Reasoning about fuzzy preferences and uncertainty

Maria Silvia Pini, Francesca Rossi, and Brent Venable

Department of Pure and Applied Mathematics, University of Padova, Padova Italy
E-mail: {mpini,frossi,kvenable}@math.unipd.it

Abstract Preferences and uncertainty occur in many real-life problems. The theory of possibility is one non-probabilistic way of dealing with uncertainty, which allows for easy integration with fuzzy preferences. In this paper we consider an existing technique to perform such an integration and, while following the same basic idea, we propose various alternative semantics which allow us to observe both the preference level and the robustness of the complete instantiations.

1 Introduction

Preferences and uncertainty occur in many real-life problems. Preferences can be represented in many ways, both qualitative and quantitative: CP-nets, soft constraints, utility theory, etc.. Uncertainty, on the other hand, has also been handled in several different ways. For example it has been treated using probabilistic approaches, and also non probabilistic approaches as possibility theory. In this paper we are concerned with the notion of uncertainty that comes from the lack of data or imprecise knowledge and with scenarios where probabilistic estimates are not available.

The theory of possibility is one non-probabilistic way of dealing with uncertainty, which allows for easy integration with fuzzy preferences. In fact, possibilities are values between 0 and 1 associated to events and which express the level of plausibility that the event will occur. In our context, we will describe a real life problem as set of variables with finite domains and a set of constraints among subsets of the variables. For us, an event will be modeled by the assignment of a value to a variable. A variable will be called uncertain if we cannot decide its value. In this case, we will associate a possibility degree to each value in its domain, which will tell how plausible it is that the variable will get that value.

Fuzzy preferences are values between 0 and 1 that are associated to single or multiple variable instantiations. In the context of fuzzy constraints, such preferences are combined using the *min* operator, and are ordered in such a way that higher values denote better preferences.

In this paper we consider an existing technique to integrate fuzzy preferences and uncertainty which uses possibility theory. This technique allows one to embed uncertainty into a fuzzy optimization engine. However, the integration is too tight since the resulting ordering over complete assignments does not allows one to discriminate between solutions which are highly preferred but assume unlikely events and solutions which are not preferred but robust with respect to uncertainty.

While following the same basic idea of translating uncertainty into fuzzy constraints, we propose various alternative semantics which allow us to observe both the preference level and the robustness of the complete instantiations.

2 Fuzzy Constraints

In [2] a *soft fuzzy constraint* C on variables $\{x_1, \ldots, x_n\}$ is associated with a $fuzzy$ relation R, i.e. a fuzzy subset of $D_1 \times \cdots \times D_n$ of values that more or less satisfy C. A membership function μ_R is associated with relation R and specifies for each tuple $(d_1, \ldots, d_n) \in D_1 \times \cdots \times D_n$ the level of satisfaction $\mu_R(d_1, \ldots, d_n)$ in a set L, which is totally ordered (e.g. [0,1]). In particular $\mu_R(d_1, \ldots, d_n) = 1$ if tuple $(d_1, \ldots, d_n)$ totally satisfies C, $\mu_R(d_1, \ldots, d_n) = 0$ if it totally violates C, and $0 < \mu_R(d_1, \ldots, d_n) < 1$ if it partially satisfies C. Moreover $\mu_R(d_1, \ldots, d_n) > \mu_R(d'_1, \ldots, d'_n)$ means that tuple $(d_1, \ldots, d_n)$ is better than the second one. Interpreting the preference degrees as membership degrees leads us to represent a fuzzy soft constraint by a fuzzy relation.

It is possible to model *prioritized constraints* using fuzzy relations with another scale V. In detail, a priority degree $Pr(C)$ is attached to each constraint indicating to what extent it's imperative that C be satisfied. If $Pr(C) = 1$ then it is imperative that C is satisfied, if $Pr(C) = 0$ it is completely possible to violate C and $Pr(C) > Pr(C')$ means that satisfying C is more important than satisfying C'.

The relation between the violation scale V and the satisfaction scale L is expressed by the following order-reversing bijection: $L = c(V) = \{c(v), v \in V\}$, where $c(0)$ is the top element of L and $c(1)$ is the bottom one. Notice that $v \geq v'$ in V implies $c(v) \leq c(v')$ in L. More precisely, a *prioritized constraint* can be represented by pair $(C, Pr(C))$ and is considered totally satisfied by a tuple if C is satisfied, and satisfied to degree c(Pr(C)) if the tuple violates C. Hence, c(V) represents a satisfaction scale and so C can be represented by a fuzzy relation such that: $\mu_R(d_1, \ldots, d_n) = c(0) = 1$ if $(d_1, \ldots, d_n)$ satisfies C, and $\mu_R(d_1, \ldots, d_n) = c(Pr(C))$ if $(d_1, \ldots, d_n)$ violates C.

The following *operations on $fuzzy$ constraints* are defined in [2]:

- The *projection* of a fuzzy constraint, represented by fuzzy relation R on variables $\{x_1, \ldots, x_k\} \subseteq V(R) = \{x_1, \ldots, x_n\}$, is a fuzzy relation $R^{\downarrow\{x_1,\ldots,x_k\}}$ defined on $\{x_1, \ldots, x_k\}$ such that:

$$\mu_{R^{\downarrow\{x_1,\ldots,x_k\}}}(d_1, \ldots, d_k) = sup_{\{d=(d_1,\ldots,d_n)|d^{\downarrow\{x_1,\ldots,x_k\}}=(d_1,\ldots,d_k)\}}\ \mu_R(d).$$

- The *conjunctive combination* of two fuzzy constraints, represented by fuzzy relations R_i and R_j, is a fuzzy relation $R_i \otimes R_j$ defined on variables $V(R_i) \cup V(R_j)$ such that:

$$\mu_{R_i \otimes R_j}(d_1, \ldots, d_k) = min(\mu_{R_i}(d_1, \ldots, d_k)^{\downarrow V(R_i)}, \mu_{R_j}(d_1, \ldots, d_k)^{\downarrow V(R_j)})$$

where $\mu_{R_i \otimes R_j}(d_1, \ldots, d_k)$ evaluates to what extent $(d_1, \ldots, d_k)$ satisfies both C_i and C_j.

3 Possibility theory: possibility as preference

In [2] it is also explained how it is possible to handle fuzzy soft and prioritized constraints using principles of possibility theory [3]. A *possibility distribution* π, attached to a variable x, is a mapping from a domain D to a totally ordered set L ([0,1] in general), which expresses to what degree is possible that $x = d$. $\pi(d) = 0$ means that it is impossible that $x = d$, $\pi(d) = 1$ means that the value d is possible for x without any conflicts and so $x = d$ cannot be excluded. In particular, $\pi(d) = 1$ for all $d \in D$ expresses the complete ignorance about x, because in this case all values d are plausible for x and so it is impossible to exclude some of them. Whereas, $\pi(\bar{d}) = 1$ for a specific value $\bar{d}$ and $\pi(d) = 0$ otherwise, expresses the complete knowledge about x, because in this case only the value $\bar{d}$ is plausible for x.

The fuzzy set of admissible values for x can be viewed as a possibility distribution indicating to what extent a value is considered suitable for x according to the constraint. Therefore the degree of possibility $\pi(d)$ can be seen as the degree of preference for choosing $x = d$. In particular, $\pi(d) = 0$ means that d is a forbidden value of x, while $\pi(d) = 1$ means that d is one of the most preferred values.

Given a possibility distribution π attached to x, you can consider events of the form $x \in A$. The occurrence of such events can be described by possibility and necessity degrees resp. defined as [2]: $\Pi(A) = sup_{d \in A}\pi(d)$, $N(A) = inf_{d \notin A}c(\pi(d))$ where c is the order reversing map on L.

The *possibility degree* of an event $x \in A$, denoted by $\Pi(A)$, evaluates the extent to which $x \in A$ is *possibly* true, whereas the dual measure of *necessity* of $x \in A$, denoted by $N(A)$, evaluates to what extent the proposition $x \in A$ is *certainly* true.

More precisely, the *possibility degree*, $\Pi(A)$, is the maximal membership degree of the elements that belongs to the intersection between the fuzzy set E such that $\pi = \mu_E$ and the set A:

$$\Pi(A) = sup_{d \in D}\, min(\pi(d), \mu_A(d)) = sup_{d \in A}\pi(d).$$

Notice that Π and N are such that $\Pi(A) = c(N(\bar{A}))$, and so $N(A) = 1 - \Pi(\bar{A})$, and that while $\Pi(A) = 1$ means that A is consistent with the constraint represented by π, $N(A) = 1$ means that the satisfaction, even partial, of the constraint represented by π implies the occurrence of A. In fact, $N(A) = 1$ means $1 - \Pi(\bar{A}) = 1$, i.e. $\Pi(\bar{A}) = 0$, which can be rewritten in the following way, using the definition of $\Pi(\bar{A})$, $sup_{d \notin A}\pi(d) = 0$. This means that $\pi(d) = 0\ \forall d \notin A$, and so the unique plausible values belong to A.

For example, if you have a possibility distribution π, attached to a variable x with domain $D_x = \{5, 6, 7, 8\}$, such that $\pi(5) = 0.9$, $\pi(6) = 0.4$, $\pi(7) = 0.7$, $\pi(8) = 0.5$, then, if $A = \{5, 6\}$ is a subset of D_x, the possibility degree of the event $x \in A$ is $\Pi(A) = sup_{d\in A}\pi(d) = sup\{0.9, 0.4\} = 0.9$, whereas the necessity degree of the same event, $x \in A$, is $N(A) = inf_{d\notin A}c(\pi(d)) = inf\{c(\pi(7)), c(\pi(8))\} = inf\{c(0.7), c(0.5)\} = inf\{0.3, 0.5\} = 0.3$. Calculating $N(A)$ using the formula $N(A) = 1 - \Pi(\bar{A})$ is the same, in fact, $N(A) = 1 - \Pi(\bar{A}) = 1 - sup_{d\in \bar{A}}\pi(d) = 1 - sup\{0.7, 0.5\} = 1 - 0.7 = 0.3$.

When considering prioritized constraints, the degree of priority α of a constraint C is considered as a degree of necessity of the subset modeling the constraint, i.e., corresponds to the higher level constraint $N(R) \geq \alpha$. Notice that $N(R) \geq \alpha$ is the same as $inf_{d\notin A}c(\pi(d)) \geq \alpha$, that is $\pi(d) \leq c(\alpha)\ \forall d \notin A$.

4 Uncertainty Parameters in Fuzzy Constraint Satisfaction Problems (FCSPs)

Whereas in usual FCSPs all the variables are assumed to be controllable, that is their value must be decided according to the constraints which relates them to other variables, in many real-world problems uncertain parameters must be used. Such parameters are associated with quantities which aren't under the user's direct control. Possibility theory [3] can also be used to handle these uncertain quantities, indicated by z, using possibility distributions π_Z. Such distributions rank the values according to their level of plausibility mapping them in a totally ordered scale U as follows: $\pi_Z : D_Z \longrightarrow U$. The interpretation of possibility distributions for uncontrollable events is different from that of preferences for controllable decisions. In fact, in the context of uncontrollable events, the possibility measure $\Pi(A)$ estimates the extent to which event A is unsurprising, while the necessity measure $N(A)$ represents the extent to which A is believed in spite of the uncertainty.

In [2] it is shown how it is possible to replace a hard constraint expressing the uncontrollability of an event, with a fuzzy constraint without uncontrollability.

Consider, for instance, uncontrollable variable z and a hard constraint C_{xz} that constrains z and controllable variable x. To satisfy C_{xz} a value for x must be chosen such that C_{xz} is satisfied *whatever the value of z turns out to be.* This means that the satisfaction degree for the value $d \in D_x$ is the necessity degree of the event $z \in (R_{xz} \cap \{d\})^{\downarrow D_z}$, given the restriction of the possible values of z defined by π_z, that is:

$$N(x = d\, satisfies\, C_{xz}) = c(sup_{a\notin (R_{xz}\cap\{d\})^{\downarrow D_z}, a\in D_z}\pi_z(a)).$$

This degree evaluates to what extent it is impossible to have a whatsoever possible value of z violating the constraint. More precisely it is equal to 0 if there is a totally plausible value a_1 for z such that the pair (a_1, d) violates the constraint, whereas it is equal to 1 if all values a such that the pair $(z, x) = (a, d)$ violates C_{xz} are impossible.

A constraint involving a decision variable x and an uncertain parameter z can be interpreted as the unary fuzzy constraint C_x on x defined by the fuzzy relation R_x:

$$\mu_{R_x}(d) = N(x = d\, satisfies\, C_{xz}).$$

Figure 1 shows an example with one decision variable, X, with domain $D_X = \{1, 2\}$ and one uncertain parameter, Z, with domain $D_Z = \{0, 1, 2, 3, 4, 5\}$. Figure 1 (a) describes the original constraint problem. There is a hard constraint, C_{XZ}, on X and Z represented by the values assigned by the membership function μ, defined by the fuzzy relation R_{XZ}, to all the possible tuples in $D_Z \times D_X$, and there is the possibility distribution π_Z which associates to each possible assignment to Z its possibility. Figure 1 (b) shows the result of transforming the uncertain information on Z into preferences on X applying for each value in D_X the formula given above, that we rewrite here for clarity:

$$N(X = d\, satisfies\, C_{XZ}) = c(sup_{a \notin (R_{XZ} \cap \{d\})^{\downarrow D_Z}, a \in D_Z} \pi_Z(a)).$$

In detail the preference for a value in D_X, e.g. $X = 1$ is obtained by the following steps:

- all the pairs in the constraint C_{XZ} containing $X = 1$ and with preference $= 1$ are considered. In this case $R_{XZ} \cap (X = 1) = \{(Z = 0, X = 1), (Z = 1, X = 1), (Z = 2, X = 1), (Z = 5, X = 1)\}$;
- the possibilities associated with the values for Z that do not belong to $R_{XZ} \cap (X = 1)$, but belong to D_Z, i.e. $Z = 3$ and $Z = 4$, are compared using *sup* operator: $sup(\pi_Z(3), \pi_Z(4)) = sup(0.8, 0) = 0.8$. This represents the maximum possibility associated to an assignment to Z that violates C_{XZ} when $X = 1$;
- the new preference that will be associated to $X = 1$, $\mu'(X = 1)$, is computed using the order-reversing function c, that here is assumed to be $c(p) = 1 - p$, on the *sup* calculated in the previous step: $c(0.8) = 0.2$.

This can be generalized to the case of a fuzzy constraint C, represented by the fuzzy relation R, which relates a set of decision variables $X = \{x_1, \ldots, x_n\}$ to a set of uncertain parameters $Z = \{z_1, \ldots, z_k\}$ with domains $A_1, \ldots, A_k$. The knowledge of the uncertain parameters is modeled with the possibility distribution π_Z defined on $A_Z = A_1 \times \cdots \times A_k$. The constraint is considered satisfied by the assignment $d = (d_1, \ldots, d_n) \in D_1 \times \cdots \times D_n$ if, *whatever the values of* $z = (z_1, \ldots, z_k)$, *these values are compatible with* d, i.e., the set of possible values for z is included in $T = (R \otimes \{(d_1, \ldots, d_n)\})^{\downarrow z}$. Obviously $\mu_T(a) = \mu_R(a, d)$ and

$$N(d\, satisfies\, C) = N(z \in T) = c(sup_{a \in A_z} min(c(\mu_T(a)), \pi_Z(a))).$$

Notice that $N(d\, satisfies\, C) = 1$ iff $\forall a, \pi_Z(a) > 0 \implies \mu_T(a) = 1$, i.e. any value of z which is whatsoever plausible leads to a total satisfaction of constraint C.

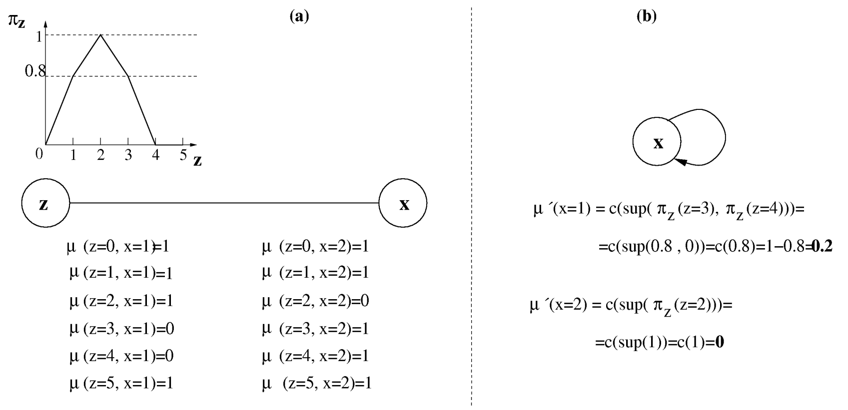

Figure1. Example with a single decision variable X connected with a single uncertain parameter Z. Figure (a) shows the original hard constraint C_{XZ} represented by membership function μ, and the possibility distribution π_Z describing the plausibility of Z. Figure (b) shows how the resulting fuzzy constraint on decision variable X.

The quantity $N(d\, satisfies\, C)$ represents the degree of satisfaction of C. If the uncertain parameters are independent then we can write distribution π_z in terms of the single distributions π_{z_j} as follows: $\pi_z(a) = min_{j=1,\ldots,k}\pi_{z_j}(a_j)$ $\forall a = (a_1, \ldots, a_k) \in A_1 \times \cdots \times A_k$. The result is a new constraint C' on the decision variables $X = \{x_1, \ldots, x_n\}$ with fuzzy relation R' defined by $R' = \overline{((\bar{R} \otimes F_z)^{\downarrow X})}$ where F_z is the fuzzy set whose membership function is π_z, and where $\bar{R}$ stands for the complement of R, i.e. $\mu(\bar{R}) = c(\mu_R)$.

In general we can use the information on uncertain parameters to change each constraint C_i into a new constraint C_i' following the procedure described above. Generally $\mu_R'(d) \geq \alpha$ means that if it is taken for granted that the actual value of z has plausibility strictly greater than $c(\alpha)$, then it is sure that the decision d satisfies C at least at level α. In fact, $\mu_R'(d) \geq \alpha$ means that $N(d\, satisfies\, C) \geq \alpha$ and then, applying the definition of $N(d\, satisfies\, C)$, it means $c(sup_{a \in A_z} min(c(\mu_T(a)), \pi_Z(a))) \geq \alpha$, where for hypothesis $\pi_Z(a) > c(\alpha)$. Applying the order reversing map c, you have $sup_{a \in A_z} min(c(\mu_T(a)), \pi_Z(a)) \leq c(\alpha)$, where for hypothesis $\pi_Z(a) > c(\alpha)$. This implies $min(c(\mu_T(a)), \pi_Z(a)) \leq c(\alpha)$ for all $a \in A_z$, then, knowing that $\pi_Z(a) > c(\alpha)$, $min(c(\mu_T(a)), \pi_Z(a))$ must be $c(\mu_T(a))$ for all $a \in A_z$. Therefore $c(\mu_T(a)) \leq c(\alpha)$ for all $a \in A_z$, and then $\mu_T(a) \geq \alpha$ for all $a \in A_z$. This means that $\mu_R(a, d) \geq \alpha$ for all $a \in A_z$, i.e. d satisfies C at least at level α.

In Figure 2 we show an example with two decision variables X with domain $D_X = \{1, 2\}$, Y with domain $D_Y = \{3, 4\}$ and an uncertain parameter Z with domain $D_Z = \{0, 1, 2, 3, 4, 5\}$. In Figure 2 (a) we have the original problem with a ternary constraint C_{XYZ} represented by membership function μ defined by relation R, and the possibility distribution on the values of D_Z. Figure 2 (b)

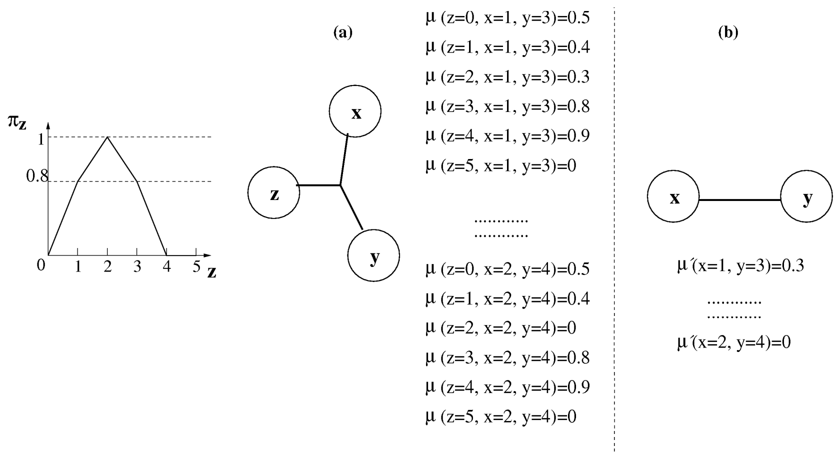

Figure2. Example with two decision variables X and Y and a single uncertain parameter Z. Figure (a) shows the original ternary fuzzy constraint C_{XYZ} represented by membership function μ, and the possibility distribution π_Z describing the plausibility of Z. Figure (b) shows how the resulting binary fuzzy constraint on decision variables X and Y.

shows the binary constraint on X and Y such that the preferences it associates to pairs of values of X and Y are obtained from the uncertain information on Z. The preference of a pair (x, y) where $x \in D_X$ and $y \in D_y$ is computed according to the formula (given before):

$$N((x, y)\, satisfies\, C_{XYZ}) = c(sup_{z \in D_Z} min(c(\mu_T(z)), \pi_Z(z))).$$

To see how it works consider the pair $(X = 1, Y = 3)$:

- since R is the set of all tuples of constraint C_{XYZ}, $R \otimes (X = 1, Y = 3)$ is the subset of R containing only tuples with $X = 1$ and $Y = 3$;
- $T = (R \otimes (X = 1, Y = 3))^{\downarrow z}$ is the projection of $R \otimes (X = 1, Y = 3)$ on Z, in this case $T = \{0, 1, 2, 3, 4, 5\}$;
- for each value z in T we compute $\mu_T(z) = \mu(z, 1, 3)$ obtaining $\mu_T(0) = 0.5$, $\mu_T(1) = 0.4$, $\mu_T(2) = 0.3$, $\mu_T(3) = 0.8$, $\mu_T(4) = 0.9$, $\mu_T(5) = 0$;
- then the order reversing function $c(p) = 1 - p$ is applied to all the $\mu_T(z)$ obtained in the previous step. This gives $c(\mu_T(0)) = 0.5$, $c(\mu_T(1)) = 0.6$, $c(\mu_T(2)) = 0.7$, $c(\mu_T(3)) = 0.2$, $c(\mu_T(4)) = 0.1$, $c(\mu_T(5)) = 1$;
- then for each z we take the minimum of $c(\mu_T(z))$ and the possibility $\pi_Z(z)$ and so for $z = 0$ we take $min(0.5, 0) = 0$, for $z = 1$ we take $min(0.6, 0.8) = 0.6$, for $z = 2$ we take $min(0.7, 1) = 0.7$, for $z = 3$ we take $min(0.2, 0.8) = 0.2$, for $z = 4$ we take $min(0.1, 0) = 0$ and for $z = 5$ we take $min(1, 0) = 0$;
- then the sup operator is applied to the values obtained at the previous step: $sup(0, 0.6, 0.7, 0.2, 0, 0) = 0.7$;

– finally we use function c again obtaining the new preference for pair $(X = 1, Y = 3)$ in the new binary constraint on decision variables X and Y, that is $\mu'(X = 1, Y = 3) = c(0.7) = 0.3$.

To see that the preference associated in the new binary fuzzy constraint to a pair of values (x, y) is related to whether such pair is consistent or not with all the plausible values of Z, consider the pair $(X = 2, Y = 4)$ in Figure 2. Notice that w.r.t. pair $(X = 1, Y = 3)$ the only difference is the preference associated to the tuple containing $Z = 2$ for which $\pi_Z(2) = 1$. In fact, whereas $\mu(2, 1, 3) = 0.3$, $\mu(2, 2, 4) = 0$. This means that pair $(X = 2, Y = 4)$ is not consistent with a value for Z, $Z = 2$, which is maximally possible. Repeating the same reasoning done before gives pair $(X = 2, Y = 4)$ a new preference equal to 0. This is correct, since it means that a pair, which is inconsistent with values of Z which have a high possibility, should have a low preference. If the possibility is 1 (maximal) for some of the values of the uncertain parameter with which the pair of values of decision variables is inconsistent, then the new preference associated with a such pair must be 0.

Summarizing, the method proposed in [2] for managing the uncertainty in a general Fuzzy CSP, is the following. It starts from a Fuzzy CSP with decisional variables, uncertain parameters, fuzzy constraints among decisional variables and constraints that link decisional variables with uncertain parameters. At the first step the original problem is reduced to another one, in which there aren't uncertain parameters. For doing so all constraints, which link uncertain parameters with decisional variables, are changed in fuzzy constraints only among these decisional variables. The new preference levels of the decisional variables in such particular constraints are computed applying the specific procedure given above in this section. At the second step the new problem is composed by only fuzzy constraints, therefore you can solve it applying usual method for solving fuzzy CSPs, i.e. using min operator to combine constraints and using max operator to project them.

An application of this method to a general Fuzzy CSP (for example Fuzzy CSP proposed in Figure 3) can be seen in Figure 4. The Figure 3 shows a particular Fuzzy CSP with an uncertain parameter. In this problem there are three decisional variables X, Y, W and an uncertain parameter Z with these domains: $D_X = D_Y = \{1, 2\}$, $D_Z = \{3, 4\}$ and $D_Z = \{5, 6\}$. Moreover we have a ternary constraint C_{XYZ} linking X and Y with Z, which is represented by membership function μ and by possibility distribution π_Z, and a fuzzy constraint linking X and W. Figure 4 shows how the method proposed in works on this particular FCSP. Figure 4 (a) shows the new fuzzy CSP problem with the previous fuzzy constraint between X and W and with the new fuzzy binary constraint on X and Y such that the preferences it associates to pairs of values of X and Y are obtained from the uncertain information on Z after applying the formula given above. Figure 4 (b) shows all assignments with their final preferences, that you can find applying usual procedure for solving Fuzzy CSP, i.e. combining constraints using operator min and projecting them using operator max. More precisely, final preference degrees are obtained applying min operator between the preference degree given by the fuzzy (F) constraint C_{XZ} in the original problem and the preference degree obtained reasoning on uncertain parameter (U). The solution is found choosing the assignment with the maximum final preference degree.

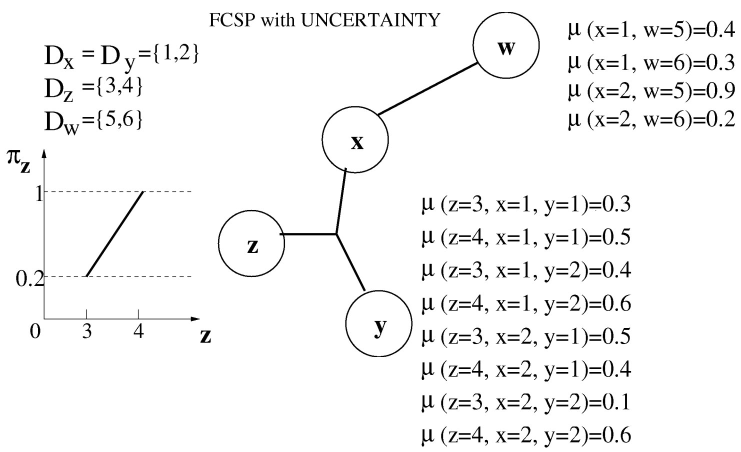

Figure3. Example of Fuzzy CSP with uncertainty.

Notice that the final preferences of the assignments may derive from reasoning on uncertainty parameters (case U) or from reasoning on preferences of the fuzzy constraints appeared in the original problem (case F). In the method of Dubois at al. this kind of information is lost, i.e. you don't know the origin of the final preferences.

For example if you have an assignment d with a pair of preferences $\langle 0.3, 0.8\rangle$ and another one d' with a pair $\langle 0.8, 0.3\rangle$, then, according to this method, d is considered equal to d', forgetting relevant information, like that in d there is a low preference $F = 0.3$, whereas in d' there is a very high preference $F = 0.8$ and that d has a high robustness $U = 0.8$, whereas d' has a low robustness $U = 0.3$.

5 New semantics: risky, safe, diplomatic

The method of Dubois at al. in [2], described in detail in section 4, incorporate uncertainty into fuzzy preferences, but in doing so part of the information is lost, how shown in the last part of the previous section. This method reduces the original problem with uncertain parameters to a fuzzy constraint problem with only decisional variables. Once a solution is found its global preference is computed only looking at such fuzzy constraints, and it is not possible to tell if the final preference of the solution is derived from reasoning on uncertainty or on actual preferences.

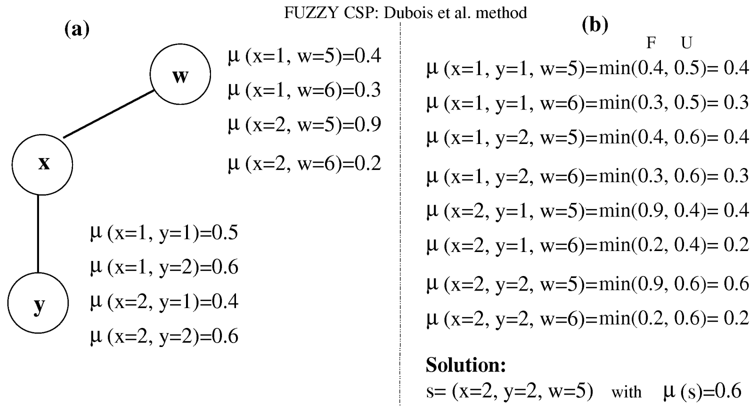

Figure4. This Figure shows how the method proposed in [2] for solving the Fuzzy CSP with an uncertain parameter, illustrated in Figure 3, works. Figure (a) shows the resulting fuzzy constraint on decisional variables X and Y represented by membership function μ obtained applying the procedure given before. Figure (b) shows all Fuzzy CSP assignments with their preference degrees obtained applying standard procedure for solving usual fuzzy CSP.

In order to avoid loss of information, we will propose a new method for solving a Fuzzy CSP with uncertain parameters. The method proposed in [2] eliminates uncertain parameters creating new fuzzy constraints, as explained in section 4, but in doing so it forgets the original preferences associated to the tuple of values of the variables connected with uncertain parameters. We believe that it's important to remember these original preferences, and so in our method, before eliminating uncertain parameters, we *project* the constraints involving uncertain parameters on the particular decisional variables connected to these uncertain parameters. Then we apply the procedure defined in section 4: creation of a new particular fuzzy constraints after elimination of uncertain parameters and resolution of the new generated Fuzzy CSP using standard procedure of solving FCSP. Moreover we think that it's better to *keep separate* all kinds of preferences: those obtained by projection, those obtained by fuzzy constraints reasoning and those gained by reasoning on uncertainty.

Figure 5 shows an application of this new method on the particular Fuzzy CSP with uncertainty illustrated in Figure 3, where there are three decisional variables X, Y, W and an uncertain parameter Z. The uncertain parameter Z is connected by a ternary constraint, C_{XYZ}, to X and Y, whereas W is linked to X by a fuzzy constraint C_{XW}. In Figure 5 (a) there is the new Fuzzy CSP obtained from Fuzzy CSP in Figure 3 after applying these steps:

– *projection* of the ternary constraint C_{XYZ} on the decisional variables X and Y, obtaining a new fuzzy constraint on X an Y, which associates to each pair of value $(x, y) \in X \times Y$ a level of preference which is the maximum

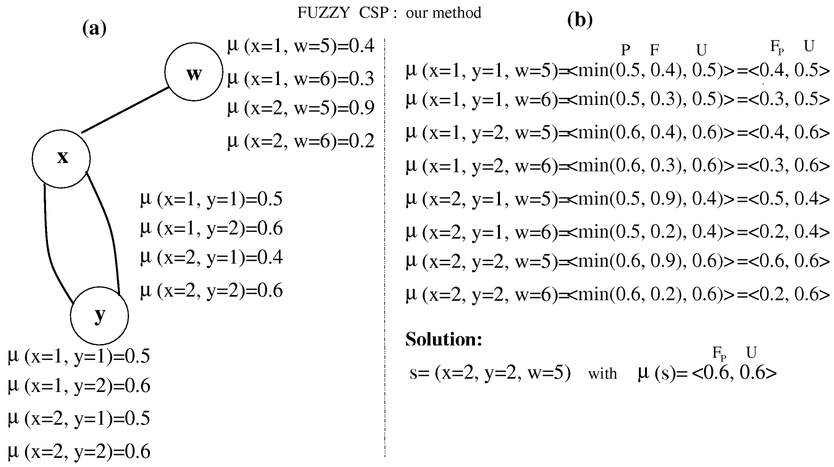

Figure5. This Figure shows the new method, which we propose in order to solve the Fuzzy CSP illustrated in Figure 3 with an uncertain parameter. Figure (a) shows the resulting fuzzy CSP obtained both applying the procedure described in [2] and projecting the original constraint with uncertain parameter Z, C_{XYZ}, on the decisional variables X and Y. Figure (b) shows all Fuzzy CSP assignments with the pair of their preference degrees.

preference of the tuples appeared in the original ternary constraint C_{XYZ}, which have $X = x$ and $Y = y$;

- application of the *procedure* in [2] for reasoning on uncertain parameter. In the new fuzzy CSP there is no longer uncertain parameter Z, but there is a new fuzzy constraint between the decisional variables connected in the original problem to Z. In this new fuzzy constraint the preferences of the pairs of values of X and Y are computed using the procedure described in detail in section 4.

In Figure 5 (b) there are all the tuples of assignments, for the FCSP considered, with a tuple of three preference degrees: the first one (P) is the preference degree obtained after projecting the constraint C_{XYZ} on the variables X and Y, the second one (F) is the preference level gained reasoning on fuzzy constraint C_{XW} of the original problem and the third one is the preference obtained reasoning on uncertain parameter Z, i.e. applying the particular procedure proposed in [2]. In the bottom to the Figure 5 (b) there is the solution of my original problem, which has as preference not a single value, like in [2], but a tuple of values. An assignment of the decisional variables is preferred to another one according to one of the semantics that will describe in the next section. In this particular case the preferred solution is the better one for all these semantics.

Consider the example shown in Figure 2 as a part of a bigger fuzzy constraint problem. Then the preference associated to tuple $(X = 1, Y = 3)$, projecting constraint C_{XYZ} on variables X and Y, is $sup_{z \in Z}\{\mu(z, 1, 3)\} = 0.9$. However the procedure given in [2] assigns to tuple $(X = 1, Y = 3)$ preference $\mu'(X = 1, Y = 3) = 0.3$. Hence any solution s of the whole problem which assigns value 1 to X and 3 to Y will have global preference ≤ 0.3 (there might be another constraint which lowers the preference even further). In this case the preference related to the uncertain information, $\mu'(X = 1, Y = 3) = 0.3$ is lower than the preference obtained projecting the preferences.

We will now prove that this holds whenever the uncertain parameter has in its domain a value z with possibility $\pi_Z(z) = 1$.

Theorem 1. *Consider a fuzzy constraint C on decision variables $X = \{x_1, \ldots, x_n\}$ with domains $D_1, \ldots, D_n$, and uncertain parameters $Z = \{z_1, \ldots, z_k\}$ with domains $A_1, \ldots, A_k$. The knowledge of the uncertain parameters is modeled with the possibility distribution π_Z defined on $A_Z = A_1 \times \cdots \times A_k$. Then if there is at least a tuple $a = (a_1, \ldots, a_k) \in A_1 \times \cdots \times A_k$, such that $\pi_Z(a_1, \ldots, a_k) = 1$ then $\forall \bar{d} = (\bar{d_1}, \ldots, \bar{d_n}) \in D_X = D_1 \times \cdots \times D_n$ we have:*

$$sup_{(a,d) \in A_Z \times D_X | d = (a,d)^{\downarrow D_X} = \bar{d}} \, \mu_R((a, d)) \geq c(sup_{a \in A_Z} min(c(\mu_R(a, \bar{d}), \pi_Z(a)))).$$

Proof. Assume that there is an instantiation $a' \in A_Z$ such that $\pi_Z(a') = 1$. Then $min(c(\mu_R(a', \bar{d})), \pi_Z(a')) = min(c(\mu_R(a', \bar{d})), 1) = c(\mu_R(a', \bar{d}))$. Let's indicate $S = sup_{a \in A_Z} min(c(\mu_R(a, \bar{d}), \pi_Z(a)))$. Then we have $S \geq c(\mu_R(a', \bar{d}))$. This implies that applying the reversing-order $c(p) = 1 - p$ we get $c(S) \leq c(c(\mu_R(a', \bar{d}))) = \mu_R(a', \bar{d})$.
But clearly $\mu_R(a', \bar{d}) \leq sup_{(a,d) \in A_Z \times D_X | d = (a,d)^{\downarrow D_X} = \bar{d}} \, \mu_R((a, d))$ and then $c(S) \leq sup_{(a,d) \in A_Z \times D_X | d = (a,d)^{\downarrow D_X} = \bar{d}} \, \mu_R((a, d))$ $\square$.

Note that this theorem holds also in the context of [2] as well as in our method. It is not redundant in our method even if we perform projection in the constraint involving uncertain parameters.

Notice that if the uncertain parameters are independent the above theorem holds if there is at least an a such that $a = (a_1, \ldots, a_k) \in A_Z$ and $\pi_{Z_j}(a_j) = 1$, $j = 1, \ldots, k$. Summarizing the theorem states that if there is at least an assignment to all the uncertain parameters that has a global possibility of 1, then the preference on the decision variables d obtained considering the uncertain information, which from now on we will indicate with $P_2(d)$, is always smaller or equal to that obtained simply projecting on the decision variables, which will be indicated by $P_1(d)$.

However if all the assignments a for the uncertain parameters have possibility $0 \leq \pi_Z(a) < 1$, $P_1(d)$ can be equal or ordered in either way with respect to $P_2(d)$, as can be seen in the following example.

For simplicity consider a single decision variable X with domain $D_X = \{1\}$ and an uncertain parameter Z with domain $D_Z = \{1, 2\}$. Assume first that the possibility distribution π_Z is such that $\pi_Z(1) = \pi_Z(2) = 0.3$ and that the constraint C_{XZ} is such that $\mu(X = 1, Z = 1) = \mu(X = 1, Z = 2) = 0.5$. Then $P_1(X = 1) = 0.5$ and $P_2(X = 1) = 0.7$. Hence we have $P_1(X = 1) < P_2(X = 1)$.

If instead $\pi_Z(1) = \pi_Z(2) = 0.7$ then we still have that $P_1(X = 1) = 0.5$ but $P_2(X = 1) = 0.5$. Hence $P_1(X = 1) = P_2(X = 1)$.

Finally if we consider the case in which $\mu(X = 1, Z = 1) = 0.3$ and $\mu(X = 1, Z = 2) = 0.4$, while $\pi_Z(1) = 0.8$ and $\pi_Z(2) = 0.3$, then $P_1(X = 1) = 0.4$ but $P_2(X = 1) = 0.3$. Hence $P_1(X = 1) > P_2(X = 1)$.

In the original semantics in [2] the final preference of an assignment d is given taking the *min* of $P_1(d)$ but $P_2(d)$. However consider assignment d_1 associated with $\langle P_1(d_1) = 0.8, P_2(d_1) = 0.3\rangle$ and assignment d_2 associated with $\langle P_1(d_2) = 0.3, P_2(d_2) = 0.8\rangle$. According to the original semantics first of all d_1 and d_2 are equally preferable. This is true even if the meaning of the pair of preferences, $\langle P_1, P_2\rangle$ is different. While for d_1 there is a value for the uncertain parameters, that has a weak possibility of occurring, but such that with d_1 gives a total preference of 0.8, for d_2 there is no such value since the global preference of any solution containing d_2 will be always $\leq P_1(d_2) = 0.3$. According to the semantics we will describe in the next sections, we establish preference orderings that all favor d_1 over d_2, because d_1 has a slight chance of giving a better overall preference of 0.8 and a high possibility of falling into a preference value of 0.3, while d_2 forces a solution with a preference no better than 0.3[1] in all cases.

Moreover the information on whether the minimum comes from P_1 (preferences) or P_2 (uncertainty) is lost.

In this paper we propose new semantics that both allow not to lose the information on where the preferences are coming from and that discriminate tuples that are ranked equal in the original semantics, as mentioned above.

We will consider fuzzy constraints as represented in the semiring based approach for soft constraints. In particular we will deal with the Fuzzy semiring:

$S_{FCSP} = \{[0, 1], max, min, 0, 1\}$, which will be used when reasoning on P_1 and P_2 separately.

6 The new semantics

In order not to lose the information derived from how the preferences are generated, from other preferences for P_1 and from uncertain parameters for P_2, we propose three new semantics that are defined on pairs of preferences of type $\langle P_1(d), P_2(d)\rangle$, where d is an assignment to the decision variables.

Our interpretation is as follows. Given an assignment d to the decision variables:

- $P_1(d)$ is the preference obtained combining (min) all the projections (max) of constraints involving the decision variables connected with the uncertain parameters;
- $P_2(d)$ is the preference applying the reasoning described above which eliminates the uncertain parameters: in particular it represents to what extent it is impossible to have a whatsoever possible value of the uncertain parameters violating the constraint (on X, Y and Z). This means that $1 - P_2(d)$

[1] The preference, in general, might even drop below 0.3.

gives an idea of the risk of hitting a value of Z that is inconsistent with d, hence $P_2(d)$ can be seen as a measure of the *robustness* of d.

We propose three different semantics that represent three different attitudes. Consider two pairs $\langle P_1(d), P_2(d)\rangle = \langle a_1, b_1\rangle$ and $\langle P_1(d'), P_2(d')\rangle = \langle a_2, b_2\rangle$.

6.1 Risky Semantics

The first semantics we propose can be seen as a Lex ordering on pairs $\langle a_i, b_i\rangle$, with the first component as the most important feature. Hence

- if $a_1 > a_2$ then $\langle a_1, b_1\rangle >_R \langle a_2, b_2\rangle$;
- if $a_2 > a_1$ then $\langle a_2, b_2\rangle >_R \langle a_1, b_1\rangle$;
- if $a_1 = a_2$ then
  - if $b_1 > b_2$ then $\langle a_1, b_1\rangle >_R \langle a_2, b_2\rangle$;
  - if $b_2 > b_1$ then $\langle a_2, b_2\rangle >_R \langle a_1, b_1\rangle$;
  - if $b_1 = b_2$ then $\langle a_1, b_1\rangle = \langle a_2, b_2\rangle$.

Informally, the idea is to give more importance to the preference level that can be reached (a higher a_i) considering less important a high risk of being inconsistent (a low robustness b_i).

6.2 Safe Semantics

This semantics follows the opposite attitude with the respect to the previous one: it can be seen as a Lex ordering on pairs $\langle a_i, b_i\rangle$, with the second component as most important feature. Hence:

- if $b_1 > b_2$ then $\langle a_1, b_1\rangle >_S \langle a_2, b_2\rangle$;
- if $b_2 > b_1$ then $\langle a_2, b_2\rangle >_S \langle a_1, b_1\rangle$;
- if $b_1 = b_2$ then
  - if $a_1 > a_2$ then $\langle a_1, b_1\rangle >_S \langle a_2, b_2\rangle$;
  - if $a_2 > a_1$ then $\langle a_2, b_2\rangle >_S \langle a_1, b_1\rangle$;
  - if $a_1 = a_2$ then $\langle a_1, b_1\rangle = \langle a_2, b_2\rangle$.

Informally, the idea is to give more importance to the robustness level that can be reached (a higher b_i) considering less important having a high preference (a high a_i).

6.3 Diplomatic Semantics

Our third semantics aims at giving the same importance to the two aspects of a solution: preference and robustness. In order to do that, it is obtained via the Pareto ordering on pairs $\langle a_i, b_i\rangle$, where all the components have the same importance. Hence:

- if $a_1 \leq a_2$ and $b_1 \leq b_2$ then $\langle a_1, b_1\rangle \leq_D \langle a_2, b_2\rangle$;
- if $a_2 \leq a_1$ and $b_2 \leq b_1$ then $\langle a_2, b_2\rangle \leq_D \langle a_1, b_1\rangle$;

– if $a_1 = a_2$ and $b_1 = b_2$ then $\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$;
– else $\langle a_1, b_1 \rangle \bowtie \langle a_2, b_2 \rangle$.

In this definition $\bowtie$ stands for incomparability. The idea is that a pair is to be preferred to another only if it wins both on preference and robustness, leaving incomparable all the pairs that have one component higher and the other lower. Contrarily to the diplomatic semantics, the other two semantics produce a total order over the solutions.

Let us now consider an example that explains the differences between our semantics and the approach of [2]. Suppose we have two complete assignments, t_1 and t_2, with preference resp. 0.3 and 0.5, and robustness resp. 0.5 and 0.3. Then the method of [2] would say that they are equally good, since it would as representative of both these solutions the minimum of their degrees, that is, 0.3. On the other hand, for our semantics we have the following ordering: $\langle 0.3, 0.5 \rangle <_R \langle 0.5, 0.3 \rangle$ according to Risky; $\langle 0.3, 0.5 \rangle >_S \langle 0.5, 0.3 \rangle$ according to Safe; $\langle 0.3, 0.5 \rangle \bowtie \langle 0.5, 0.3 \rangle$ according to Diplomatic.

7 Future work

We want to understand if, and under which conditions, our approach is generalizable to other classes of soft constraints. We also want to understand if these semantics can be modeled via a semiring structure, thus allowing to remain within the framework of soft constraints also for modeling uncertainty.

References

1. S. Bistarelli, U. Montanari and F. Rossi. Semiring-based Constraint Solving and Optimization. Journal of the ACM, Vol.44, n.2, March 1997.
2. D. Dubois, H. Fargier, H. Prade. Possibility theory in constraint satisfaction problems: handling priority, preference and uncertainty. Applied Intelligence, 6, 287 309, 1996.
3. L.A. Zadeh. Fuzzy sets as a basis for the theory of possibility. Fuzzy Sets and Systems, 13-28, 1978.