Numer Algor (2014) 65:745–757
DOI 10.1007/s11075-013-9712-0

ORIGINAL PAPER

Adaptive Arnoldi-Tikhonov regularization for image restoration

Paolo Novati · Maria Rosaria Russo

Received: 7 September 2012 / Accepted: 8 April 2013 / Published online: 24 April 2013

Abstract In the framework of the numerical solution of linear systems arising from image restoration, in this paper we present an adaptive approach based on the reordering of the image approximations obtained with the Arnoldi-Tikhonov method. The reordering results in a modified regularization operator, so that the corresponding regularization can be interpreted as problem dependent. Numerical experiments are presented.

Keywords Linear discrete ill-posed problem · Image restoration · Tikhonov regularization · Arnoldi algorithm · Krylov methods

1 Introduction

Given a vector $b \in \mathbb{R}^N$ representing a blurred and noisy observed image, rearranged columnwise, the problem of restoring the original image can often be modeled by means of a linear system of equations

$$Ax = b, \tag{1}$$

in which $A \in \mathbb{R}^{N \times N}$ models the blurring operator and x is an approximation of the unknown original image $\widehat{x}$, solution of $A\widehat{x} = \widehat{b}$, where $\widehat{b}$ is the noise-free right-hand side. In this sense, we assume $b = \widehat{b} + e_b$, where e_b is the noise vector.

P. Novati (✉) · M. R. Russo
Department of Pure Mathematics, University of Padova, Via Trieste 63, 35121 Padova, Italy
e-mail: novati@math.unipd.it

M. R. Russo
e-mail: mrrusso@math.unipd.it

For this kind of problem, the matrix A is typically very large (N is at least the number of pixels of the image) and ill-conditioned, so that some regularization technique is generally employed for solving (1). In this paper we use the popular Tikhonov regularization method, based on the minimization

$$\min_{x\in\mathbb{R}^N} \|Ax-b\|_2^2 + \lambda\,\|Lx\|_2^2\,, \tag{2}$$

where $\lambda > 0$ is a given parameter and L is a regularization matrix (see, e.g., [6] and [8] for a background). The solution x_λ of the minimization problem is, hopefully, a meaningful approximation of the exact solution $\widehat{x}$ of the error-free problem. For what concerns the choice of L, which plays the role of a penalizing filter, it should be made exploiting (when it is possible) information on the solution $\widehat{x}$, keeping in mind that the ideal situation would be to have $\widehat{x} \in \ker(L)$, that is, $L\widehat{x} = 0$.

Generally, the most popular choices for L are the identity matrix I_N (hence looking for the solution of minimum norm) or L representing a discretization of a differential operator such as the first or the second derivative, eventually rearranged in order to take into account that an image is a two-dimensional object (see, e.g., [11] for a discussion). Whenever the image to restore does not involve high frequencies, as for instance in a jpeg image, in which high frequencies are already filtered out via the Discrete Cosine Transform (DCT), taking L as a derivative operator generally produces good results. On the other side, since a derivative operator is a high-pass filter, if the image has well marked edges (high-frequencies involved) the use of such regularization operators typically smooths the edges and the quality of the restoration is extremely sensitive to the choice of the parameter λ.

These considerations hold in particular with operators related to the second derivative. Indeed, the discrete first order derivative operator (variation) is generally considered rather good to filter out noise, preserving edges at the same time. A number of methods for noise removal of 1D or 2D signals, based on the minimization of functionals involving the total variation (TV) have been presented (see the original paper [12]). For 1D signals the total variation is defined as $\|L_1 x\|_1$ where

$$L_1 = \begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{pmatrix} \in \mathbb{R}^{(N-1)\times N}. \tag{3}$$

Notwithstanding its one-dimensional nature, the operator L_1 is widely used also for 2D signals, inside Tikhonov regularization (2), (see, e.g., [9]), since the image is reshaped as a vector and hence treated as a 1D signal. The results are generally rather good but typically with visible horizontal discontinuities.

The aim of this paper is to show that it is possible to improve the quality of the restoration obtained with L_1 in Tikhonov regularization (2), using information about the order of the pixel values of the image to restore. We show that if we know the permutation matrix P_{opt} such that $P_{opt}\widehat{x}$ is sorted (an ideal situation because $\widehat{x}$ is not known), then it is possible to restore $\widehat{x}$ with high quality using just the product $L_1 P_{opt}$ as regularization operator. The reason is that the high frequencies components of $\widehat{x}$ are damped in $P_{opt}\widehat{x}$, so that an high-pass filter L does not heavily reduce these components anymore. Note that using permutations, we actually forget the original nature of the problem. From a practical point of view, since $\widehat{x}$ is unknown, what we can do is to restart a reliable method for (2) and use the arising approximations to approximate P_{opt} by means of a sequence of permutation matrices $\{P_k\}_{k\geq 0}$, starting from $P_0 = I_N$. If the method is iterative, we can even try to update the approximations of P_{opt} step by step. In this paper we consider these two approaches, that is, restarted and update, working with a generalized version of the Arnoldi-Tikhonov (AT) method introduced in [3], that can be regarded to as a regularization version of the GMRES which provides Krylov subspace approximations to the solution working in small dimensions.

We need to mention that a similar idea has been studied in [1], where the authors try to approximate P_{opt} using an undersampling approach in the framework of image reconstruction of MRI data.

The paper is organized as follows. In Section 2 we explain the idea of using permutation matrices to construct more reliable regularization operators. In Section 3 we outline the basic features of the AT method and we introduce a generalized version able to work with an arbitrary regularization operator. In Section 4 we present a restarted and an adaptive version of the method. Finally, in Section 5 we show the behavior of the methods on two classical test problems.

2 Image dependent regularization

Give a permutation matrix $P \in \mathbb{R}^{N\times N}$, let $y = Px$. With respect to y, the Tikhonov regularization (2) for solving (1) leads to

$$\min_{y} \left\| AP^T y - b \right\|_2^2 + \lambda \left\| Ly \right\|_2^2 . \tag{4}$$

The problem (4) is equivalent to the linear systems

$$\begin{aligned} \left(PA^T AP^T + \lambda L^T L \right) y &= PA^T b, \\ \left(A^T A + \lambda P^T L^T LP \right) x &= A^T b, \end{aligned}$$

that is, to the minimization problem

$$\min_{x} \|Ax - b\|_2^2 + \lambda \|LPx\|_2^2 . \tag{5}$$

Let $\mathbf{P}$ be the set of the $N \times N$ permutation matrices. In light of (5), the basic idea is to define P such that for a given L and x

$$\|LPx\|_2 = \min_{Q\in\mathbf{P}} \|LQx\|_2 . \tag{6}$$

While the idea can be applied independently of the choice of L, in this paper we are mainly interested in the case of L_1 defined by (3), for which the solution of (6) is clearly the permutation matrix P which sorts the vector x in increasing or decreasing order. As already mentioned, the ideal situation would be to know the permutation matrix P_{opt} of (6) corresponding to the exact solution $\widehat{x}$ of the problem. This ideal situation is represented in Figs. 1 and 2, where we consider the restoration of a 35×35 subimage of `mri.tif` from `Matlab`'s `Image Processing Toolbox`. The matrix A, representing the blurring operator, comes from the discretization of the Gaussian Point Spread Function (PSF) with half-bandwidth $q = 7$ and variance $\sigma = 2$. Additional details about this kind of matrix are given Section 5. In both experiments, in which we change the noise level in the observed image b, we consider the restoration obtained with $L_1 P_{opt} \in \mathbb{R}^{N-1 \times N}$ and the matrix

$$L_{1,2D} := \begin{pmatrix} I_n \otimes L_1 \\ L_1 \otimes I_n \end{pmatrix} \in \mathbb{R}^{2n(n-1) \times N}, \quad n^2 = N,$$

(here $L_1 \in \mathbb{R}^{n-1 \times n}$) introduced in [9] in order to extend the use of L_1 to the 2D case. In each experiment, the value of the regularization parameter λ is set using the L-curve criterion.

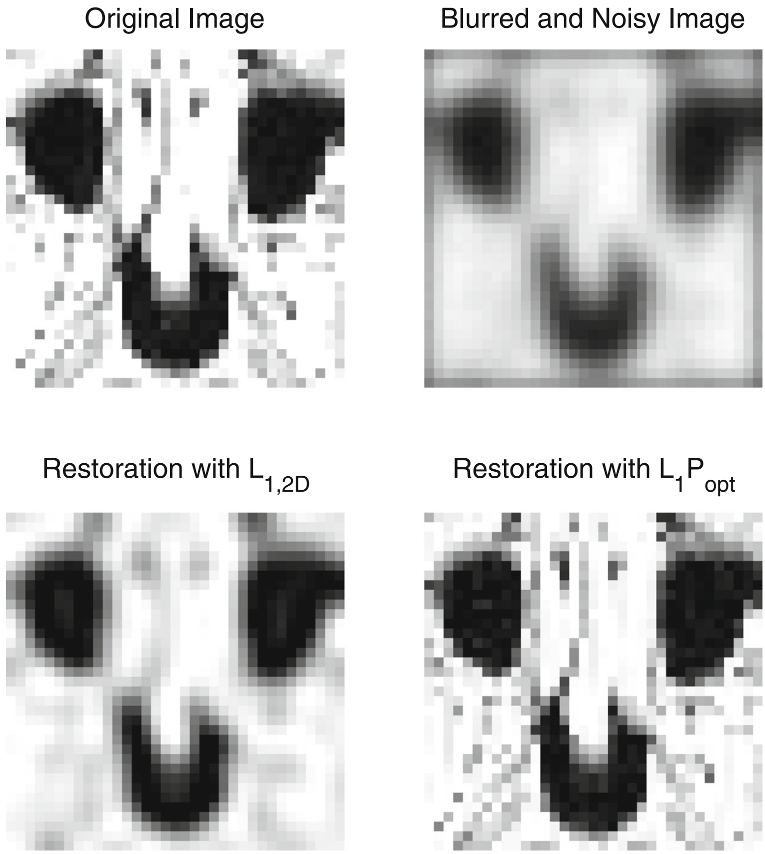

Fig. 1 Restoration of a 35x35 subimage of mri.tif, with *blurring parameters* $q = 7$ and $\sigma = 2$, and 1 % Gaussian noise

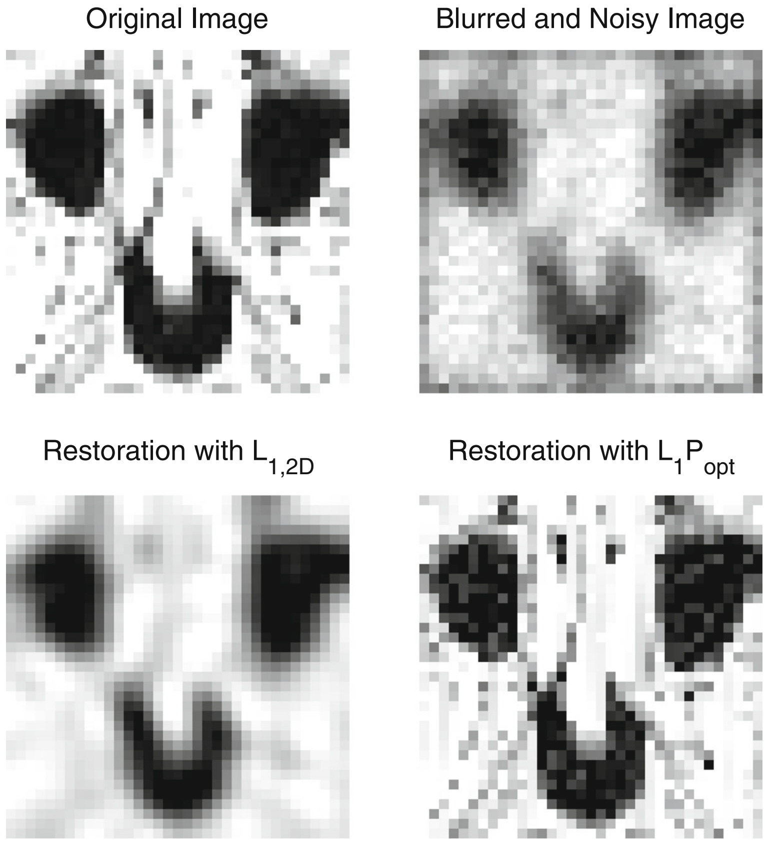

Fig. 2 Restoration of a 35x35 subimage of mri.tif, with *blurring parameters* $q = 7$ and $\sigma = 2$, and 10 % Gaussian noise

The results do not need many comments. Of course we are in an ideal situation but the results lead us to consider the possibility of applying the Tikhonov regularization iteratively using at each step the approximate solution to define a permutation hopefully close to the ideal one.

It is important to remark that reducing $\|Lx\|_2$ in Tikhonov regularization has also the important effect of reducing the dependence on the choice of λ for having a good reconstruction. In both experiments, denoting by x_λ the solution with $L_{1,2D}$, and with $\overline{x}_\lambda$ the solution with $L_1 P_{opt}$, we have obtained $\left\|L_1 P_{opt}\overline{x}_\lambda\right\|_2 \approx 0.1 \cdot \left\|L_{1,2D}x_\lambda\right\|_2$. This consideration is particularly important for large scale problems, where the existing parameter choice techniques may be expensive and sometimes not much reliable.

3 The extension of the Arnoldi-Tikhonov method

Image deblurring has of course to be regarded to as a large scale problem so that suitable methods need to be used to solve the Tikhonov minimization (2). In this framework, the Arnoldi-Tikhonov (AT) method has been introduced in [3] with the aim of reducing the problem

$$\min_{x\in\mathbb{R}^N} \|Ax-b\|_2^2 + \lambda \|Lx\|_2^2, \tag{7}$$

in the case of $L = I_N$, to a problem of much smaller dimension. The idea is to project the matrix A onto the Krylov subspaces generated by A and the vector b, that is, $K_m(A,b) = span\{b, Ab, ..., A^{m-1}b\}$, with $m \ll N$. The method was basically introduced to avoid the matrix-vector multiplication with A^T used by Lanczos type schemes (see e.g. [2], [5]). For the construction of the Krylov subspaces the AT method uses the Arnoldi algorithm, that leads to the decomposition

$$AV_m = V_{m+1}H_{m+1}, \tag{8}$$

where $V_{m+1} = [v_1, ..., v_{m+1}] \in \mathbb{R}^{N\times(m+1)}$ has orthonormal columns which span the Krylov subspace $K_m(A,b)$ defining $v_1 = b/\|b\|$. The matrix $H_{m+1} \in \mathbb{R}^{(m+1)\times m}$ is an upper Hessenberg matrix. Substituting $x = V_m y_m$, $y_m \in \mathbb{R}^m$, into (7) and using (8) yields the reduced minimization

$$\min_{y_m\in\mathbb{R}^m} \left\| H_{m+1}y_m - V_{m+1}^T b \right\|_2^2 + \lambda \|y_m\|_2^2 . \tag{9}$$

Since the method starts with $v_1 = b/\|b\|$, we have

$$V_{m+1}^T b = \|b\|_2 e_1, \quad e_1 = (1, 0, ..., 0)^T \in \mathbb{R}^{m+1}.$$

In this sense, the AT method can be interpreted as a regularized GMRES with starting approximation $x_0 = 0$, that is, with an initial residual $r_0 = b$. Besides, in 2009, Lewis and Reichel [10] introduced and studied the 'range-restricted' variant to this method, RRAT (Range Restricted Arnoldi Tikhonov) which assumes to start the Arnoldi process with $v_1 = Ab/\|Ab\|$, that is, to work with the Krylov subspaces $K_m(A, Ab) = span\{Ab, A^2b, ..., A^m b\}$. This approach leads again to (9) but with a different H_{m+1} and V_{m+1}. For both methods the solution of (7) with $L = I_N$, is then approximated by $x_m = V_m y_m$.

The method considered in this paper is an extension of the Arnoldi-Tikhonov method, able to work with a general regularization operator $L \neq I_N$ and an arbitrary starting vector x_0. We consider the minimization problem

$$\min_{x\in\mathbb{R}^N} \|Ax-b\|_2^2 + \lambda \|L(x-x_0)\|_2^2, \tag{10}$$

where x_0 is an approximate solution (eventually 0 if not available). We seek for approximations of the type

$$x_m = x_0 + V_m y_m, \tag{11}$$

where V_m spans the Krylov subspace $K_m(A, r_0)$, and $r_0 = b - Ax_0$. Substituting (11) into (10) yields the reduced minimization

$$\min_{y_m\in\mathbb{R}^m} \|H_{m+1}y_m - \|r_0\|_2 e_1\|_2^2 + \lambda \|LV_m y_m\|_2^2,$$

that is,

$$\min_{y_m \in \mathbb{R}^m} \left\| \begin{pmatrix} H_{m+1} \\ \sqrt{\lambda} L V_m \end{pmatrix} y_m - \begin{pmatrix} \|r_0\| e_1 \\ 0 \end{pmatrix} \right\|_2^2 . \tag{12}$$

With respect to the AT method as introduced in [3], the least squares problem (12) has a matrix coefficient of dimension $(m + 1 + P) \times m$, if $L \in \mathbb{R}^{P \times N}$, instead of $(2m + 1) \times m$ as in the AT method, where in (12) LV_m is replaced by I_m (cf. (9)). Of course this is a computational disadvantage, but it is absolutely balanced by the effect that L may have on noisy problems. To avoid confusion with the standard AT method, and for simplicity, we denote by GAT (Generalized Arnoldi Tikhonov) the reduced minimization (12). Whenever y_m has been computed, the norm of the residual r_m of the corresponding approximation (11) is given by

$$\|r_m\|_2 = \|H_{m+1} y_m - \|r_0\|_2 e_1\|_2 . \tag{13}$$

For what follows in this paper, it is very important to observe that the GAT method (but in general each iterative method for solving (10)), may be very fast if x_0 is close to the solution. Of course, with the word 'fast' we just mean that the approximations rapidly achieve best attainable approximation, since for this kind of problem an iterative method typically shows semiconvergence, or, in the best case, it stagnates.

4 The restarted and the adaptive regularization

As already observed, since is not possible to compute the ideal permutation matrix P_{opt}, we can try to approximate this matrix by solving more than once the problem (5) with a certain method, adapting at each step the matrix P_{opt} to the new approximation. Since in principle, this restarted approach could be quite expansive, the basic idea is to find the minimum of (6) iterating the GAT method. Basically, starting from $x_0^{(0)} = 0$, we define initially $P^{(0)} = I_N$ and solve (12) with $L^{(0)} = L_1$, so that

$$\begin{pmatrix} H_{m+1} \\ \sqrt{\lambda} L^{(0)} V_m \end{pmatrix} \in \mathbb{R}^{(m+N)\times m}.$$

Then sort the approximate solution $x^{(1)}$ ($= x_m^{(0)}$ for a certain m) in increasing or decreasing order by means of the permutation matrix $P^{(1)}$. Restart the GAT method with $x_0^{(1)} = x^{(1)}$ and $L^{(1)} = L_1 P^{(1)}$, and so on.

Assuming to know the relative noise level $\epsilon = \left\| b - \widehat{b} \right\|_2 / \left\| \widehat{b} \right\|_2$, where $\widehat{b}$ denotes the noise-free blurred image, we stop the GAT method using the discrepancy principle. With this criterion, (12) is solved until the residual (13) fulfils

$$\|r_m\|_2 \le \eta \epsilon \, \|b\|_2 , \tag{14}$$

where $\eta \ge 1$ is a given parameter. Actually, whenever (14) is satisfied, we let the method run for a couple of further iterations (see e.g. [10] for a discussion). For what concerns the number of restarts, denoting by $x^{(j)}$ the final approximation arising from the j-th application of the GAT method, and with $r^{(j)}$ the corresponding residual, the whole procedure ends when

$$\left| \frac{\left\| r^{(j)} \right\|_2 - \left\| r^{(j-1)} \right\|_2}{\left\| r^{(j-1)} \right\|_2} \right| < \varepsilon, \quad or \quad \left\| r^{(j)} \right\|_2 > \left\| r^{(j-1)} \right\|_2, \tag{15}$$

for a given ε.

For the definition of the parameter λ, we employ the procedure described in [4]. Since the residual (discrepancy) depends on λ, i.e., $r_m = r_m(\lambda)$, at each step we approximate the solution (with respect to λ) of the equation

$$\|r_m(\lambda)\|_2 = \eta\epsilon \, \|b\|_2 \, ,$$

using a zero finder based on the secant method. In particular, we consider the linear function

$$y(\lambda) = \|r_m(0)\|_2 + \lambda \left(\frac{\|r_m(\lambda_{m-1})\|_2 - \|r_m(0)\|_2}{\lambda_{m-1}} \right), \tag{16}$$

where λ_{m-1} is the parameter coming from the previous step and $\|r_m(0)\|_2$ is just the GMRES residual. The function $y(\lambda)$ interpolates $\|r_m(\lambda)\|_2$ at 0 and λ_{m-1}, and the new parameter λ_m is obtained by solving $y(\lambda) = \eta\epsilon$, which leads to

$$\lambda_m = \left| \frac{\eta\epsilon - \|r_m(0)\|_2}{\|r_m(\lambda_{m-1})\|_2 - \|r_m(0)\|_2} \right| \lambda_{m-1}. \tag{17}$$

We again refer to [4] for details.

In summary, the algorithm can be written as follows.

Algorithm 1 Restarted GAT (RGAT)

1. define $x^{(0)} = 0$ and $P^{(0)} = I_N$
2. set η, ε, $\lambda_0^{(0)}$, $m_{\max}$ and $j_{\max}$
3. while (15) does not hold and $j \leq j_{\max}$
   - (a) while (14) does not hold and $m \leq m_{\max}$
     - (i) solve (12) with $r_0 = r^{(j)} = b - Ax^{(j)}$, $L = L^{(j)} = L_1 P^{(j)}$ and $\lambda = \lambda_{m-1}^{(j+1)}$ to define $y_m^{(j+1)}$
     - (ii) compute $\left\| r_m^{(j+1)}(\lambda_{m-1}^{(j+1)}) \right\|_2 = \left\| H_{m+1} y_m^{(j+1)} - \left\| r^{(j)} \right\|_2 e_1 \right\|_2$ and $\left\| r_m^{(j+1)}(0) \right\|_2$
     - (iii) compute the new parameter $\lambda_m^{(j+1)}$ by (17)
   - (b) define $x^{(j+1)} = x^{(j)} + V_m y_m^{(j+1)}$
   - (c) define $P^{(j+1)}$ reordering $x^{(j+1)}$

Alternatively, it is even possible to modify the GAT method, updating the regularization matrix L inside the Arnoldi iteration, that is, using the $(m-1)$-th Arnoldi approximation to define $L^{(m-1)} = L_1 P^{(m-1)}$ and then solve (12) with $L = L^{(m-1)}$. The algorithm can be written as follows.

Algorithm 2 Adaptive GAT (AGAT)

1. define $x_0 = 0$ and $P^{(0)} = I_N$
2. set η, $m_{\max}$ and λ_0
3. while (14) does not hold and $m \leq m_{\max}$
   - (a) solve (12) with $r_0 = b$, $L = L^{(m-1)} = L_1 P^{(m-1)}$ and $\lambda = \lambda_{m-1}$ to define y_m
   - (b) compute the corresponding residual $\|r_m(\lambda_{m-1})\|_2 = \|H_{m+1}y_m - \|b\|_2 e_1\|_2$ and $\|r_m(0)\|_2$
   - (c) compute the new parameter λ_m by (17)
   - (d) define $x_m = V_m y_m$
   - (e) define $P^{(m)}$ reordering x_m

5 Numerical experiments

In this section we compare the behavior of the GAT method implemented with $L = L_{1,2D}$, and the methods RGAT and AGAT described by Algorithms 1 and 2, on two classical deblurring problems. For what concerns the undefined parameters of Algorithms 1 and 2, in all experiments we set $\eta = 1.01$ (cf. (14)), $m_{\max} = 100$ for GAT and AGAT. For the RGAT we set $m_{\max} = 40$ since it is a restarted method, and $j_{\max} = 6$ as the maximum number of restarts. For each method we set $\lambda_0 = 1$ as the initial value of the regularization parameter. The experiments have been made using `Matlab` on a single processor computer (Intel Core i5). Regarding the computation of the permutation matrices corresponding to an approximate solution, we have used the instructions

```
[unused,pos] = sort(x);
P = sparse([1:N^2]', pos, ones(N^2,1));
```

Example 1 We consider the matrix $A \in \mathbb{R}^{N\times N}$ representing the blurring operator arising from the discretization of the Gaussian Point Spread Function (PSF). For a given image $\widehat{x}$, the vector $\widehat{b} = A\widehat{x}$ represents the associated blurred and noise-free image. We generate a blurred and noisy image $b = \widehat{b} + e_b$, where e_b is a noise vector defined by

$$e_b = \frac{\epsilon\,\|b\|}{\sqrt{N}}c, \tag{18}$$

where ϵ is the relative noise level, and $c = \text{randn}(N, 1)$, that in `Matlab` notation is a vector of N random components with normal distribution with mean 0 and standard deviation 1 . The matrix A is a symmetric Toeplitz matrix given by

$$A = (2\pi\sigma^2)^{-1} T \otimes T,$$

where T is a $n \times n$ symmetric banded Toeplitz matrix whose first row is a vector v whose elements are

$$v_j := \begin{cases} \frac{e^{-(j-1)^2}}{2\sigma^2}, & \text{for } j = 1, ..., q \\ 0, & \text{for } j = q+1, ..., n \end{cases} .$$

The parameter q is the half-bandwidth of the matrix T, and the parameter σ controls the width of the underlying Gaussian Point Spread function

$$h(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right),$$

which models the degradation of the image. We define $q = 7$ and $\sigma = 2$, so that the condition number of A is around 10^{10}.

We consider the deblurring of `testpat2.tif`, which is a 256×256 image taken from the `Image Processing Toolbox`. In Table 1, for the noise levels $\epsilon = 10^{-1}, 10^{-2}, 10^{-3}$, we report the results. In particular, we have considered the relative error of the final approximation (ERR), the number of iterations (IT), the final value of the regularization parameter (λ_{final}) and the elapsed time in seconds (SEC).

For a better view of the behavior of the methods, in Fig. 3 we show the reconstruction of `testpat2.tif` obtained with the GAT and the RGAT methods, with blurring parameters $q = 12$ and $\sigma = 4$, and noise level $\epsilon = 10^{-3}$. The reconstruction attained with the AGAT method is similar to the one of the GAT method, and hence not reported.

Example 2 We consider the block Toeplitz matrix $A \in \mathbb{R}^{N\times N}$, $N = n^2$, representing motion blur along the x-axis. Given a positive integer q,

$$A = I_n \otimes S,$$

where $S \in \mathbb{R}^{n\times n}$ is a symmetric banded matrix of half-bandwidth q, in which the non-zero entries are $1/(2q-1)$, that is,

$$S_{ij} = \begin{cases} 1/(2q-1) & |i-j| \leq q \\ 0 & \text{elsewhere} \end{cases}$$

Table 1 Result of the restoration of Example 1

ϵ		ERR	IT	λ_{final}	SEC
	GAT	3.90e-1	6	7.44e-3	0.7
10^{-1}	RGAT	3.65e-1	17	3.65e-1	1.4
	AGAT	3.94e-1	6	4.97e-2	0.6
	GAT	3.39e-1	8	1.06e-4	0.8
10^{-2}	RGAT	3.09e-1	35	5.31e-2	2.2
	AGAT	3.40e-1	8	3.14e-4	0.7
	GAT	2.92e-1	20	1.97e-6	1.6
10^{-3}	RGAT	2.42e-1	65	1.22e-3	3.5
	AGAT	2.90e-1	20	1.72e-5	1.4

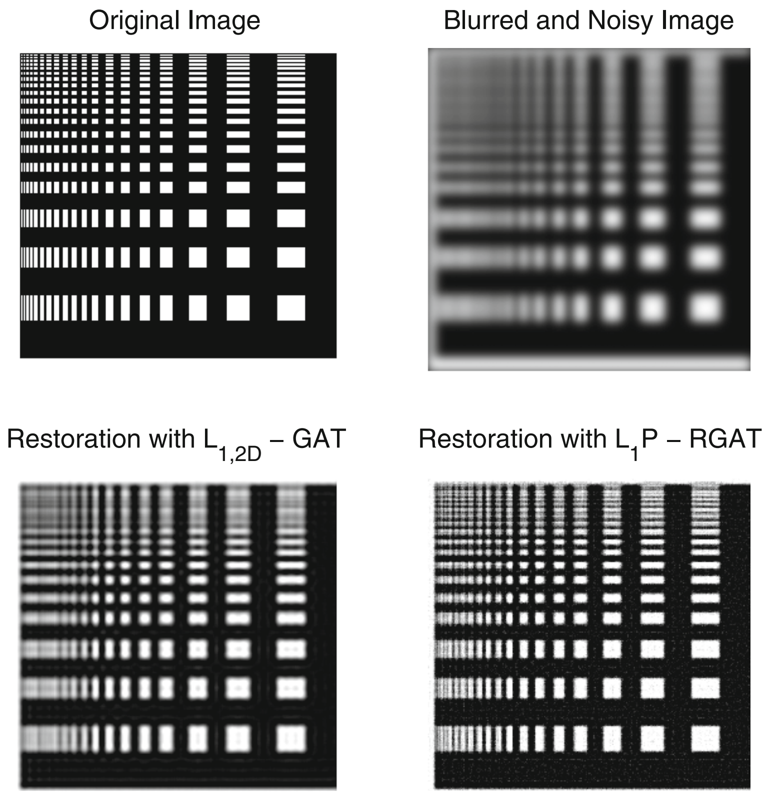

Fig. 3 Reconstruction of `testpat2.tif` obtained with the GAT and the RGAT methods, with $q = 12$, $\sigma = 4$, and $\epsilon = 10^{-3}$

see [7]. The test image is `mri.tif`, a 128×128 image taken from the `Image Processing Toolbox`. The results are reported in Table 2, and refer to the choice of $q = 15$. Moreover in Fig. 4 an example reconstruction is reported.

Table 2 Result of the restoration of Example 2

ϵ		ERR	IT	λ_{final}	SEC
	GAT	3.52e-1	6	6.02e-2	0.3
10^{-1}	RGAT	3.32e-1	16	3.52e-0	0.8
	AGAT	3.50e-1	7	5.80e-2	0.3
	GAT	1.90e-1	15	1.01e-3	0.6
10^{-2}	RGAT	1.61e-1	44	4.29e-1	2.3
	AGAT	1.86e-1	15	2.94e-3	0.5
	GAT	9.41e-2	32	8.34e-7	1.3
10^{-3}	RGAT	5.35e-2	96	1.54e-2	3.8
	AGAT	9.20e-2	32	1.62e-5	1.1

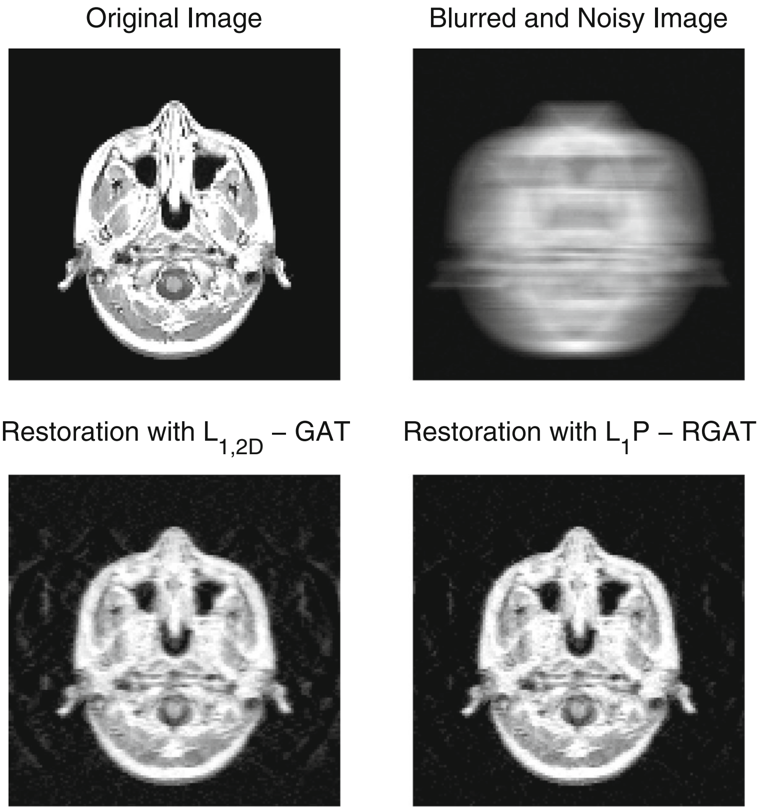

Fig. 4 Reconstruction of `mri.tif` affected by *motion blur*, obtained with the GAT and the RGAT methods, with $q = 15$ and $\epsilon = 10^{-3}$

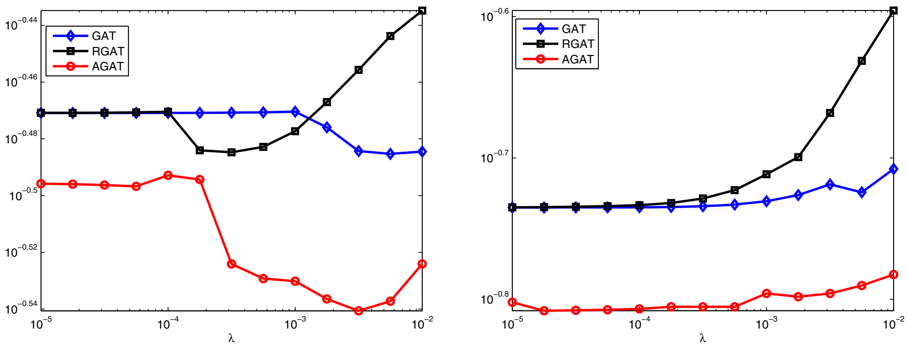

Fig. 5 Relative errors for the restoration with a fixed value of the parameter lambda. On the *left*: results for Example 1 with $q = 7$ $\sigma = 2$. On the *right*: results for Example 2 with $q = 15$. In both cases $\epsilon = 10^{-2}$

Finally, for both examples, we want to show that the two algorithm are less sensitive with respect to the choice of the regularization parameter. While the choice of this parameter depends on the algorithm used to define it, and the choice of the regularization operator, we consider the situation in which this parameter is a-priori fixed and not updated during the Arnoldi algorithm. The results are reported in Fig. 5, where the minimum attainable error versus λ is plotted.

6 Conclusions

After generalizing the Arnoldi-Tikhonov method as presented in [3], in this paper we have exposed two algorithms based on the reordering of the approximate solution in a restarted and an adaptive way. In both cases the methods seem able to detect a more effective regularization operator, as confirmed by the selection of the regularization parameters during the algorithms (cf. Tables 1 and 2). This makes both methods less sensitive to the choice of these parameters as showed in Fig. 5. The restarted method, RGAT, typically shows a substantial improvement in the quality of restoration, with a computational cost which remains comparable with the one of the standard procedure. On the other side, the quality of the reconstruction attainable with the AGAT method is generally quite close to the one of the GAT method. Some advantage can be observed in terms of the computational cost, since $L_1 P \in \mathbb{R}^{(N-1)\times N}$ while $L_{1,2D} \in \mathbb{R}^{2n(n-1)\times n^2}$ where $n^2 = N$.

References

1. Adluru, G., DiBella, E.V.R.: Reordering for improved constrained reconstruction from Undersampled k-Space data. Int. J. Biomed. Imaging **28**, Article ID 341684, 12 pages (2008)
2. Bjorck, A.: A bidiagonalization algorithm for solving large and sparse ill-posed systems of linear equations. BIT **28**, 659–670 (1988)
3. Calvetti, D., Morigi, S., Reichel, L., Sgallari, F.: Tikhonov regularization and the L-curve for large discrete ill-posed problems. J. Comput. Appl. Math. **123**, 423–446 (2000)
4. Gazzola, S., Novati, P.: Automatic parameter setting for Arnoldi-Tikhonov methods. Submitted (2012)
5. Hanke, M.: On Lanczos based methods for the regularization of discrete ill-posed problems. BIT **41**, 1008–1018 (2001)
6. Hansen, P.C.: Rank-deficient and discrete ill-posed problems: numerical aspects of linear inversion. SIAM, Philadelphia (1998)
7. Hansen, P.C.: Regularization Tools Version 4.0 for Matlab 7.3. Numer. Algorithms **46**, 189–194 (2007)
8. Hanke, M., Hansen, P.C.: Regularization methods for large-scale problems. Surv. Math. Ind. **3**, 253–315 (1993)
9. Kilmer, M.E., Hansen, P.C., Español, M.I.: A projection-based approach to general-form Tikhonov regularization. SIAM J. Sci. Comput. **29**, 315–330 (2007)
10. Lewis, B., Reichel, L.: Arnoldi-Tikhonov regularization methods. J. Comput. Appl. Math. **226**, 92–102 (2009)
11. Nagy, J.G., Palmer, K., Perrone, L.: Iterative methods for image deblurring: a matlab object oriented approach. Numer. Algorithms **36**, 73–93 (2004)
12. Rudin, L.I., Osher, S., Fatemi, E.: Nonlinear total variation based noise removal algorithms. Phys. D. **60**, 259–268 (1992)