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# Uncertainty in soft constraint problems

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**Abstract** Preferences and uncertainty occur in many real-life problems. The theory of possibility is one way of dealing with uncertainty, which allows for easy integration with fuzzy preferences. In this paper we consider an existing technique to perform such an integration and, while following the same basic idea, we propose to generalize it to other classes of preferences and to probabilistic uncertainty while maintaining certain desirable properties.

## 1 Introduction

Preferences and uncertainty occur in many real-life problems. In this paper we are concerned with the coexistence of such concepts in the same problem. In particular, we consider uncertainty that comes from lack of data or imprecise knowledge.

The theory of possibility [6, 10] is one non-probabilistic way of dealing with uncertainty, which allows for easy integration with fuzzy preferences [3]. In fact, both possibilities and fuzzy preferences are values between 0 and 1 associated to events and express the level of plausibility that the event will occur, or its preference.

In our context, we will describe a real-life problem as set of variables with finite domains and a set of soft constraints among subsets of the variables. A variable will be said to be uncertain if we cannot decide its value. In this case, we will associate a possibility degree to each value in its domain, which will tell how plausible it is that the variable will get that value.

Soft constraints [2] allow to express preferences over the instantiations of the variables of the constraints. In particular, fuzzy preferences are values between 0 and 1, which are combined using the *min* operator, and are ordered in such a way that higher values denote better preferences. Probabilistic preferences are similar to fuzzy ones, except that they are combined via multiplication: the goal is to maximize the product of the preferences. Weighted constraints use instead preferences representing costs which are combined by summing them: the goal is to minimize the sum of the weights.

In this paper we consider existing techniques to integrate fuzzy preferences and uncertainty, which use possibility theory [3, 9]. In particular, the approach in [9] allows one to handle uncertainty within a fuzzy optimization engine, and at the same time to observe separately the preference level and the robustness of the complete instantiations. This approach has certain desirable properties, that we describe formally.

We then generalize the approach, and its properties, in order to use it also for other classes of soft constraints, not necessarily fuzzy, by pointing out a sufficient condition for the properties to hold. This allows us to handle the coexistence of preferences and uncertainty in a more general setting. We also consider uncertainty expressed by a probabilistic distribution rather than a possibilistic one, and we show that in this setting the same properties hold.

## 2 Soft constraints

Soft constraints [2] are a very general formalism to describe quantitative preferences. In general, a soft constraint is just pair $\langle def, con \rangle$, where *con* is the set of variables of the constraint (that is, its scope), and $def$ is a function from the Cartesian product of the domains of the variables in *con* to a preference set, say $A$. Therefore $def$ defines the constraint, by associating a level of preference from $A$ to each assignment of values to the variables of the constraint.

Set $A$ can be totally or partially ordered, and its ordering, denoted by $\leq$, can be used to order the assignments of values to variables: assignments corresponding to higher preferences are more preferred. Moreover, a combination operation $\times$ should be defined over $A$, to combine different constraints and generate the preference level of an assignment of values to variables which range over the scopes of several constraints. More precisely, $A$ should have properties similar to a semiring. We will therefore say that a soft constraint is defined *over semiring* $A$. For more details on semiring-based soft constraints, see [2].

A soft constraint problem is usually denoted by a tuple $\langle S, V, C \rangle$ where $S$ is a semiring, $V$ is a set of variables, and $C$ is a set of soft constraints over $S$ whose scopes are subsets of these variables. An optimal solution of a soft constraint problem is an assignment of its variables which is optimal according to the ordering associated to the semiring.

This general description of soft constraints instantiates to several classes of concrete constraints:

- *Fuzzy constraints*: when $A = [0, 1]$, $\leq$ is derived by the *max* operator, and the combination operator is *min*. This means that a fuzzy constraint associates an element between 0 and 1 to each instantiation of its variables, that values closer to 1 denote a higher preference, and that the preferences of two or more constraints are combined by taking their minimum value.
- *Hard constraints*: they can also be described by this framework, by just choosing $A = \{true, false\}$, $\leq$ derived by logical *or* (thus 1 is better than 0), and combination is logical *and*.
- *Weighted constraints*: they are soft constraints where each assignment of values to variables has a weight, and the goal is to minimize the sum of the weights: this can be cast by choosing $A$ as the set of possible weights, by deriving the ordering by the *min* operator, and by using the *sum* as the combination operator.

– *Probabilistic constraints*: they are soft constraints where each assignment is associated to a probability, which informally represents the chance for the assignment to satify the constraint in the real problem. Constraints are then combined by multiplying the associated probabilities, and *max* is used to induce the ordering over preferences: the best solutions have the highest probability.

The concept of fuzzy constraint, as defined above, was originally based on the notion of fuzzy set [5, 7, 10]. A fuzzy set $A$ is a subset of a referential set $U$ whose boundaries are gradual. More formally: the *membership function* $\mu_A$ of a fuzzy set $A$ assigns to each element $u \in U$ its degree of membership $\mu_A(u)$ usually taking values in $[0, 1]$. If $\mu_A(u) = 1$, it means that $u$ belongs to $A$, while $\mu_A(u) = 0$ means that $u$ does not belong to $A$. If $\mu_A(u)$ is between 0 and 1, then it means that $u \in A$ with degree $\mu_A(u)$.

Fuzzy constraints use the notion of fuzzy sets to describe the level of preference of a certain assignment of values to variables. More precisely, a *soft fuzzy constraint* [3] C on variables $\{x_1, \ldots, x_n\}$ is associated with a $fuzzy$ relation R, i.e. a fuzzy subset of $D_1 \times \cdots \times D_n$ of values that more or less satisfy C. A membership function $\mu_R$ is associated with relation R and specifies for each tuple $(d_1, \ldots, d_n) \in D_1 \times \cdots \times D_n$ the level of satisfaction $\mu_R(d_1, \ldots, d_n)$ in a set L, which is totally ordered (e.g. [0,1]). In particular, $\mu_R(d_1, \ldots, d_n) = 1$ if tuple $(d_1, \ldots, d_n)$ totally satisfies C, $\mu_R(d_1, \ldots, d_n) = 0$ if it totally violates C, and $0 < \mu_R(d_1, \ldots, d_n) < 1$ if it partially satisfies C. Moreover, $\mu_R(d_1, \ldots, d_n) > \mu_R(d'_1, \ldots, d'_n)$ means that tuple $(d_1, \ldots, d_n)$ is better than tuple $(d'_1, \ldots, d'_n)$.

In the following we will use two operations on fuzzy constraints [3]: projection and combination. The *projection* of a fuzzy constraint, represented by fuzzy relation $R$ on variables $\{x_1, \ldots, x_k\} \subseteq V(R) = \{x_1, \ldots, x_n\}$, is a fuzzy relation $R^{\downarrow\{x_1,\ldots,x_k\}}$ defined on $\{x_1, \ldots, x_k\}$ such that: $\mu_{R^{\downarrow\{x_1,\ldots,x_k\}}}(d_1, \ldots, d_k) = sup_{\{d=(d_1,\ldots,d_n) | d^{\downarrow\{x_1,\ldots,x_k\}}=(d_1,\ldots,d_k)\}}\ \mu_R(d)$. The *conjunctive combination* of two fuzzy constraints, represented by fuzzy relations $R_i$ and $R_j$, is a fuzzy relation $R_i \otimes R_j$ defined on variables $V(R_i) \cup V(R_j)$ such that: $\mu_{R_i \otimes R_j}(d_1, \ldots, d_k) = min(\mu_{R_i}(d_1, \ldots, d_k)^{\downarrow V(R_i)}, \mu_{R_j}(d_1, \ldots, d_k)^{\downarrow V(R_j)})$ where $\mu_{R_i \otimes R_j}(d_1, \ldots, d_k)$ evaluates to what extent $(d_1, \ldots, d_k)$ satisfies both $C_i$ and $C_j$.

## 3 Possibility theory

A *possibility distribution* [10] is the membership function of a fuzzy set $A$ attached to a single-valued variable $x$. It is denoted $\pi_x = \mu_A$ and represents the set of more or less plausible, mutually exclusive values of $x$. A possibility distribution is similar to a probability density. However, $\pi_x(u) = 1$ only means that $x = u$ is a plausible situation, which cannot be excluded. Thus, a degree of possibility can be viewed as an upper bound of a degree of probability.

Possibility theory encodes incomplete knowledge while probability accounts for random and accurately observed phenomena. In particular, the complete ignorance about $x$ is expressed by $\pi_x(u) = 1$, for all $u \in U$, since in this case all values $u$ are plausible for $x$ and so it is impossible to exclude any of them. Whereas, $\pi_x(\bar{u}) = 1$ for a specific value $\bar{u}$ and $\pi_x(u) = 0$ otherwise, expresses the complete knowledge about $x$, because in this case only the value $\bar{u}$ is plausible for $x$.

The *possibility* of an event "$x \in E$", $E \subseteq U$, denoted by $\Pi(x \in E)$, is formally $\Pi(x \in E) = sup_u min(\pi_x(u), \mu_E(u)) = sup_{u \in E} \pi_x(u)$.

If an event has possibility equal to 1, it means that it is totally possible. However, it could also not happen. Therefore it means that we are completely ignorant about its occurrence. On the contrary, having a possibility equal to 0 means that the event will not happen.

The dual measure of *necessity* of "$x \in E$", denoted by $N(x \in E)$, evaluates the extent to which "$x \in E$" is *certainly* true, i.e. to what extent the proposition "$x \in E$" is implied by the item of information "$x \in A$": $N(x \in E) = inf_u max(c(\pi_x(u)), \mu_E(u)) = inf_{u \notin E}(c(\pi_x(u))) = 1 - \Pi(x \in E^C)$, where $c$ is the order reversing map such that $c(p) = 1 - p$ and $E^C$ is the complement of $E$ in $U$.

$N(x \in E) = 1$ when it is certain that $x \in E$. On the contrary, having necessity equal to 0 means that the event is not necessary at all, although it may happen. In fact, $N(x \in E) = 0$ iff $\Pi(x \in E^C) = 1$.

## 4 Uncertainty in soft constraints

Whereas in usual soft constraint problems all the variables are assumed to be controllable, that is, their value can be decided according to the constraints which relate them to other variables, in many real-world problems uncertain parameters must be used. Such parameters are associated with variables which are not under the user's direct control and thus cannot be assigned. Only Nature will assign them.

Formally, we can define an *uncertain soft constraint problem* as a tuple $\langle S, V_c, V_u, C \rangle$, where $S$ is a semiring, $V_c$ is the set of controllable variables, $V_u$ is the set of uncontrollable variables, and $C$ is the set of soft constraints. The soft constraints in $C$ may involve any subset of variables of $V_c \cup V_u$.

While in a classical soft constraint problem we can decide how to assign the variables to make the assignment optimal, in the presence of uncertain parameters we must assign values to the controllable variables guessing what Nature will do with the uncontrollable variables. So, in this paper an optimal solution for an uncertain soft constraint problem is an assignment of values to the variables in $V_c$ such that, *whatever* Nature will decide for the variables in $V_u$, the overall assignment will be optimal. This is a pessimistic view and other definitions of solutions can be considered [1].

Moreover, the uncontrollable variables can be equipped with additional information on the likelyhood of their values. Such information can be given in several ways, depending on the amount and precision of knowledge we have. In this paper we will consider two ways of expressing such information: possibilities and probabilities. This information can be used to infer new soft constraints over the controllable variables, expressing the compatibility of the controllable parts of the problem with the uncertain parameters, and can be used to change the notion of optimal solution.

The next section describes two existing approaches [3,9] for integrating fuzzy constraints and uncertainty given by possibilities. In both, the original problem is replaced by another one without uncontrollable variables and with new soft constraints depending on the possibilistic distributions. In [9] the two sets of constraints are kept separate, thus allowing for a fine discrimination between preferences and robustness to uncertainty.

## 5 Unifying fuzzy preferences and uncertainty via possibility theory

In [3] it is shown how it is possible to replace a fuzzy constraint involving at least one uncontrollable variable with a fuzzy constraint over controllable variables only. Consider a fuzzy constraint $C$, represented by the fuzzy relation R, which relates a set of controllable variables $X = \{x_1, \ldots, x_n\}$ to a set of uncertain parameters $Z = \{z_1, \ldots, z_k\}$ with domains $A_1, \ldots, A_k$. The knowledge of the uncertain parameters is modeled with the possibility distribution $\pi_Z$ defined on $A_Z = A_1 \times \cdots \times A_k$. The constraint $C$ is considered satisfied by the assignment $d = (d_1, \ldots, d_n) \in D_1 \times \cdots \times D_n$ if, *whatever the values of* $Z$, $z = (z_1, \ldots, z_k)$, $d$ *is compatible with* $z$, i.e., the set of possible values for $z$ is included in $T = (R \otimes \{(d_1, \ldots, d_n)\})^{\downarrow z}$. Therefore $\mu_T(a) = \mu_R(a, d)$ and $\mu'(d) = \mu'_R(d) = N(d\, satisfies\, C) = N(z \in T) = inf_{a \in A_Z} max(\mu_T(a), c(\pi_Z(a)))$ $= c(sup_{a \in A_Z} min(c(\mu_T(a)), \pi_Z(a)))$. If $C$ is a hard constraint, then the formula above still applies, and becomes the following one: $N(d\, satisfies\, C) = inf_{a \notin T = (R \cap \{d\})^{\downarrow D_Z}} c(\pi_Z(a))$.

The method proposed in [3], which we call *Algorithm DFP* (by the name of the authors), for managing uncertainty in a fuzzy CSP, is the following: It starts from an uncertain fuzzy CSP, say $P$. $P$ is then reduced to a fuzzy constraint problem $P'$: all the constraints which link uncertain parameters to decision variables are replaced by fuzzy constraints only among the decision variables. The new preference levels of the decision variables in such new constraints are computed by applying the specific procedure defining $\mu'$. $P'$ has only fuzzy constraints, therefore it can be solved by applying the usual method for solving fuzzy CSPs, i.e. using the *min* operator to combine the constraints and choosing the complete assignments with the highest preference.

In [3] the following property is given:

**Property 1**. $\mu'(d) \geq \alpha$ iff, when $\pi_Z(a) > c(\alpha)$ then $\mu_R(d, a) \geq \alpha$, where $a$ is the actual value of $Z$.

Moreover, from the definition of $\mu'$, the following two properties can also be proven [9].

**Property 2.** Given the possibilities of uncertain parameters, defined by $\pi_Z$, an assignment $d$ to the decision variables $X_1, \ldots, X_k$, and two preference functions $\mu_1$ and $\mu_2$ such that $\mu_1(d,a) \leq \mu_2(d,a)$ for all $a$ assignments to $Z$, then $\mu'_1(d) \leq \mu'_2(d)$.

**Property 3.** Given a preference function $\mu$, an assignment $d$ to the decision variables $X_1, \ldots, X_k$, and two possibility distributions $\pi_1$ and $\pi_2$ on $Z$, such that $\pi_1(a) \geq \pi_2(a)$ for all $a$, then $\mu'_1(d) \leq \mu'_2(d)$, where $\mu'_i$ is the preference function obtained considering $\pi_i$, i=1,2.

By using algorithm DFP, the preference of a complete assignment is the minimum value among all the preferences of the constraints, both the original fuzzy constraints and those obtained via the transformation which eliminates the uncontrollable variables. In other words, the overall preference for a solution is $min(F, U)$, where $F$ is the minimum of the preferences in the initially given fuzzy constraints only on decision variables, and $U$ is the minimum of the preferences of the new fuzzy constraints. This means that a low overall preference may be caused from a low preference in some of the new fuzzy constraints (when $U$ is less than $F$), that is, a low compatibility with the uncertain events, or also from a low preference on some fuzzy constraint initially given only on decision variables (when $F$ is less than $U$).

In [9] these two components ($F$ and $U$) are kept separate, rather than combined with *min*. This is done by performing, for each constraint $c$ involving both decision and uncertain variables ($X$ and $Z$), a projection over the decision variables. This will create a new constraint $c''$ over $X$ where, for each assignment of values to its variables, the preference is computed by assuming the best in the uncertain parameters. Since preferences are combined via the *min* operator, this new constraint will force the overall preference to be no higher than its best preference. Given an assignment to decision variables, we denote with $P$ the minimum preference over these new *projection constraints*. Such a value $P$, combined with preference $F$ given by the initial constraints, defines the new preference $F_P$.

The algorithm presented in [9], called *algorithm* **SP** (from *separation* and *projection*), is the following:

1. It starts from an uncertain fuzzy CSP with fuzzy constraints $C$.
2. All the constraints which link uncertain parameters to decision variables are replaced by fuzzy constraints only among the decision variables. Let us call $C_u$ such new constraints.
3. It computes the projection constraints, say $C_p$.
4. For each complete assignment, it computes its overall preference as a pair $\langle F_P, U\rangle$, where $F_P = min(F, P)$ and $F$, $P$, and $U$ are, respectively, the minimum preference over $C$, $C_p$, and $C_u$.

Let us consider the following example, where we have a complete assignment $d$ with $F = 0.3$, $P = 0.9$, and $U = 0.9$, and another one $d'$ with $F = 0.9$, $P = 0.9$, and $U = 0.3$. According to algorithm DFP, $d$ is considered equally preferred to $d'$ since $d$ and $d'$ have the same preference $min(F, U) = 0.3$. However, $d$ and $d'$ differ on both preference and robustness.

The approach based on algorithm **SP** preserves such a difference, by defining various semantics which exploit both elements of the pair $\langle F_P = min(F, P), U \rangle$ to deduce a solution ordering.

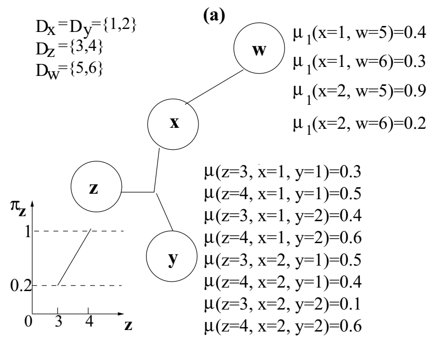


**Figure1.** An uncertain soft CSP.

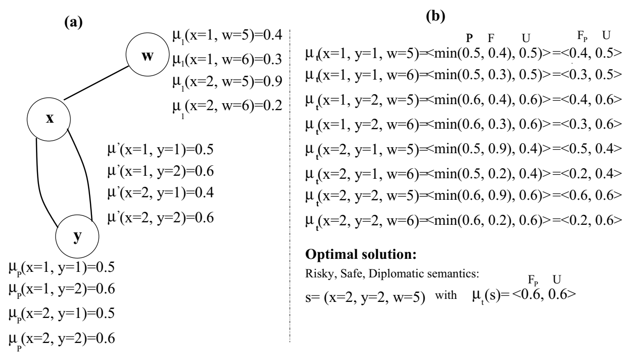


**Figure2.** Result of algorithm **SP** applied to the uncertain soft CSP in Figure 1, seen as an uncertain fuzzy CSP.

Figure 1 shows a soft CSP with uncertainty. There are three decision variables $(X, Y, W)$, one uncertain variable $(Z)$, and two constraints: $C_{XYZ}$ with function $\mu$ and $C_{XW}$ with function $\mu_1$. The possibility distribution $\pi_Z$ describes the plausibility of $Z$.

Figure 2 (a) shows the fuzzy CSP obtained from the soft CSP in Figure 1 seen as an uncertain fuzzy CSP, after applying step 2 and step 2 of algorithm **SP**. In Figure 2 (b) there are all the complete assignments to decision variables of the fuzzy CSP, with the overall preference $\mu_t$.

Given an assignment $d$ to the decision variables and the pair $\langle F_P, U\rangle$ computed as described above, $F_P$ tells us how much $d$ is preferred by the constraints, while $U$ represents to what extent it is impossible to have a possible value of the uncertain parameters violating the constraints involving them. This means that $1 - U$ gives an idea of the risk of hitting a value of uncertain parameter that is inconsistent with $d$, hence $U$ can be seen as a measure of the *certainty* (or *robustness*) of $d$.

$U$ is computed as in [3], so Properties 1, 2 and 3 still hold. We recall that these properties state that $U$ can increase in the following two cases: when the possibilities of the uncertain parameters remain fixed and the preferences of the constraints involving them increase, or when preferences are fixed but possibilities decrease.

Consider two solutions $d$ and $d'$ and the corresponding pairs of values $\langle F_P(d), U(d)\rangle = \langle a_1, b_1\rangle$ and $\langle F_P(d'), U(d')\rangle = \langle a_2, b_2\rangle$. The first semantics proposed in [9], called *Risky*, can be seen as a Lex ordering on pairs $\langle a_i, b_i\rangle$, with the first component as the most important feature. Informally, the idea is to give more importance to the preference level that can be reached in the best case (a higher $a_i$) considering less important a high risk of being inconsistent (a low certainty $b_i$).

The second semantics, called *Safe*, follows the opposite attitude with the respect to the previous one: it can be seen as a Lex ordering on pairs $\langle a_i, b_i\rangle$, with the second component as most important feature. Informally, the idea is to give more importance to the certainty level that can be reached (a higher $b_i$) considering less important having a high preference (a high $a_i$).

The third semantics, called *Diplomatic*, aims at giving the same importance to the two aspects of a solution: preference and certainty. This is obtained via the Pareto ordering on pairs $\langle a_i, b_i\rangle$, where both components have the same importance. The idea is that a pair is to be preferred to another only if it wins both on preference and certainty, leaving incomparable all the pairs that have one component higher and the other lower. Contrarily to the diplomatic semantics, the other two semantics produce a total order over the solutions.

In the example in Figure 2, solution $(x = 2, y = 2, w = 5)$ is optimal for all three semantics. In general, the comparison among the orders induced by the three semantics of [9] and the one in [3] can be seen in Table 1.

**Table1.** Comparison between the ordering in [3] and in [9].

| [3] | Risky | Safe | Dipl. |
|---|---|---|---|
| $=$ | $<,>,=$ | $<,>,=$ | $<,>,=,\bowtie$ |
| $>$ | $<,>$ | $<,>$ | $>,\bowtie$ |
| $<$ | $<,>$ | $<,>$ | $<,\bowtie$ |

## 6 A generalized approach

The methods described in the previous sections can handle only uncertainty in fuzzy constraints. In the following of the paper we will extend the method in [9] to other classes of soft constraints and to probabilistic uncertainty. In particular, we will consider the combinations of fuzzy, probabilistic, and weighted soft constraints with either possibilistic or probabilistic uncertainty.

To do this, we now redefine **SP** for generic soft constraints and generic uncertainty. Then we will define the extensions above as instances of this general framework.

We recall that in Fuzzy CSPs with uncertainty the preferences of the new constraints obtained removing the uncertain parameters are computed via the formula $\mu'(d) = inf_{a \in A_Z} max(\mu_R(d,a), c(\pi_Z(a)))$, where $c$ is the order reversing map such that $c(p) = 1 - p$ and $\pi_Z$ is the possibilistic distribution.

Let us consider any semiring $S = \{A, +, \times, 0, 1\}$, where $\leq_S$ is the semiring ordering on $A$ (we denote incomparability with $\bowtie_S$). The formula above can be generalized to deal with any semiring as follows:

$$\mu'(d) = inf_{a \in A_Z}(\mu_R(d,a) + c(\pi_Z(a)))$$

where

- $+$ refer to the one of the semiring operations, and $inf$ is one of the bottom elements of $A_Z$, i.e. an element such that $\forall a' \in A_Z$ with $a' \neq a$ then $a' >_S a$ or $a' \bowtie_S a$.
- $[0, 1] \subseteq A$;
- $c$ is an bijection from [0,1] to [0,1] such that, for each $a_1, a_2 \in [0, 1]$, $a_1 \leq a_2$ if and only if $c(a_1) \geq_S c(a_2)$; moreover, $c(c(a)) = a$ for all $a$. We will say that $c$ is an order-reversing map w.r.t. semiring $S$;
- $\pi_Z$ is a possibilistic or probabilistic distribution.

Notice that, by generalizing the formula, we do not change the set of values for the possibilities, which remains $[0, 1]$. When working with other classes of soft constraints rather than fuzzy CSPs, functions $\mu_R$ associates a preference from set $A$ to an assignment. In particular, $\mu_R$ is the preference function of the soft constraint $R$.

The algorithm **SP** presented in [9], can be generalized as follows:

1. It starts from an uncertain soft CSP $\langle S, V_c, V_u, C\rangle$;
2. All the constraints which link variables in $V_u$ to variables in $V_c$ are replaced by soft constraints defined by $\mu'$, described above, only among variables in $V_c$. Let us call $C_u$ such new constraints.
3. All the constraints which link variables in $V_u$ to variables in $V_c$ are used to compute their projection over variables in $V_c$. The constraints obtained in this way are called $C_p$.
4. For each assignment of the variables in $V_c$, it computes its overall preference as the pair $\langle F_P, U\rangle$, where $F_P = F \times P$ and $F$, $P$, and $U$ are, respectively, the preference of the assignment over the constraints in $C$, $C_p$, and $C_u$.

We recall that, given a soft constraint $R$, defined on variables $X$ and $Z$ its projection over $X$ is defined by $\mu_P(d) = \sum_{a \in A_Z} \mu_R(d,a)$. In the following we show that properties $1,2,3$ hold for generic soft constraints where the set $A$ of the semiring is *totally ordered*.

**General Property 1:** $\mu'(d) \geq_S \alpha$ if and only if, when $\pi_Z(a) > c(\alpha)$, then $\mu_R(d,a) \geq_S \alpha$.

*Proof.* Let us recall that $\mu'(d) \geq_S \alpha$ iff, by definition of $\mu'$, $inf_{a \in A_Z}(\mu_R(d,a) + c(\pi_Z(a))) \geq_S \alpha$.

($\Rightarrow$) Assume that $inf_{a \in A_Z}(\mu_R(d,a)+c(\pi_Z(a))) \geq_S \alpha$. Since $inf_{a \in A_Z}(\mu_R(d,a)+c(\pi_Z(a))) \leq_S (\mu_R(d,a)+c(\pi_Z(a)))$ for all $a \in A_Z$, then $(\mu_R(d,a)+c(\pi_Z(a))) \geq_S \alpha$ for all $a \in A_Z$. By hypothesis $\pi_Z(a) > c(\alpha)$. Since $c$ is an order-reversing map such that $c(c(p)) = p$, we have $c(\pi_Z(a)) <_S \alpha$. Since $A$ is totally ordered, for any two elements of the semiring we have $a + b = a$ or $b$, then $\mu_R(d,a) = (\mu_R(d,a) + c(\pi_Z(a))) \geq_S \alpha$.

($\Leftarrow$) Assume that, for every $a$ such that $\pi_Z(a) > c(\alpha)$, we have $\mu_R(d,a) \geq_S \alpha$. Then, since $c(\pi_Z(a)) <_S \alpha$ and $\mu_R(d,a) \geq_S \alpha$ for such $a$, then $(\mu_R(d,a) + c(\pi_Z(a))) \geq_S \alpha$. For all $a$ such that $\pi_Z(a) \leq c(\alpha)$, we have $c(\pi_Z(a)) \geq_S \alpha$ and so $(\mu_R(d,a) + c(\pi_Z(a))) \geq_S \alpha$. Thus for all $a$, $(\mu_R(d,a) + c(\pi_Z(a))) \geq_S \alpha$. Therefore, since the $inf$ among the elements of the semiring is one of these elements $inf_{a \in A_Z}(\mu_R(d,a) + c(\pi_Z(a))) \geq_S \alpha$, i.e., $\mu'(d) \geq_S \alpha$. $\square$

Notice that, if $A$ is *partially ordered*, this property doesn't hold. However, two slightly weaker properties can be proved.

**Weak general Property 1 ($\not\leq$):** $\mu'(d) \not\leq_S \alpha$ if and only if, when $\pi_Z(a) \geq c(\alpha)$, then $\mu_R(d,a) \not\leq_S \alpha$.

*Proof.* Let us recall that $\mu'(d) \not\leq_S \alpha$ iff, by definition of $\mu'$, $inf_{a \in A_Z}(\mu_R(d,a) + c(\pi_Z(a))) \not\leq_S \alpha$.

($\Rightarrow$) Assume that $inf_{a \in A_Z}(\mu_R(d,a) + c(\pi_Z(a))) \not\leq_S \alpha$, where $\not\leq_S$ means that can be $>_S$ or incomparable $\bowtie_S$, then $(\mu_R(d,a) + c(\pi_Z(a))) \not\leq_S \alpha$ for all $a \in A_Z$. In fact, if $(\mu_R(d,a) + c(\pi_Z(a))) \leq_S \alpha$ for some $a$ then $inf_{a \in A_Z}(\mu_R(d,a) + c(\pi_Z(a))) \leq_S \alpha$, that is a contradiction. By hypothesis $\pi_Z(a) \geq c(\alpha)$. Since $c$ is an order-reversing map such that $c(c(p)) = p$, we have $c(\pi_Z(a)) \leq_S \alpha$ and so, since $(\mu_R(d,a) + c(\pi_Z(a))) \not\leq_S \alpha$ for all $a \in A_Z$, $\mu_R(d,a) \not\leq_S \alpha$. In fact, suppose $\mu_R(d,a) \leq_S \alpha$. Then $(\mu_R(d,a) + c(\pi_Z(a))) \leq_S \alpha + \alpha$ for monotonicity of $+$, and since, for idempotency of $+$, $\alpha + \alpha = \alpha$, and so $(\mu_R(d,a) + c(\pi_Z(a))) \leq_S \alpha$, that is a contradiction since for hypothesis $\mu_R(d,a) \not\leq_S \alpha$.

($\Leftarrow$) Assume that, for every $a$ such that $\pi_Z(a) \geq c(\alpha)$, we have $\mu_R(d,a) \not\leq_S \alpha$. Then, since $c(\pi_Z(a)) \leq_S \alpha$ and $\mu_R(d,a) \not\leq_S \alpha$ for such $a$, then $(\mu_R(d,a) + c(\pi_Z(a))) \not\leq_S \alpha$. In fact, if $(\mu_R(d,a) + c(\pi_Z(a))) \leq_S \alpha$, then we have for monotonicity and idempotency of $+$, $\mu_R(d,a) \leq_S (\mu_R(d,a)+c(\pi_Z(a))) \leq_S \alpha+\alpha \leq_S \alpha$, that is a contradiction since $\mu_R(d,a) \not\leq_S \alpha$. Moreover, for all $a$ such that $\pi_Z(a) < c(\alpha)$, we have $c(\pi_Z(a)) >_S \alpha$ and so $(\mu_R(d,a) + c(\pi_Z(a))) >_S \alpha$. Thus for all $a$, $(\mu_R(d,a) + c(\pi_Z(a))) \not\leq_S \alpha$. Therefore, since every $inf$ among the elements of the semiring is one of these elements, $inf_{a \in A_Z}(\mu_R(d,a) + c(\pi_Z(a))) \not\leq_S \alpha$, i.e., $\mu'(d) \not\leq_S \alpha$. $\square$

**Weak general Property 1 ($\not<$):** $\mu'(d) \not<_S \alpha$ if and only if, when $\pi_Z(a) > c(\alpha)$, then $\mu_R(d,a) \not<_S \alpha$.

*Proof.* Very similar to the proof above.

Let us recall that $\mu'(d) \not<_S \alpha$ iff, by definition of $\mu'$, $inf_{a \in A_Z}(\mu_R(d,a) + c(\pi_Z(a))) \not<_S \alpha$.

($\Rightarrow$) Assume that $inf_{a \in A_Z}(\mu_R(d,a) + c(\pi_Z(a))) \not<_S \alpha$, where $\not<_S$ means that can be $\geq_S$ or incomparable $\bowtie_S$, then $(\mu_R(d,a) + c(\pi_Z(a))) \not<_S \alpha$ for all $a \in A_Z$. In fact, if $(\mu_R(d,a) + c(\pi_Z(a))) <_S \alpha$ for some $a$ then $inf_{a \in A_Z}(\mu_R(d,a) + c(\pi_Z(a))) <_S \alpha$, that is a contradiction. By hypothesis $\pi_Z(a) > c(\alpha)$. Since $c$ is an order-reversing map such that $c(c(p)) = p$, we have $c(\pi_Z(a)) <_S \alpha$ and so, since $(\mu_R(d,a) + c(\pi_Z(a))) \not<_S \alpha$, $\mu_R(d,a) \not<_S \alpha$. In fact, suppose $\mu_R(d,a) <_S \alpha$. Then $(\mu_R(d,a) + c(\pi_Z(a))) <_S \alpha + \alpha$ for monotonicity of $+$, and for idempotency of $+$, $\alpha + \alpha = \alpha$ and so $(\mu_R(d,a) + c(\pi_Z(a))) <_S \alpha$, that is a contradiction since for hypothesis $\mu_R(d,a) \not<_S \alpha$.

($\Leftarrow$) Assume that, for every $a$ such that $\pi_Z(a) > c(\alpha)$, we have $\mu_R(d,a) \not<_S \alpha$. Then, since $c(\pi_Z(a)) <_S \alpha$ and $\mu_R(d,a) \not<_S \alpha$ for such $a$, then $(\mu_R(d,a) + c(\pi_Z(a))) \not<_S \alpha$. In fact, if $(\mu_R(d,a) + c(\pi_Z(a))) <_S \alpha$, then we have for monotonicity and idempotency of $+$, $\mu_R(d,a) <_S (\mu_R(d,a) + c(\pi_Z(a))) <_S \alpha + \alpha = \alpha$, that is a contradiction since $\mu_R(d,a) \not<_S \alpha$. Moreover, for all $a$ such that $\pi_Z(a) \leq c(\alpha)$, we have $c(\pi_Z(a)) \geq_S \alpha$ and so $(\mu_R(d,a) + c(\pi_Z(a))) \geq_S \alpha$. Thus for all $a$, $(\mu_R(d,a) + c(\pi_Z(a))) \not<_S \alpha$. Therefore, since every $inf$ among the elements of the semiring is one of these elements, $inf_{a \in A_Z}(\mu_R(d,a) + c(\pi_Z(a))) \not<_S \alpha$, i.e., $\mu'(d) \not<_S \alpha$. $\Box$

In the following we will show that properties $2, 3$ hold for problems with generic soft constraints and uncertainty described by $\pi_Z$ that can be a possibilistic or a probabilistic distribution.

**General Property 2:** Given $\pi_Z$ and two preference functions $\mu_1$ and $\mu_2$ such that $\mu_1(d,a) \leq_S \mu_2(d,a)$ for all $a$ assignments to $Z$, then $\mu'_1(d) \leq_S \mu'_2(d)$, where $\mu'_i$ is the preference function obtained considering $\mu_i$, i=1,2.

*Proof.* We recall that $\mu'_1(d) = inf_{a \in A_z}(\mu_1(d,a) + c(\pi_Z(a)))$ and $\mu'_2(d) = inf_{a \in A_z}(\mu_2(d,a) + c(\pi_Z(a)))$. Since $\mu_1(d,a) \leq_S \mu_2(d,a)$, $(\mu_1(d,a) + c(\pi_Z(a))) \leq_S (\mu_2(d,a) + c(\pi_Z(a)))$ by monotonicity of $+$. Then we have $inf_{a \in A_z}(\mu_1(d,a) + c(\pi_Z(a))) \leq_S inf_{a \in A_z}(\mu_2(d,a) + c(\pi_Z(a)))$, i.e. $D \leq_S E$ if we call $D = inf_{a \in A_z} D_a$, where $D_a = (\mu_1(d,a) + c(\pi_Z(a)))$ and $E = inf_{a \in A_z} E_a$, where $E_a = (\mu_2(d,a) + c(\pi_Z(a)))$. If we suppose that $D >_S E$, then since $E \geq_S D_a$ for some $a$, $D >_S E \geq_S D_a$ for some $a$, that is a contradiction since $D \leq_S D_a$, for all $a$ since it is the $inf$. Thus $\mu'_1(d) \leq_S \mu'_2(d)$. $\Box$

**General Property 3:** Given a preference function $\mu$ and two distributions $\pi_1$ and $\pi_2$ on $Z$, such that $\pi_1(a) \geq \pi_2(a)$ for all $a$, then $\mu'_1(d) \leq_S \mu'_2(d)$, where $\mu'_i$ is the preference function obtained considering $\pi_i$, i=1,2.

*Proof.* We recall that $\mu'_1(d) = inf_{a \in A_Z}(\mu(d,a) + c(\pi_1(a)))$, $\mu'_2(d) = inf_{a \in A_Z}(\mu(d,a) + c(\pi_2(a)))$ and that $c$ is an order-reversing map, that is, if $\pi_1(a) \geq \pi_2(a)$, then if $c(\pi_1(a)) \leq_S c(\pi_2(a))$. By monotonicity of $+$, we can conclude that $(\mu(d,a) + c(\pi_1(a))) \leq_S (\mu(d,a) + c(\pi_2(a)))$. For the same reasoning explained in the proof above $inf_{a\in A_Z}(\mu(d,a)+c(\pi_1(a))) \leq_S inf_{a\in A_Z}(\mu(d,a)+c(\pi_2(a)))$. Thus $\mu'_1(d) \leq_S \mu'_2(d)$. □

Notice that, if $A$ is *partially ordered*, general property 2 and 3 assume a slightly weaker form. The only change in the statement is that $\mu'_1(d) \leq_S \mu'_2(d)$ is replaced with $\mu'_1(d) \not>_S \mu'_2(d)$. The proofs of these properties are equal to the corresponding ones in the total order case, where the only difference is that you have replaced $inf_{a\in A_Z}(\mu(d,a)+c(\pi_1(a))) \leq_S inf_{a\in A_Z}(\mu(d,a)+c(\pi_2(a)))$ with $inf_{a\in A_Z}(\mu(d,a)+c(\pi_1(a))) \not>_S inf_{a\in A_Z}(\mu(d,a)+c(\pi_2(a)))$.

Since these properties hold in general, then we can safely and effectively handle problems with many kinds of soft constraints as well as uncertain possibilistic or probabilistic variables.

The generic semantics are defined like those ones in [9], except that $\leq$ is replaced by $\leq_S$.

## 7 Uncertain Probabilistic CSPs

In several real-life scenarios, fuzzy constraints are not the ideal setting. In fact, they suffer for the well-known drawning effect which makes solutions with the same minimum preference but very different higher preferences not distinguished.

To avoid this problem, it can be useful to combine preferences by multiplying them rather than taking their minimum value. This is what happens in probabilistic CSPs (PCSPs) [8]. In a PCSP, variable assignments have associated preferences, and the goal is to maximize the product of all such preferences. Therefore, the semiring to be used is $S = \{[0,1], max, \times, 0, 1\}$.

To make sure that the desired properties 1, 2, and 3 hold in this setting, we just need to check whether the assumptions we made are met: $[0,1] \subseteq A$ and $c$ is an bijection from [0,1] to [0,1] such that, for each $a_1, a_2 \in [0,1]$, $a_1 \leq a_2$ if and only if $c(a_1) \geq_S c(a_2)$; moreover, $c(c(a)) = a$ for all $a$.

The first assumption is trivially true since $A = [0,1]$. As for the second one, we consider $c(x) = 1 - x$ for all $x$, which satisfies the order-reversing property. In fact, given $a_1, a_2 \in [0,1]$, with $a_1 \leq a_2$ we have $1 - a_1 \geq_S 1 - a_2$, since $max(1-a_1, 1-a_2) = 1 - a_1$.

In this setting, we have

$$\mu'(d) = min_{a\in A_Z} max(\mu_R(d,a), 1 - \pi_Z(a)).$$

Figure 3 shows the result of applying **SP** to the problem in Figure 1, seen as an uncertain PCSP. In this case, the algorithm is instantiated with $\times_S = \times$, $+_S = max$, and $c(p) = 1-p$. In particular, Figure 3 (a) shows the resulting probabilistic CSP obtained by the algorithm, while Figure 3 (b) shows all complete assignments, together with the associated pair.

## 8 Uncertain Weighted CSPs

In several situations where neither fuzzy nor probabilistic constraints are ideal, weighted constraints can be useful to model preferences. For example, this is usually done when dealing with costs which are naturally combined by a sum.

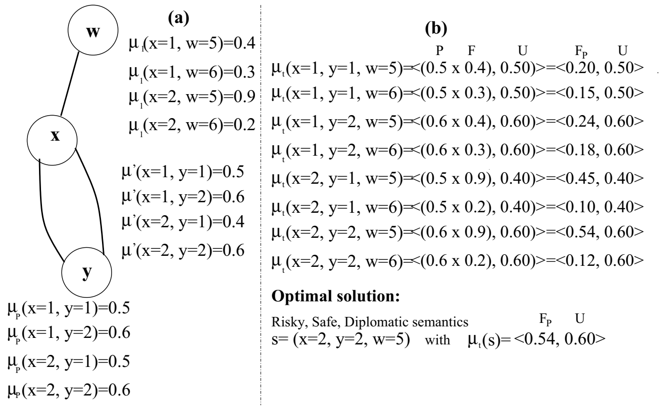


**Figure3.** Result of algorithm **SP** for the uncertain soft CSP in Figure 1, seen as an uncertain PCSP.

In this setting, preferences are penalties (or costs) to be added, and the best solutions are those with the smallest preference. Thus operators $+$ and $min$ are the instantiation of the operators $\times$ and $+$ of the general case. Therefore, the semiring to be used is $S = \{\mathcal{R}^+, min, +, +\infty, 0\}$.

The three desired property hold if we choose $c$ as the identity map. In fact, $[0,1] \subseteq \mathcal{R}^+$. Moreover, $c$ is an order-reversing map w.r.t semiring $S$. In fact, given $a_1, a_2 \in [0,1]$ such that $a_1 \leq a_2$, we have $a_1 \geq_S a_2$ since $min(a_1, a_2) = a_1$.

The instantiated formula for $\mu'$ is then

$$\mu'(d) = max_{a \in A_z} min(\mu_R(d,a), \pi_Z(a))).$$

Figure 4 shows how algorithm **SP** works on weighted CSP with possibilitic uncertainty, as the one in Figure 1 (where preferences are interpreted as costs). Figure 3 (a) shows the resulting weighted CSP, while Figure 3 (b) shows all complete assignments, together with the associated pair. In this problem the optimal solution obtained using the *Risky* semantics is different from the one obtained using the *Safe* semantics, whereas *Diplomatic* considers them both optimal.

## 9 Probabilistic or possibilistic uncertainty

In the previous sections we have considered different classes of soft constraints with uncertainty described via a possibilitic or a probabilistic distribution. Here we consider the case where a possibilistic distribution is available only for some uncertain variables, while for others we have a probabilistic distribution.

If there are no constraints connecting both possibilistic and probabilistic variables, then we can apply the same algorithm as above, and the three properties hold. However, consider the situation where a constraint involves uncertain variables over which a possibility distribution is available, and also uncertain

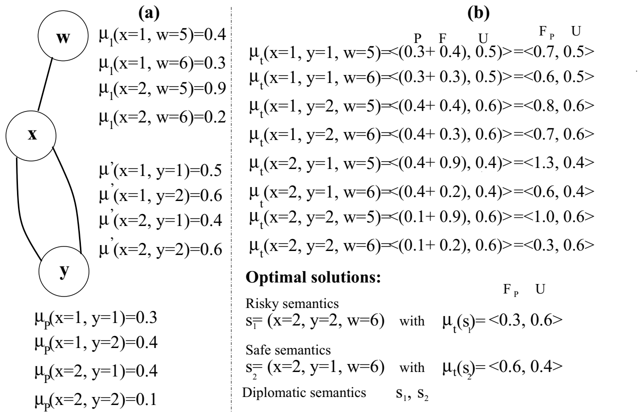


**Figure4.** Result of algorithm **SP** for the uncertain soft CSP in Figure 1, seen as an uncertain weighted CSP.

variables over which we have a probabilistic distribution. In this case, we could get to the usual setting by replacing possibilities with probabilities, or vice versa. In [4] it is presented a way to do this. Thus we could use such a method to obtain only one kind of distribution and then use our approach.

However, transforming a probability into a possibility distribution we loose information, and solutions have a lower robustness. In fact, using property 3, it is possible to see that, if we use possibilities, which are higher than probabilities, we get a smaller robustness value. Thus we can say that the robustness value obtained in this way is a lower bound to the certainty that the values of the decision variables are compatible with the uncertain variables. On the other hand, if we transform possibilities into probabilities, we get smaller values, and thus by property 3 a higher robustness value, which can be seen as an upper bound to the certainty degree of a solution.

Thus, in the presence of constraints involving both possibilistic and probabilistic uncertainty, we can still use the same approach except that we work with upper and lower bounds for the robustness of the solutions.

To be able to use the same three semantics defined above, we need to define an ordering over intervals, since now the robustness is described by an interval. A possible ordering defines two intervals incomparable if one is strictly contained in the other one, and both lower and upper bounds are different. For example, $\langle 0.2, 0.5\rangle$ is incomparable with $\langle 0.1, 0.6\rangle$ but it is not incomparable with $\langle 0.1, 0.4\rangle$, nor with $\langle 0.2, 0.6\rangle$. In all other cases, the two intervals are ordered. More precisely, $\langle l_1, u_1\rangle$ is better than $\langle l_2, u_2\rangle$ if they are different and $l_1 \geq l_2$ and $u_1 \geq u_2$.

Notice that this ordering is partial over the robustness values, while before we had a total order. This yields more incomparability in the ordering over solutions induced by each of the semantics. Thus also the Risky and Safe semantics will induce partial orders.

## 10 Future work

We plan to develop a solver that can handle problems with several classes of soft constraints, together with uncertainty expressed via possibility or probability distributions. The solver will be able to generate orderings according the three semantics proposed in this paper as well as others that we will define by following different optimistic or pessimistic approaches.

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