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EQUITY, FAIRNESS AND MULTICRITERIA OPTIMIZATION*

Introduction

Equity or fairness issues appear in many decision models of Operations Research. Especially models dealing with allocation of resources try to achieve some fairness of allocation patterns [9]. More generally, the models related to the evaluation of various systems which serve many users and the quality of service for every individual user defines the criteria. This applies among others to networking where a central issue is how to allocate bandwidth to flows efficiently and fairly [1]. The issue of equity is widely recognized in location analysis of public services, where the clients of a system are entitled to fair treatment according to community regulations. In such problems, the decisions often concern the placement of a service center or other facility in a position so that the users are treated in an equitable way, relative to certain criteria [13]. Moreover, uniform individual outcomes may be associated with some events rather than physical users, like in many dynamic optimization problems where uniform individual criteria represent a similar event in various periods and all they are equally important.

Fairness is, essentially, an abstract socio–political concept that implies impartiality, justice and equity [23]. Nevertheless, fairness was usually quantified with the so–called inequality measures to be minimized [18]. Unfortunately, direct minimization of typical inequality measures (especially relative ones) contradicts the maximization of individual outcomes and it may lead to inferior decisions. Recently, several research publications relating the fairness and equity concepts to the multiple criteria optimization methodology have appeared [4,7,9,10,13]. Finally, the novel and distinct mathematical approach denoted by equitable efficiency has been developed to provide solutions to these examples of multiple criteria optimization [6]. The concept of equitably efficient solution is a specific refinement of the Pareto-optimality. This paper deals with generation techniques for equitably efficient solutions to multiple criteria optimization problems.

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1. Equity and fairness

The generic decision problem, we consider, may be stated as follows. There is given a set I of m services (users, clients). There is also given a set Q of feasible decisions. For each service $i \in I$ a function $f_i(\mathbf{x})$ of the decision $\mathbf{x}$ has been defined. This function, called the individual objective function, measures the outcome (effect) $y_i = f_i(\mathbf{x})$ of the decision for service i. An outcome usually expresses the service quality. However, outcomes can be measured (modeled) as service time, service costs, service delays as well as in a more subjective way. In typical formulations a larger value of the outcome means a better effect (higher service quality or client satisfaction). Otherwise, the outcomes can be replaced with their complements to some large number. Therefore, without loss of generality, we can assume that each individual outcome y_i is to be maximized which results in a multiple criteria maximization model.

$$\max \{\mathbf{f}(\mathbf{x}) \; : \; \mathbf{x} \in Q\} \tag{1}$$

where

$\mathbf{f}(\mathbf{x})$	is a vector–function that maps the decision space $X = R^n$ into the criterion space $Y = R^m$,
$Q \subset X$	denotes the feasible set,
$\mathbf{x} \in X$	denotes the vector of decision variables.

Model (1) only specifies that we are interested in maximization of all objective functions f_i for $i \in I = \{1, 2, \ldots, m\}$. In order to make it operational, one needs to assume some solution concept specifying what it means to maximize multiple objective functions.

Typical solution concepts for multiple criteria problems are defined by aggregation (or utility) functions $g : Y \to R$ to be maximized. Thus the multiple criteria problem (1) is replaced with the maximization problem

$$\max \{g(\mathbf{f}(\mathbf{x})) \; : \; \mathbf{x} \in Q\} \tag{2}$$

In order to guarantee the consistency of the aggregated problem (2) with the maximization of all individual objective functions in the original multiple criteria problem (or Pareto-optimality of the solution), the aggregation function must be strictly increasing with respect to every coordinate, i.e., for all $i \in I$,

$$g(y_1, \ldots, y_{i-1}, y'_i, y_{i+1}, \ldots, y_m) < g(y_1, y_2, \ldots, y_m) \tag{3}$$

whenever $y'_i < y_i$.

The simplest aggregation functions commonly used for the multiple criteria problem (1) are defined as the sum of outcomes

$$g(\mathbf{y}) = \sum_{i=1}^{m} y_i \tag{4}$$

or the worst outcome

$$g(\mathbf{y}) = \min_{i=1,\dots,m} y_i. \tag{5}$$

The sum (4) is a strictly increasing function while the minimum (5) is only nondecreasing. Therefore, the aggregation (2) using the sum of outcomes always generates a Pareto-optimal solution while the maximization of the worst outcome may need some additional refinement.

Equity is, essentially, an abstract socio–political concept, but it is usually quantified with the so–called inequality measures to be minimized. Inequality measures were primarily studied in economics [18] while recently they become very popular tools in Operations Research. For instance, Marsh and Schilling [10] describe twenty different measures proposed in the literature to gauge the level of equity in facility location alternatives. The simplest inequality measures are based on the absolute measurement of the spread of outcomes, like the mean (absolute) difference

$$D(\mathbf{y}) = \frac{1}{2m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} |y_i - y_j| \tag{6}$$

or the maximum (absolute) difference

$$R(\mathbf{y}) = \frac{1}{2} \max_{i,j=1,\dots,m} |y_i - y_j|. \tag{7}$$

In most application frameworks better intuitive appeal may have inequality measures related to deviations from the mean outcome like the mean (absolute) deviation

$$\delta(\mathbf{y}) = \frac{1}{2m} \sum_{i=1}^{m} |y_i - \mu(\mathbf{y})|. \tag{8}$$

In economics one usually considers relative inequality measures normalized by mean outcome. Among many inequality measures perhaps the most commonly accepted by economists is the Gini coefficient, which is the relative mean difference. One can easily notice that direct minimization of typical inequality measures (especially the relative ones) may contradict the optimization of individual outcomes. As pointed out by Erkut [2], it is rather a common flaw of all the relative inequality measures that while moving away from the spatial units to be serviced one gets better values of the measure as the relative distances become closer to one–another. As an extreme, one may consider an unconstrained continuous (single–facility) location problem and find that the facility located at (or near) infinity will provide (almost) perfectly equal service (in fact, rather lack of service) to all the spatial units. Unfortunately, these flaws of the inequality measure minimization remains also valid when the inequality measure is added as an additional criterion [13].

In order to guarantee fairness (equitability) of the solution concept (2), additional requirements on the class of aggregation (utility) functions may be introduced. In particular, the aggregation function must be additionally symmetric (impartial), i.e. for any permutation τ of I,

$$g(y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(m)}) = g(y_1, y_2, \ldots, y_m) \tag{9}$$

as well as be equitable (to satisfy the principle of transfers)

$$g(y_1, \ldots, y_{i'} - \varepsilon, \ldots, y_{i''} + \varepsilon, \ldots, y_m) > g(y_1, y_2, \ldots, y_m) \tag{10}$$

for any $0 < \varepsilon < y_{i'} - y_{i''}$. In the case of an aggregation function satisfying all the requirements (3), (9) and (10), we call the corresponding problem (2) a *fair (equitable) aggregation* of problem (1). Every optimal solution to the fair aggregation (2) of a multiple criteria problem (1) defines some fair (equitable) solution.

Note that symmetric functions satisfying the requirement

$$g(y_1, \ldots, y_{i'} - \varepsilon, \ldots, y_{i''} + \varepsilon, \ldots, y_m) \geq g(y_1, y_2, \ldots, y_m) \tag{11}$$

for $0 < \varepsilon < y_{i'} - y_{i''}$ are called (weakly) Schur-concave [11] while the stronger requirement of equitability (10), we consider, is related to strictly Schur-concave functions. In other words, an aggregation (2) is fair if it is defined by a strictly increasing and strictly Schur-concave function g.

Note that both the simplest aggregation functions, the sum (4) and the minimum (5), are symmetric and satisfy the requirement (11), although they do not satisfy the equitability requirement (10). Hence, they are Schur-concave but not strictly Schur-concave. To guarantee the fairness of solutions, some enforcement of concave properties is required.

For any strictly concave, increasing utility function $s : R \to R$, the function

$$g(\mathbf{y}) = \sum_{i=1}^{m} s(y_i) \tag{12}$$

is a strictly monotonic and strictly Schur-concave function [11]. This defines a family of the fair aggregations according to the following proposition [12].

Proposition 1 *For any strictly convex, increasing function $s : R \to R$, the optimal solution of the problem*

$$\max \{\sum_{i=1}^{m} s(f_i(\mathbf{x})) \ : \ \mathbf{x} \in Q\} \tag{13}$$

is a fair solution for decision problem (1).

Various concave functions utility s can be used to define fair aggregations (13) and the resulting fair solution concepts. In the case of the outcomes restricted to positive values, one may use logarithmic function thus resulting in the so-called proportional fairness model [5]. A parametric class of utility functions:

$$s(y_i,\alpha) = \begin{cases} y_i^{1-\alpha}/(1-\alpha) & \text{if } \alpha \neq 1 \\ \log(y_i) & \text{if } \alpha = 1 \end{cases}$$

may be used for this purpose generating various solution concepts for $\alpha \geq 0$. In particular, for $\alpha = 0$ one gets the total output maximization which is the only linear criterion within the entire class. For $\alpha = 1$, it represents the Proportional Fairness approach [5] that maximizes the sum of logarithms of the flows while with α tending to the infinity it converges to the lexicographic max-min optimization which represents the Rawlsian [17] concept of justice. However, every such approach requires to build (or to guess) a utility function prior to the analysis and later it gives only one possible compromise solution. It is very difficult to identify and formalize the preferences at the beginning of the decision process. Moreover, apart from the trivial case of the total output maximization all the utility functions that really take into account any fairness preferences are nonlinear. Many decisions models considered with fair outcomes are originally LP or MILP models. Nonlinear objective functions applied to such models may results in computationally hard optimization problems. In the following, we shall describe an approach that allows to search for such compromise solutions with multiple linear criteria rather than the use nonlinear objective functions.

2. Ordered outcomes

Multiple criteria optimization defines the dominance relation by the standard vector inequality. The theory of majorization [11] includes the results which allow us to express the relation of fair (equitable) dominance as a vector inequality on the cumulative ordered outcomes [6]. This can be mathematically formalized as follows. First, introduce the ordering map $\Theta : R^m \rightarrow R^m$ such that $\Theta(\mathbf{y}) = (\theta_1(\mathbf{y}), \theta_2(\mathbf{y}), \ldots, \theta_m(\mathbf{y}))$, where $\theta_1(\mathbf{y}) \leq \theta_2(\mathbf{y}) \leq \cdots \leq \theta_m(\mathbf{y})$ and there exists a permutation τ of set I such that $\theta_i(\mathbf{y}) = y_{\tau(i)}$ for $i = 1, \ldots, m$. Next, apply to ordered outcomes $\Theta(\mathbf{y})$, a linear cumulative map thus resulting in the cumulative ordering map $\bar{\Theta}(\mathbf{y}) = (\bar{\theta}_1(\mathbf{y}), \bar{\theta}_2(\mathbf{y}), \ldots, \bar{\theta}_m(\mathbf{y}))$ defined as

$$\bar{\theta}_i(\mathbf{y}) = \sum_{j=1}^{i} \theta_j(\mathbf{y}) \quad \text{for } i = 1, \ldots, m \tag{14}$$

The coefficients of vector $\bar{\Theta}(\mathbf{y})$ express, respectively: the smallest outcome, the total of the two smallest outcomes, the total of the three smallest outcomes, etc.

Note that fair solutions to problem (1) can be expressed as Pareto-optimal solutions for the multiple criteria problem with objectives $\bar{\Theta}(\mathbf{f}(\mathbf{x}))$

$$\max \{(\bar{\theta}_1(\mathbf{f}(\mathbf{x})), \bar{\theta}_2(\mathbf{f}(\mathbf{x})), \ldots, \bar{\theta}_m(\mathbf{f}(\mathbf{x}))) \;:\; \mathbf{x} \in Q\} \tag{15}$$

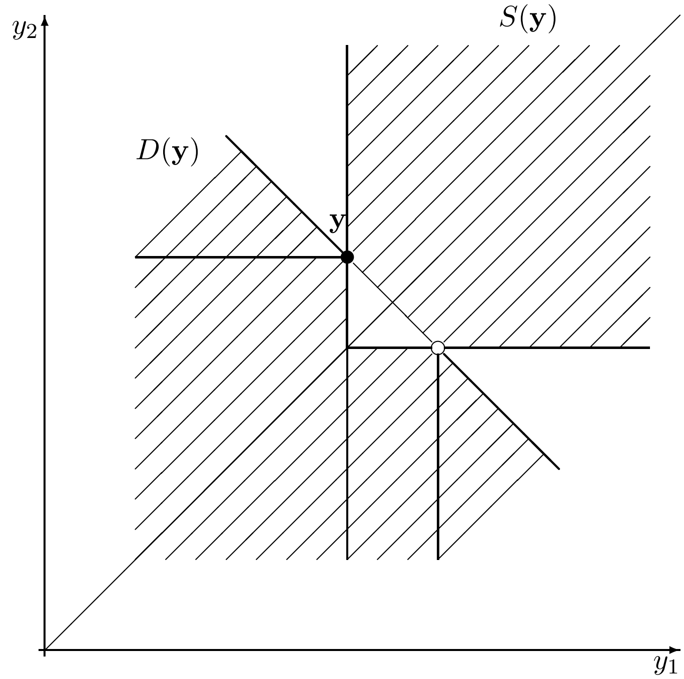

Figure 1: Structure of the equitable dominance.

Proposition 2 *A feasible solution $\mathbf{x} \in Q$ is a fair solution of the problem (1), iff it is a Pareto-optimal solution of the multiple criteria problem (15).*

Proposition 2 provides the relationship between fair solutions and the standard Pareto-optimality. One may notice that the set $D(\mathbf{y})$ of directions leading to outcome vectors being dominated by a given $\mathbf{y}$ is, in general, not a cone and it is not convex. Although, when we consider the set $S(\mathbf{y})$ of directions leading to outcome vectors dominating given $\mathbf{y}$ we get a convex set. Figure 1 shows both $S(\mathbf{y})$ and $D(\mathbf{y})$ fixed at $\mathbf{y}$.

Hence, the multiple criteria problem (15) may serve as a source of fair solution concepts. Although the definitions of quantities $\bar{\theta}_k(\mathbf{y})$, used as criteria in (15), are very complicated, the quantities themselves can be modeled with simple auxiliary variables and constraints. It is commonly known that the smallest outcome may be defined by the following optimization: $\bar{\theta}_1(\mathbf{y}) = \max\{t \ : \ t \leq y_i \quad \text{for } i = 1,\ldots,m\}$, where t is an unrestricted variable. It turns out that this can be generalized to provide an effective modeling technique for quantities $\bar{\theta}_k(\mathbf{y})$ with arbitrary k [16]. Let us notice that for any given vector $\mathbf{y}$, the quantity $\bar{\theta}_k(\mathbf{y})$ is defined by the following LP:

$$\begin{aligned}\bar{\theta}_k(\mathbf{y}) &= \min \sum_{i=1}^{m} y_i u_{ki} \\ &\quad \text{s.t.} \quad \sum_{i=1}^{m} u_{ki} = k,\ 0 \le u_{ki} \le 1 \quad \text{for } i = 1, \dots, m.\end{aligned} \tag{16}$$

Exactly, the above problem is an LP for a given outcome vector $\mathbf{y}$ while it begins nonlinear for a variable $\mathbf{y}$. This difficulty can be overcome by taking advantages of the LP dual to (16):

$$\begin{aligned}\bar{\theta}_k(\mathbf{y}) &= \max\ kt_k - \sum_{i=1}^{m} d_{ik} \\ &\quad \text{s.t.} \quad t_k - y_i \le d_{ik},\ d_{ik} \ge 0 \quad \text{for } i = 1, \dots, m\end{aligned} \tag{17}$$

where t_k is an unrestricted variable while nonnegative variables d_{ik} represent, for several outcome values y_i, their downside deviations from the value of t [16].

3. Multicriteria approaches

Proposition 2 allows one to generate equitably efficient solutions of (1) as efficient solutions of multicriteria problem:

$$\max\ (\eta_1, \eta_2, \dots, \eta_m) \tag{18}$$

$$\text{subject to} \quad \mathbf{x} \in Q$$

$$\eta_k = kt_k - \sum_{i=1}^{m} d_{ik} \quad \text{for } k = 1, \dots, m \tag{19}$$

$$t_k - d_{ik} \le x_i, \quad d_{ik} \ge 0 \quad \text{for } i, k = 1, \dots, m \tag{20}$$

The aggregation maximizing the sum of outcomes, corresponds to maximization of the last (m–th) objective (η_m) in problem (18)–(20). Similar, the maximin scalarization corresponds to maximization of the first objective (η_1). For modeling various fair preferences one may use some combinations the criteria. In particular, for the weighted sum $\sum_{i=1}^{m} w_i \eta_i$ on gets equivalent combination of the cumulative ordered outcomes $\bar{\theta}_i(\mathbf{y})$:

$$\sum_{i=1}^{m} w_i \bar{\theta}_i(\mathbf{y}). \tag{21}$$

Note that, due to the definition of map $\bar{\Theta}$ with (14), the above function can be expressed in the form with weights $v_i = \sum_{j=i}^{m} w_j$ $(i = 1, \dots, m)$ allocated to coordinates of the ordered outcome vector. Such an approach to aggregation

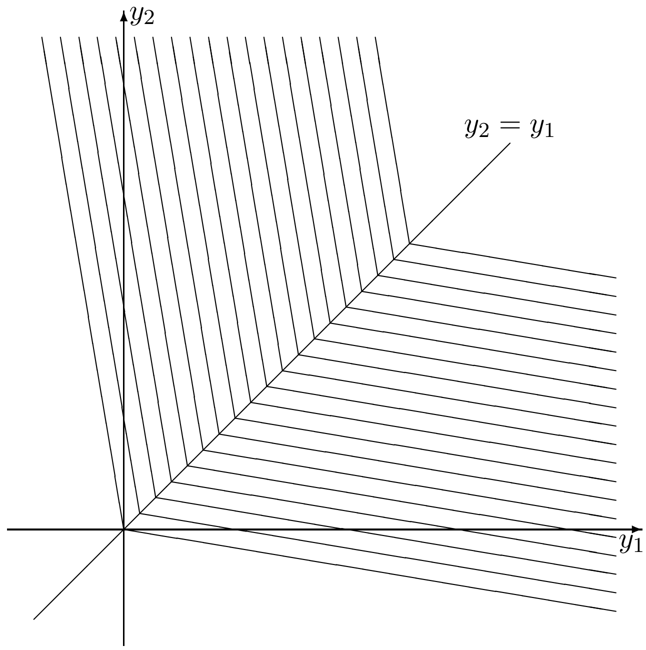

Figure 2: Isoline contours for an equitable OWA aggregation.

of outcomes was introduced by Yager [22] as the so–called Ordered Weighted Averaging (OWA). When applying OWA to problem (1) we get

$$\max \{\sum_{i=1}^{m} v_i \theta_i(\mathbf{f}(\mathbf{x})) \ : \ \mathbf{x} \in Q\} \tag{22}$$

The OWA aggregation is obviously a piece wise linear function since it remains linear within every area of the fixed order of arguments. If weights v_i are strictly decreasing and positive, i.e. $v_1 > v_2 > \cdots > v_{m-1} > v_m > 0$, then each optimal solution of the OWA problem (22) is a fair solution of (1).

While equal weights define the linear aggregation, several decreasing sequences of weights lead to various strictly Schur-concave and strictly monotonic aggregation functions. Thus, the monotonic OWA aggregations provide a family of piece wise linear aggregations filling out the space between the piece wise linear aggregation functions (4) and (5) as shown in Fig. 3. Actually, formulas (21) and (17) allow us to formulate any monotonic (not necessarily strictly) OWA problem (22) as the following LP extension of the original multiple criteria problem:

$$\begin{aligned} &\max \sum_{k=1}^{m} w_k \eta_k && (23)\\ &\text{subject to} \quad \mathbf{x} \in Q \end{aligned}$$

$$\eta_k = kt_k - \sum_{i=1}^{m} d_{ik} \quad \text{for } k = 1, \dots, m \tag{24}$$

$$t_k - d_{ik} \le f_i(\mathbf{x}), \quad d_{ik} \ge 0 \quad \text{for } i, k = 1, \dots, m \tag{25}$$

where $w_m = v_m$ and $w_k = v_k - v_{k+1}$ for $k = 1, \dots, m-1$.

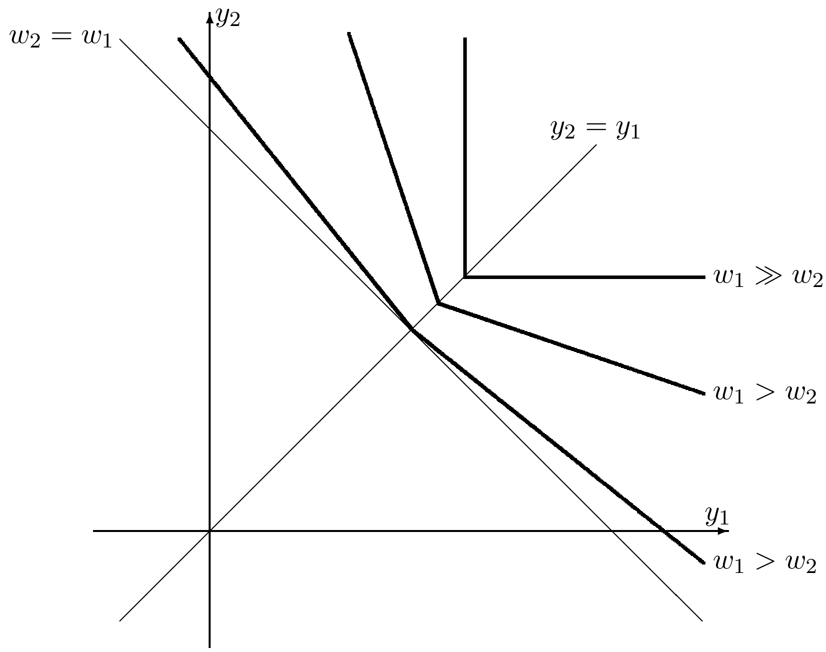

Figure 3: Isoline contours for various equitable OWA aggregations.

When differences among weights tend to infinity, the OWA aggregation approximates the lexicographic ranking of the ordered outcome vectors [13]. That means, as the limiting case of the OWA problem (22), we get the lexicographic problem:

$$\text{lexmax}\,\{\Theta(\mathbf{f}(\mathbf{x}))\ :\ \mathbf{x} \in Q\} \tag{26}$$

which represents the lexicographic maximin ordering approach to the original problem (1). Problem (26) is a regularization of the standard maximin optimization (5), but in the former, in addition to the worst outcome, we maximize also the second worst outcome (provided that the smallest one remains as large as possible), maximize the third worst (provided that the two smallest remain as large as possible), and so on. Due to (14), the MMF problem (26) is equivalent to the problem:

$$\text{lexmax}\,\{\bar{\Theta}(\mathbf{f}(\mathbf{x}))\ :\ \mathbf{x} \in Q\}$$

which leads us to a standard lexicographic optimization with predefined linear criteria defined according to (17).

Moreover, in the case of LP models, every fair solution can be identified as an optimal solution to some OWA problem with appropriate monotonic weights [6] but such a search process is usually difficult to control. Better controllability and the complete parameterization of nondominated solutions even for non-convex, discrete problems can be achieved with the direct use of the reference point methodology introduced by Wierzbicki [20] and later extended leading to efficient implementations of the so-called aspiration/reservation based decision support (ARBDS) approach with many successful applications [8]. The ARBDS approach is an interactive technique allowing the DM to specify the requirements in terms of aspiration and reservation levels, i.e., by introducing acceptable and required values for several criteria. Depending on the specified aspiration and reservation levels, a special scalarizing achievement function is built which may be directly interpreted as expressing utility to be maximized. Maximization of the scalarizing achievement function generates an efficient solution to the multiple criteria problem. The solution is accepted by the DM or some modifications of the aspiration and reservation levels are introduced to continue the search for a better solution. The ARBDS approach provides a complete parameterization of the efficient set to multi-criteria optimization. Hence, when applying the ARBDS methodology to the ordered cumulated criteria in (15), one may generate all (fairly) equitably efficient solutions of the original problem (1).

While building the scalarizing achievement function the following properties of the preference model are assumed. First of all, for any individual outcome η_k more is preferred to less (maximization). To meet this requirement the function must be strictly increasing with respect to each outcome. Second, a solution with all individual outcomes η_k satisfying the corresponding reservation levels is preferred to any solution with at least one individual outcome worse (smaller) than its reservation level. Next, provided that all the reservation levels are satisfied, a solution with all individual outcomes η_k equal to the corresponding aspiration levels is preferred to any solution with at least one individual outcome worse (smaller) than its aspiration level. That means, the scalarizing achievement function maximization must enforce reaching the reservation levels prior to further improving of criteria. In other words, the reservation levels represent some soft lower bounds on the maximized criteria. When all these lower bounds are satisfied, then the optimization process attempts to reach the aspiration levels.

The generic scalarizing achievement function takes the following form [20]:

$$\sigma(\eta) = \min_{k=1,\dots,m} \{\sigma_k(\eta_k)\} + \varepsilon \sum_{k=1}^{m} \sigma_k(\eta_k) \qquad (27)$$

where ε is an arbitrary small positive number and σ_k, for $k = 1, \dots, m$, are the partial achievement functions measuring actual achievement of the individual outcome η_k with respect to the corresponding aspiration and reservation levels (η_k^a and η_k^r, respectively). Thus the scalarizing achievement function is, essentially, defined by the worst partial (individual) achievement but additionally regularized with the sum of all partial achievements. The regularization term is introduced only to guarantee the solution efficiency in the case when the maximization of the main term (the worst partial achievement) results in a non-unique optimal solution.

The partial achievement function σ_k can be interpreted as a measure of the DM's satisfaction with the current value (outcome) of the k-th criterion. It is a strictly increasing function of outcome η_k with value $\sigma_k = 1$ if $\eta_k = \eta_k^a$, and $\sigma_k = 0$ for $\eta_k = \eta_k^r$. Thus the partial achievement functions map the outcomes values onto a normalized scale of the DM's satisfaction. Various functions can be built meeting those requirements [21]. We use the piece wise linear partial achievement function introduced in [12]. It is given by

$$\sigma_k(\eta_k) = \begin{cases} \gamma(\eta_k - \eta_k^r)/(\eta_k^a - \eta_k^r), & \text{for } \eta_k \le \eta_k^r \\ (\eta_k - \eta_k^r)/(\eta_k^a - \eta_k^r), & \text{for } \eta_k^r < \eta_k < \eta_k^a \\ \beta(\eta_k - \eta_k^a)/(\eta_k^a - \eta_k^r) + 1, & \text{for } \eta_k \ge \eta_k^a \end{cases} \tag{28}$$

where β and γ are arbitrarily defined parameters satisfying $0 < \beta < 1 < \gamma$. This partial achievement function is strictly increasing and concave which guarantees its LP computability with respect to outcomes η_k.

Recall that in our model outcomes η_k represent cumulative ordered outcomes, i.e. $\eta_k = \sum_{i=1}^{k} \theta_i(\mathbf{y})$. Hence, the reference vectors (aspiration and reservation) represent, in fact, some reference distributions of outcomes. Moreover, due to the cumulation of outcomes, while considering equal outcomes ϕ as the reference (aspiration or reservation) distribution, one needs to set the corresponding levels as $\eta_k = k\phi$. Certainly, one may specify any desired reference distribution in terms of the ordered values of the outcomes (quantiles in the probability language) $\phi_1 \le \phi_2 \le \ldots \le \phi_m$ and cumulating them automatically get the reference values for the outcomes η_k representing the cumulated ordered values. However, such rich modeling technique may be too complicated to control effectively the search for a compromise solution.

Although defined with simple linear constraints the auxiliary conditions (17) introduces m^2 additional variables and inequalities into the original model. This may cause a serious computational burden for real-life problems containing numerous outcomes. In order to reduce the problem size one may attempt the restrict the number of criteria in the problem (15).

Let us consider a sequence of indices $K = \{k_1, k_2, \ldots, k_q\}$, where $1 = k_1 < k_2 < \ldots < k_{q-1} < k_q = m$, and the corresponding restricted form of the multiple criteria model (15):

$$\max \{(\eta_{k_1}, \eta_{k_2}, \ldots, \eta_{k_q}) \; : \; \eta_k = \bar{\theta}_k(\mathbf{f}(\mathbf{x})) \quad \text{for } k \in K, \quad \mathbf{x} \in Q\} \tag{29}$$

with only $q < m$ criteria. Following Proposition 2, multiple criteria model (15) allows us to generate any fairly efficient solution of problem (1). Reducing the number of criteria we restrict these opportunities. Nevertheless, one may still generate reasonable compromise solutions. First of all the following assertion is valid.

Theorem 1 *If $\mathbf{x}^o$ is an efficient solution of the restricted problem (29), then it is an efficient (Pareto-optimal) solution of the multiple criteria problem (1) and it can be fairly dominated only by another efficient solution $\mathbf{x}'$ of (29) with exactly the same values of criteria: $\bar{\theta}_k(\mathbf{f}(\mathbf{x}')) = \bar{\theta}_k(\mathbf{f}(\mathbf{x}^o))$ for all $k \in K$.*

Proof. Suppose, there exists $\mathbf{x}' \in Q$ which dominates $\mathbf{x}^o$. This means, $y'_i = f_i(\mathbf{x}') \geq y^o_i = f_i(\mathbf{x}^0)$ for all $i \in I$ with at least one inequality strict. Hence, $\bar{\theta}_k(\mathbf{y}') \geq \bar{\theta}_k(\mathbf{y}^o)$ for all $k \in K$ and $\bar{\theta}_{k_q}(\mathbf{y}') > \bar{\theta}_{k_q}(\mathbf{y}^o)$ which contradicts efficiency of $\mathbf{x}^o$ within the restricted problem (29).

Suppose now that $\mathbf{x}' \in Q$ fairly dominates $\mathbf{x}^o$. Due to Proposition 2, this means that $\bar{\theta}_i(\mathbf{y}') \geq \bar{\theta}_i(\mathbf{y}^o)$ for all $i \in I$ with at least one inequality strict. Hence, $\bar{\theta}_k(\mathbf{y}') \geq \bar{\theta}_k(\mathbf{y}^o)$ for all $k \in K$ and any strict inequality would contradict efficiency of $\mathbf{y}^o$ within the restricted problem (29). Thus, $\bar{\theta}_k(\mathbf{y}') = \bar{\theta}_k(\mathbf{y}^o)$ for all $k \in K$ which completes the proof. □

It follows from Theorem 1 that while restricting the number of criteria in the multiple criteria model (15) we can essentially still expect reasonably fair efficient solution and only *unfairness* may be related to the distribution of flows within classes of skipped criteria. In other words we have guaranteed some rough fairness while it can be possibly improved by redistribution of flows within the intervals $(\theta_{k_j}(\mathbf{y}), \theta_{k_{j+1}}(\mathbf{y})]$ for $j = 1, 2, \ldots, q-1$. Since the fairness preferences are usually very sensitive for the smallest flows, one may introduce a grid of criteria $1 = k_1 < k_2 < \ldots < k_{q-1} < k_q = m$ which is dense for smaller indices while sparser for lager indices and expect solution offering some reasonable compromise between fairness and throughput maximization.

Conclusions

Due to additional requirements on the utility functions the fairly efficient solutions represent a specific subset of all the Pareto-optimal solutions. However, they can be expressed as Pareto-optimal solutions to the problem with modified (ordered and cumulated) criteria. Hence, the simplest way to model a large gamut of fairly efficient decisions may depend on the use some combinations of the ordered criteria, i.e. the so–called Ordered Weighted Averaging (OWA) aggreagtions. If the weights are strictly decreasing each optimal solution corresponding to the OWA maximization is a fair (fairly efficient) solution. Moreover, in the case of LP models every fairly efficient solution can be identified as an OWA optimal solution with appropriate strictly monotonic weights. Several decreasing sequences of weights provide us with various aggregations. Better controllability and the complete parameterization of nondominated solutions even for non-convex, discrete problems can be achieved with the use of the reference point methodology.

Bibliography

1. Denda R., Banchs A., Effelsberg W. (2000). The fairness challenge in computer networks. Lect. Notes in Comp. Sci., 1922, 208–220.
2. Erkut E. (1993). Inequality Measures for Location Problems, Location Science, 1, 199–217.
3. Fandel G., Gal T. (2001). Redistribution of funds for teaching and research among universities: The case of North Rhine-Westphalia, European J. Opnl. Research, 130, 111–120.
4. Kaliszewski I. (2004). On non-interactive selection of the winner in multiple criteria decision making. In: Modelowanie preferencji a ryzyko '03, T.Trzaskalik (ed.), 195–211.
5. Kelly F., Mauloo A., Tan D. (1997). Rate control for communication networks: shadow prices, proportional fairness and stability. J. Oper. Res. Soc., 49, 206–217.
6. Kostreva M.M., Ogryczak W. (1999). Linear optimization with multiple equitable criteria. RAIRO Oper. Res., 33, 275–297.
7. Kostreva M.M., Ogryczak W., Wierzbicki A. (2004). Equitable aggregations and multiple criteria analysis, European J. Opnl. Research, 158, 362–367.
8. Lewandowski A., Wierzbicki A.P. (1989). Aspiration Based Decision Support Systems — Theory, Software and Applications. Berlin: Springer.
9. Luss H. (1999). On equitable resource allocation problems: A lexicographic minimax approach. Oper. Res., 47, 361–378.
10. Marsh M.T., Schilling D.A. (1994). Equity measurement in facility location analysis: A review and framework. European J. Opnl. Research, 74, 1–17.
11. Marshall A.W., Olkin I. (1979). Inequalities: Theory of Majorization and Its Applications. New York: Academic Press.
12. Ogryczak W. (1997). Linear and Discrete Optimization with Multiple Criteria: Preference Models and Applications to Decision Support (in Polish). Warsaw: Warsaw Univ Press.

13. Ogryczak W. (2000). Inequality measures and equitable approaches to location problems, European J. Opnl. Research, 122, 374–391.

14. Ogryczak W., Śliwiński T. (2002). On equitable approaches to resource allocation problems: the conditional minimax solution. J. Telecommunication and Info. Tech., 3, 40–48.

15. Ogryczak W., Śliwiński T., Wierzbicki A. (2003). Fair resource allocation schemes and network dimensioning problems. J. Telecommunication and Info. Tech., 3, 34–42.

16. Ogryczak W., Tamir A. (2003). Minimizing the sum of the k largest functions in linear time. Information Proc. Letters, 85, 117–122.

17. Rawls J. (1971). The Theory of Justice. Cambridge: Harvard Univ. Press.

18. Sen, A. (1973), On Economic Inequality. Clarendon Press, Oxford.

19. Steuer R.E. (1986). Multiple Criteria Optimization: Theory, Computation & Applications. New York: Wiley.

20. Wierzbicki A.P. (1982). A mathematical basis for satisficing decision making. Math. Modelling, 3, 391–405.

21. Wierzbicki A.P., Makowski M., Wessels J. (2000). Model Based Decision Support Methodology with Environmental Applications. Dordrecht: Kluwer.

22. Yager R.R. (1988). On ordered weighted averaging aggregation operators in multicriteria decision making. IEEE Trans. Systems, Man and Cybernetics, 18, 183–190.

23. Young, H.P. (1994), Equity in Theory and Practice. Princeton University Press, Princeton.