable marriage model is the desirability by the opposite sex. Horizontal heterogeneity [5] implies that each agent has a different most preferred mate. We show that it is possible to extend these sufficient conditions to SM with partially ordered preferences. However, for the vertical heterogeneity property, we need to consider uniqueness up to indifference and incomparability.

A brief overview of some of the theoretical results shown in this paper is contained in [7].

2 Background

2.1 Stable matching problems

Definition 1 (profile). *Given n men and n women, a profile is a sequence of $2n$ strict total orders (i.e., transitive and complete binary relations), n over the men and n over the women.*

Given a profile, the stable marriage problem (SM) [6] is the problem of finding a matching between men and women so that there are no two people of opposite sex who would both rather be married to each other than their current partners. If there are no such people, the matching is said to be stable.

Definition 2 (feasible partner). *Given an SM P, a feasible partner for a man m (resp., a woman w) is a woman w (resp., a man m) such that there is a stable marriage for P where m and w are married.*

The set of the stable marriages for an SM forms a lattice w.r.t. the men's (or women's) preferences. This is a graph where vertices correspond bijectively to the stable marriages and a marriage is above another if every man (resp., every woman) is married with a woman (resp., man) is at least as happy with the first marriage as with the second. The top of this lattice is the stable matching, called male-optimal (resp., female optimal), where men (resp., women) are mostly satisfied. Conversely, the bottom is the stable matching where men's (resp., women's) preferences are least satisfied [8].

Definition 3 (male (resp., female) optimal matching). *Given an SM P, a matching is male (resp., female) optimal iff every man (resp., woman) is paired with his (resp., her) highest ranked feasible partner in P.*

2.2 Gale-Shapley algorithm

The *Gale-Shapley* (GS) *algorithm* [6] is a well-known algorithm to solve the SM problem. At the start of the algorithm, each person is free and becomes engaged during the execution of the algorithm. Once a woman is engaged, she never becomes free again (although to whom she is engaged may change), but men can alternate between being free and being engaged. The following step is iterated until all men are engaged: choose a free man m, and let m propose to the most preferred woman w on his preference list, such that w has not already rejected m. If w is free, then w and m become engaged. If w is engaged to man m', then she rejects the man (m or m') that she least prefers, and becomes, or remains, engaged to the other man. The rejected man becomes, or remains, free. When all men are engaged, the engaged pairs are a male optimal stable matching.

This algorithm needs a number of steps that is quadratic in n (that is, the number of men), and it guarantees that, if the number of men and women coincide, and all participants express a strict order over all the members of the other group, everyone gets married, and the returned matching is stable. Since the input includes the profiles, the algorithm is linear in the size of the input.

Example 1. Assume $n = 3$. Let $W = \{w_1, w_2, w_3\}$ and $M = \{m_1, m_2, m_3\}$ be respectively the set of women and men. The following sequence of strict total orders defines a profile: $\{m_1 : w_1 > w_2 > w_3$ (i.e., man m_1 prefers woman w_1 to w_2 to w_3); $m_2 : w_2 > w_1 > w_3$; $m_3 : w_3 > w_2 > w_1\}$ $\{w_1 : m_1 > m_2 > m_3$; $w_2 : m_3 > m_1 > m_2$; $w_3 : m_2 > m_1 > m_3\}$. For this profile, the Gale-Shapley algorithm returns the male optimal solution $\{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$. On the other hand, the female optimal solution is $\{(w_1, m_1), (w_2, m_3), (w_3, m_2)\}$. □

The *Extended Gale-Shapely algorithm* [8] is the GS algorithm [6] where, whenever the proposal of a man m to a woman w is accepted, in w's preference list all men less desirable than m are deleted, and w is deleted from the preference lists of all such men. This means that, every time that a woman receives a proposal from a man, she accepts since only most preferred men can propose to her.

3 Stable matching problems with partial orders

We assume now that men and women express their preferences via partial orders. The notions given in Section 2 can be generalized as follows.

Definition 4 (partially ordered profile). *Given n men and n women, a profile is a sequence of $2n$ partial orders (i.e., reflexive, antisymmetric and transitive binary relations), n over the men and n over the women.*

Definition 5 (SMP). *A stable matching problem with partial orders (SMP) is just a SM where men's preferences and women's preference are partially ordered.*

Definition 6 (linearization of an SMP). *A linearization of an SMP is an SM that is obtained by giving a strict ordering to all the pairs that are not strictly ordered such that the resulting ordering is transitive.*

Definition 7 (weakly-stable matching in SMP). *A matching in an SMP is weakly-stable if there is no pair (x, y) such that each one strictly prefers the other to his/her current partner.*

Definition 8 (feasible partner in SMP). *Given an SMP P, a feasible partner for a man m (resp., woman w) is a woman w (resp., man m) such that there is a weakly stable marriage for P where m and w are married.*

A weakly stable matching is male optimal if there is no man that can get a strictly better partner in some other weakly-stable matching.

Definition 9 (male optimal weakly-stable matching). *Given an SMP P, a weakly stable matching of P is male optimal iff there is no man that prefers to be married with another feasible partner of P.*

In SMs there is always exactly one male-optimal stable matching. In SMPs, however, we can have zero or more male-optimal weakly stable matchings. Moreover, given an SMP P, all the stable matchings of the linearizations of P are weakly-stable matchings. However, not all these matchings are male optimal.

Example 2. In a setting with 2 men and 2 women, consider the profile P: $\{m_1 : w_1 \bowtie w_2;\ m_2 : w_2 > w_1;\}$ $\{w_1 : m_1 \bowtie m_2;\ w_2 : m_1 \bowtie m_2;\}$. Then consider the following linearization of P, say Q: $\{m_1 : w_2 > w_1;\ m_2 : w_2 > w_1;\}$ $\{w_1 : m_2 > m_1;\ w_2 : m_1 > m_2;\}$. If we apply the men-proposing GS to Q, then we obtain the weakly-stable matching μ_1 where m_1 marries w_2 and m_2 marries w_1. However, w_1 is not the most optimal woman for m_2 amongst all weakly-stable marriages. In fact, if we consider the linearization Q', obtained from Q, by changing m_1's preferences as follows: $m_1 : w_1 > w_2$, and if we apply the men-proposing GS, then we obtain the weakly-stable matching μ_2, where m_1 is married to w_1 and m_2 to w_2, i.e., m_2 is married to a woman who he prefers more than w_1, that is, his partner in μ_1. Notice that μ_2 is male-optimal, while μ_1 is not. □

4 Finding male optimal weakly-stable matchings

We now present an algorithm, called *MaleWeaklyStable* 1, that takes as input an SMP P and, either returns a male optimal weakly-stable marriage for P, or the string 'I don't know'. This algorithm is sound but not complete: if the algorithm returns a marriage, then it is weakly stable and male-optimal; however, it may fail to return a male-optimal marriage even if there is one.

We assume that the women express strict total orders over the men. If they don't, we simply pick any linearization. The algorithm exploits the extended GS algorithm [8], and at every step orders the free men by increasing number of their current top choices (i.e., the alternatives that are undominated). List L contains the current ordered sequence of free men. More precisely, our algorithm works as follows. It takes in input an SMP P, and it computes the list L of free men. At the beginning all the men are unmarried, and thus L contains them all. Then, we continue to check the following cases on the man m which is the first element of L, until they do not occur any longer:

– If the set of top choices of m contains exactly one unmarried woman, say w, m proposes to w and, since we are using the extended GS algorithm, the proposal is accepted. Then, all men that are strictly worse than m in w's preferences are removed from w's preference list, and w is removed from the preference lists of these men. Then, m is removed from L and L is ordered again, since the top choices of some men may now be smaller.
– If m has a single top choice, say w, that is already married, m proposes to w and w accepts the proposal. Then, she breaks the engagement with her current partner, say m', and all men that are strictly worse than m in w's preferences are removed from w's preference list, and w is removed from the preference lists of these men. Then, m is removed from L, m' becomes free and is put back in L, and L is ordered again.

When we exit from this cycle, we check if L is empty or not:

– if L is empty, the algorithm returns the current marriage. Notice that the current marriage, say (m_i, w_i), for $i = 1, \ldots, n$, is weakly stable since it is the

Algorithm 1: *MaleWeaklyStable*

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Input: p: a profile;
Output: μ: a weakly stable marriage or the string 'I don't know';
μ ← ∅;
L ← list of the men of p;
L ← ComputeOrderedList(L);
while Top((first(L)) contains exactly one unmarried woman) or (first(L) has a single
top choice already married) do
    m ← first(L);
    if Top(m) contains exactly one unmarried woman then
        w ← UnmarriedTop(m);
        Add the pair (m, w) to μ;
        foreach strict successor m* of m on w's preferences do
            delete m* from w's preferences and w from m's preferences ;
        L ← L \ {m};
        L ← ComputeOrderedList(L);
    if m has a single top choice already married then
        w ← Top(m);
        m' ← μ(w);
        Remove the pair (m', w) from μ;
        Add the pair (m, w) to μ;
        foreach strict successor m* of m on w's preferences do
            delete m* from w's preferences and w from m's preferences;
        L ← L ∪ {m'} \ {m};
        L ← ComputeOrderedList(L);
if (L = ∅) or (AllDiffUnmarried(L)=true) then
    Add to μAllDiffUnmarriedMatching(L);
    return μ
else
    foreach pair of men m and m' in L with Top(m) ∩ Top(m') ≠ ∅ do
        if m > m' for every w ∈ Top(m) then
            for every w ∈ Top(m)∩ Top(m') remove w from the preferences of m'
        L ← ComputeOrderedList(L);
        if AllDiffUnmarried(L)=true then
            add to μAllDiffUnmarriedMatching(L);
            return μ;
    return 'I don't know'
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solution obtained by applying the extended GS algorithm on a linearization of P where, for every m_i with ties or incomparability in current set of top choices, we have set w_i strictly better than all the other women in the top choice. Moreover, it is male-optimal, since no man is married with a woman that is strictly better for him in some other weakly stable marriage.

- If L is not empty, it means that the next free man in L has several current top choices and more than one is unmarried.
  - If there is a way to assign to the men currently in L different unmarried women from their current top choices then these men make these proposals, that are certainly accepted by the women, since every woman receives a proposal from a different man. Therefore, we add to the current marriage these new pairs and we return the resulting marriage. Such a marriage is weakly stable and male optimal for the reason above.
  - If it is not possible to make the above assignment, the algorithm removes unfeasible women from the current top choices of the men until it is possible to make the assignment or until all unfeasible women have been

removed. More precisely, it considers a pair of men m and m' such that Top$(m) \cap$ Top$(m') \neq \emptyset)$, and it checks if there are unfeasible women in Top(m') to remove. If every woman in $Top(m)$ prefers m to m', then all the women in Top$(m)\cap$ Top(m') are unfeasible for m', and thus all these women are removed from Top(m'). This could make now possible to make the assignment. If so, the algorithm adds to the current marriage these new pairs and returns the resulting marriage; otherwise, it performs the same reasoning for another pair of men that have some woman in common in their current top choices until all such pairs of men have been considered and no marriage has been returned. At this point the algorithm stops returning the string 'I don't know'.

Example 3. Consider the profile $\{m_1 : w_1 \bowtie w_2 > w_3 > w_4 > w_5;\ m_2 : w_1 \bowtie w_2 > w_3 > w_4 > w_5;\ m_3 : w_3 > w_5 > w_4 > w_2 > w_1;\ m_4 : w_1 \bowtie w_2 > w_3 > w_5 > w_4;\ m_5 : w_4 > w_5 > w_3 > w_2 > w_1\}$ $\{w_1 : m_1 > m_2 > m_4 > m_3 > m_5;\ w_2 : m_5 > m_3 > m_1 > m_2 > m_4;\ w_3 : m_5 > m_1 > m_2 > m_4 > m_3;\ w_4 : m_4 > m_3 > m_1 > m_2 > m_5;\ w_5 : m_1 > m_2 > m_3 > m_4 > m_5\}$. The algorithm first computes the ordered list $L = [m_3, m_5, m_1, m_2, m_4]$. Then, m_3 makes a proposal to w_3, who accepts. Thus m_3 is removed from L. Since m_3 is at the bottom of w_3's preferences, no man is removed from the preference list of w_3 or from the male preference lists. Therefore, $L = [m_5, m_1, m_2, m_4]$. m_5 next makes a proposal to w_4, who accepts and m_5 is removed from L. The new ordered list $L = [m_1, m_2, m_4]$. The remaining elements of L are men with more than one top choice. All these top choices are unmarried, but there is no way to assign them with different women from their top choices as the three men have only two top choices. However, in every linearization, m_4 is not matched with w_1 or w_2, due to w_1 and w_2's preferences. In fact, m_1 and m_2 choose between $\{w_1, w_2\}$, while m_4 proposes to his next choice, i.e., w_3. Hence, the considered profile is one in which only two of the three men with multiple top choices are feasible with w_1 and w_2, i.e. m_1 and m_2, and there is a way to assign to these men different unmarried women in their top choices. Our algorithm returns one of the two male optimal weakly stable solutions, i.e., $\{(m_3, w_5), (m_5, w_4)(m_1, w_2)(m_2, w_1)(m_4, w_3)\}$, or $\{(m_3, w_5), (m_5, w_4)(m_1, w_1)(m_2, w_2)(m_4, w_3)\}$. □

The *MaleWeaklyStable Algorithm* has a time complexity which is $O(n^{\frac{5}{2}})$. In fact, the first part has the same complexity of the extended GS algorithm, which is $O(n^2)$. The second part requires performing an all-different check between the current set of free men and their top choices. Since there are at most n free men and n top choices for each man, we can build a bipartite graph where nodes are men and women, and each arc connects a man with one of his unmarried top choices. We need to find a perfect matching in this graph. This can be done in $O(m\sqrt{n})$ where m is the number of edges, which is $O(n^2)$.

The *MaleWeaklyStable Algorithm* is sound, but not complete, i.e., if it returns a matching, then such a matching is male optimal and weakly-stable, but if it returns the string 'I don't know', we don't know if there is a weakly-stable matching that is male optimal. A case where our algorithm returns the string 'I don't know' is when L is *not empty* and there is a free man with more than one top choice and *all his top choices are already married.* We conjecture that in this case there is no male optimal weakly stable matching, since it seems there are some very specific circumstances for our algorithm to mot return a male optimal weakly stable matching (i.e., it has to pass through all the conditions we test) when one exists.

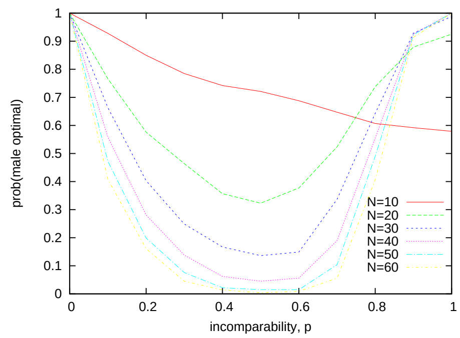

Fig. 1. Probability that the *MaleWeaklyStable* algorithm returns a male optimal stable marriage as we vary the amount of incomparability.

In Figure 1, we tested the *MaleWeaklyStable* algorithm on some SMPs generated randomly: each woman totally orders the men uniformly at random; each man totally orders the women uniformly at random; with probability p, we made any neighbouring pair in this total order incomparable. Hence, $p = 0$ means no incomparability, whilst $p = 1$ means that all woman are ranked incomparable by all men. In Figure 1, N is the total number of agents. We tested 1000 stable marriage problems for N=10 to 50 in steps of 10, and p=0 to 1 in steps of 0.1. For small instances, our algorithm has a good chance of returning a male optimal weakly stable marriage. For larger instances, the probability that we return a male optimal weakly stable marriage drops as the amount of incomparability increases. However, the probability turns around for large p, as the chance that the top choices are all different increases.

As we noticed above, there are SMPs with no male optimal weakly stable marriages. We now want to identify a class of SMPs where it is always possible to find a linearization which has a male optimal stable marriage.

Definition 10 (male-alldifference property). *An SMP P satisfies the male-alldifference property iff men's preferences satisfy the following conditions:*

- *all the men with a single top choice have top choices that are different;*
- *it is possible to assign to all men with multiple top choices an alternative in their top choices that is different from the one of all the other men of P.*

Theorem 1. *If an SMP is male-alldifferent, then there is a weakly stable matching that is male optimal and we can find it in polynomial time.*

Proof. If an SMP satisfies the male-alldifference property, then, by Definition 10, we can easily build the matching μ where all the men with a single top choice are married with their first top choice and all the men with more than one top choice are married with that alternative in their top choices that satisfies the second hypothesis of Definition 10. This matching is both weakly-stable and male optimal. It is is weakly stable, since it can be obtained by applying men-proposing GS on one of the linearizations where, for all the men m_i with more than one top choice, we put $\mu(m_i)$ strictly better than all the other alternatives. Moreover, by construction, μ is male optimal, i.e., there is no other weakly stable matching where a man can obtain a strictly better partner. In fact, all the men with a single top choice are already married in μ with their best woman, and thus they cannot obtain a better result, and, by construction, all the men with more than one alternative in their top are married with one of these alternatives, that is better than, in a ties with or incomparable to all the other alternatives, and thus also these men cannot obtain a strictly better woman in any other matching. □

The *MaleWeaklyStable Algorithm* exploits this same sufficient condition, plus some other sufficient condition. Notice that if an SMP satisfies the male-alldifference property, then, not only is there at least one weakly stable matching that is male-optimal, but there is an unique stable matching up to ties and incomparability.

5 On the uniqueness of weakly stable matching in SMPs

For strict total orders, [5] gives sufficient conditions on preference for the uniqueness of the stable matching. We now extend these results to partial orders. Notice that, if there is an unique stable matching, then it is clearly male optimal. A class of preference profiles in [5] giving an unique stable matching, when the preferences are strict total orders, is defined as follows. The set of the men and the set of the women are ordered sets, the preferences require that no man or woman prefers the mate of the opposite sex with the same rank order below his/her own order. Given such a preference ordering, by a recursive argument starting at the highest ranked mates, any other stable matching would be blocked by the identity matching, i.e., the matching in which we match mates of the same rank.

Theorem 2. [5] *Consider two ordered sets $M = (m_i)$ and $W = (w_i)$. If the profile satisfies the following conditions:*

$$\forall w_i \in W: \; m_i >_{w_i} m_j, \; \forall j > i \tag{1}$$

$$\forall m_i \in M: \; w_i >_{m_i} w_j, \; \forall j > i \tag{2}$$

then there is a unique stable matching $\mu^*(w_i) = m_i$, $\forall i \in \{1, 2, \ldots, \frac{N}{2}\}$.

Notice that the condition above is also necessary when the economies are small, i.e., $N = 4$ and $N = 6$.

There are two particular classes of preference profiles that generate a unique stable matching, and that are commonly assumed in economic applications [5].

The first assumes that all the women have identical preferences over the men, and that all the men have identical preferences over the women. In such a case there is a common (objective) ranking over the other sex.

Definition 11 (vertical heterogeneity). [5] *Consider two ordered sets $M = (m_i)$ and $W = (w_i)$. A profile satisfies the vertical heterogeneity property iff it satisfies the following conditions:*

- $\forall w_i \in W : m_k >_{w_i} m_j, \ \forall k < j$
- $\forall m_i \in M : w_k >_{m_i} w_j, \ \forall k < j$

Example 4. An example of a profile that satisfies vertical heterogeneity for $N = 6$ is the following. $\{m_1 : w_1 > w_2 > w_3;\ m_2 : w_1 > w_2 > w_3;\ m_3 : w_1 > w_2 > w_3;\}$ $\{w_1 : m_2 > m_3 > m_1;\ w_2 : m_2 > m_3 > m_1;\ w_3 : m_2 > m_3 > m_1.\}$ □

Corollary 1. [5] *Consider two ordered sets $M = (m_i)$ and $W = (w_i)$ and a profile P. If P satisfies the vertical heterogeneity property, then there is a unique stable matching $\mu^*(w_i) = m_i$.*

When agents have different preferences over the other sex, but each agent has a different most preferred mate and in addition is the most preferred by the mate, then the preference profile satisfies horizontal heterogeneity. In this situation there is a subjective ranking over the other sex.

Definition 12 (horizontal heterogeneity). [5] *Consider two ordered sets $M = (m_i)$ and $W = (w_i)$. A profile satisfies the horizontal heterogeneity property iff it satisfies the following conditions:*

- $\forall w_i \in W : m_i >_{w_i} m_j, \ \forall j$
- $\forall m_i \in M : w_i >_{m_i} w_j, \ \forall j$

Example 5. The following profile over 3 men and 3 women satisfies horizontal heterogeneity. $\{m_1 : w_1 > \ldots;\ m_2 : w_2 > \ldots;\ m_3 : w_3 > \ldots\}$ $\{w_1 : m_1 > \ldots;\ w_2 : m_2 > \ldots;\ w_3 : m_3 > \ldots.\}$ □

Corollary 2. [5] *Consider two ordered sets $M = (m_i)$ and $W = (w_i)$ and a profile P. If P satisfies the horizontal heterogeneity property, then there is a unique stable matching $\mu^*(w_i) = m_i$.*

We now check if the results given above for strictly ordered preferences can be generalized to the case of partially ordered preferences. Theorem 2 holds also when the men's preferences and/or women's preferences are partially ordered.

Theorem 3. *Consider two ordered sets $M = (m_i)$ and $W = (w_i)$ and a partially ordered profile P. If P satisfies the conditions 1 and 2 of Theorem 2, then there is a unique weakly stable matching $\mu(w_i) = m_i$, $\forall i \in \{1, 2, \ldots, \frac{N}{2}\}$.*

Proof. The proof is similar to the proof of Theorem 4. More precisely, to show that the matching μ is unique, we show that in any linearization the male optimal and the female optimal matching coincide with μ. For any linearization of p, we can compute the male optimal matching by using the men-proposing extending GS. First, m_1 makes the proposal to w_1 that accepts, since m_1 is her best man in her preferences all the other men are removed from w's preference ranking and w is removed from the list of these men. Therefore, w will not receive any other proposal and thus she remains with m_1 until the end of the algorithm. Hence, $\mu(m_1) = w_1$. Similarly, we can show that $\mu(m_2) = w_2$ and so on. Hence, the male optimal stable matching in every linearization of p is $\mu(m_i) = w_i$, $\forall i \in \{1, 2, \ldots, \frac{N}{2}\}$. To conclude that μ is unique, we can show with a reasoning similar to the one performed above, but using the women-proposing extended GS, instead of the men-proposing extended GS, that for every linearization of p also the female optimal stable matching is μ. □

Notice that the condition above is also necessary when the economies are small. For example, this holds when $N = 6$ (that is, three men and three women).

We now check if the *vertical heterogeneity* result (Corollary 1) holds also when the preferences are partially ordered. We recall that vertical heterogeneity assumes that all the agents of the same sex have the same strict preference ordering over the mates of the opposite sex. It is possible to see that, even if there is only one incomparable element in the ordering given by the men (or the women), then vertical heterogeneity does not hold and there may be more than one weakly stable marriage, as shown in the following example.

Example 6. Consider the following profile: $\{m_1 : w_1 > w_2 \bowtie w_3;\ \ m_2 : w_1 > w_2 \bowtie w_3;\ \ m_3 : w_1 > w_2 \bowtie w_3;\}$ $\{w_1 : m_1 > m_2 > m_3;\ \ w_2 : m_1 > m_2 > m_3;\ \ w_3 : m_1 > m_2 > m_3\}$. In this profile all the agents of the same sex have the same preference ordering over the mates of the opposite sex, however, there are two weakly stable matchings, i.e., $\mu_1 = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$ and $\mu_2 = \{(m_1, w_1), (m_2, w_3), (m_3, w_2)\}$. Notice however that these two weakly stable matchings differ only for incomparable or tied partners. □

It is possible to show that if all the agents of the same sex have the same preference ordering over the mates of the opposite sex and there is at least one incomparable or tied pair, then there is a unique weakly stable matching up to ties and incomparability.

Let us consider now Corollary 2 regarding the *horizontal heterogeneity* property. From Theorem 3, it follows immediately that Corollary 2 holds also when partially ordered preferences are allowed.

Corollary 3. *Consider two ordered sets $M = (m_i)$ and $W = (w_i)$ and a partially ordered profile P. If P satisfies the horizontal heterogeneity property, then there is a unique weakly stable matching $\mu(w_i) = m_i$, $\forall i \in \{1, 2, \ldots, \frac{N}{2}\}$*

Proof. It follows immediately from Theorem 3. □

For partially ordered preferences, we can guarantee uniqueness of weakly stable marriages by generalizing the horizontal heterogeneity property.

Definition 13 (p-horizontal heterogeneity). *Consider two ordered sets $M = (m_i)$ and $W = (w_i)$ that are ordered according to the number of their top choices. Let us denote with m_k the first man in the ordered list with more than one top choice, if he exists. A partially ordered profile satisfies the p-horizontal heterogeneity iff it satisfies the following conditions:*

- $\forall m_i \in M$ *with* $m_i < m_k$, $m_i : w_i >_{m_i} w_j$, $\forall j$;
- $\forall m_i \in M$ *with* $m_i \geq m_k$,
  - $m_i : w_i >_{m_i}$ (*or* $\bowtie_{m_i}$) w_j, $\forall j < i$;
  - $m_i : w_i >_{m_i} w_j$, $\forall j > i$;
- $\forall w_i \in W$, *with* $w_i < w_k$, $m_i >_{w_i} m_j$, $\forall j$;
- $\forall w_i \in W$, *with* $w_i \geq w_k$,
  - $w_i : m_i >_{w_i} m_j$, $\forall j > i$;
  - $w_i : m_i >_{w_i}$ (*or* $\bowtie_{w_i}$) m_j, $\forall j < i$,

In words, the conditions above require that every man m_i (resp., woman w_i) with a single alternative has as unique top choice w_i (resp., m_i), and that every m_i (resp., w_i) with more than one top choice has exactly one unmarried alternative, i.e., w_i (resp., m_i).

Corollary 4. *Consider two ordered sets $M = (m_i)$ and $W = (w_i)$ that are ordered according to the number of their top choices and a partially ordered profile P. If P satisfies the p-horizontal heterogeneity, then there is a unique weakly stable matching $\mu(w_i) = m_i$, $\forall i$.*

Proof. In order to show that that the matching $\mu(w_i) = m_i$, $\forall i$, is the unique weakly stable matching that can be obtained for any linearization of the given profile, say p, we will show that in every linearization of p, the male optimal matching and the female optimal matching coincide with the matching μ.

To show that in every linearization of p the male optimal matching is μ, we apply the extended version of the men-proposing GS to p and we show that the result is μ independently of how the non-ordered pairs are ordered. If we apply the extended men-proposing GS algorithm, every man m_i, for $i < k$, makes a proposal to his best woman w_i, that accepts since for her m_i is her best man, and all the other men of w_i are deleted from her preference list and w_i is removed from these men's preference ranking. This means, that when w_i accepts the proposal from m_i, then all the remaining men cannot propose to w_i and thus w_i remains with m_i, and so, for every men m_i, with $i < k$, $\mu(m_i) = w_i$. Since we are using the extended version of GS and the sets of men and women are ordered, every man m_i, for $i \geq k$, will not have in his preference ranking any woman w_j for $j < i$, and thus, since the profile satisfies the p-horizontal heterogeneity, $m_i : w_i >_{m_i} w_j$, $\forall j > i$. Therefore, every man m_i, for $i \geq k$, has in his top choice only the woman w_i and similarly every woman w_i has in his top choice m_i. Hence, for every linearization of the p, the matching returned by the men-proposing GS, that, as we know, is male optimal, is $\mu(m_i) = w_i$, $\forall i$.

Similarly, we can show that for every linearization of p the matching returned by the women-proposing GS, that, as we know, is female optimal, is $\mu(w_i) = m_i$, $\forall i$. Hence, we can conclude that μ is the unique weakly stable matching. □

6 Related work

In this paper, as in [10, 11], we permit non-strictly ordered preferences (i.e., preferences may contain ties and incomparable pairs) and we focus on weakly stable matchings. However, while in [10, 11], an algorithm is given that finds a weakly stable matching by solving a specific linearization obtained by breaking arbitrarily the ties, we present an algorithm that looks for weakly stable matchings that are male optimal, i.e., we look for those linearizations that favor one gender over the other one. Moreover, since there is no guarantee that a male optimal weakly stable matching exists, we give a sufficient condition on the preference profile that guarantees to find a weakly stable matching that is male optimal, and we show how to find such a matching. Other work focusses on providing sufficient conditions when a certain property is not assured for all matchings. For example, in [3] a sufficient condition is given for the existence of a stable roommate matching when we have preferences with ties.

7 Conclusions

We have given an algorithm to find male-optimal weakly-stable solutions when the men's preferences are partially ordered. The algorithm is sound but not complete. We conjecture, however, that incompleteness is rare since very specific circumstances are required for our algorithm not to return a male optimal weakly stable matching when one exists. We have then provided a sufficient condition, which is polynomial to check, for the existence of male-optimal weakly-stable matchings. We have also analyzed the issue of uniqueness of weakly-stable matchings, providing sufficient conditions, which are likely to occur in real life problems, that are also necessary in special cases.

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